Math 485, HW4

(1) Let $T : S^1 \to S^1$ be defined by $\theta_1 = \theta + 2\pi\alpha \ mod(2\pi)$ where α is a real number. Prove that T admits periodic orbit if and only if α is a rational number.

(2) Let $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $x_1 = 1 - 4x^2$. (a) Find all fixed points. (b) Show that $|x_n| \to \infty$ for all $x_0 > 1$ where $x_n = T^n(x_0)$.

(3) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $(x_1, y_1)^T = A(x, y)^T$ where A is a 2×2 matrix. We say F is area preserving if for any given region $D \subset \mathbb{R}^2$, we have $m(D) = m(F(D))$ where $m(D)$ is the area of D. Prove that F is area preserving if and only if $|\det(A)| = 1$.

(4) Find the time-1 map of the differential equation

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = x + 2y.$$

(5) For the differential equation

$$\begin{aligned}\frac{dx}{dt} &= x + y - x(x^2 + y^2)\\ \frac{dy}{dt} &= -x + y - y(x^2 + y^2),\end{aligned}$$

(a) find a periodic solution, and (b) derive the Poincaré return map around this periodic solution.

(6) For the differential equation

$$\begin{aligned}\frac{dr}{dt} &= -r + \cos\theta \sum_{n=-\infty}^{+\infty} \delta(t - nT)\\ \frac{d\theta}{dt} &= 2 + r\end{aligned}$$

where $\delta(t)$ is the standard δ-function, find the time-T map.

(7) Discuss the stability of the fixed points for the map $T(x) = \mu x(1 - x)$ for $2 < \mu < 6$.

(8) Let $x_0 \in [0, 1]$ be a periodic solution of period n for $T(x) = 7x(1 - x)$. Is x_0 stable? Why?

(9) Let $T : [0, 1] \to [0, 1]$ be defined by letting $T(x) = 2x$ for $x \in [0, 1/2]$ but $T(x) = 2 - 2x$ for $x \in (1/2, 1]$. Prove

(a) T has exactly 2^n-many periodic orbits of period n; and

(b) the set of periodic orbits is dense in $[0, 1]$.

Solution on (4): Let $X = (x, y)^T$. For the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

The eigenvalues are

$$\lambda_1 = \frac{3 + \sqrt{5}}{2}, \qquad \lambda_2 = \frac{3 - \sqrt{5}}{2}.$$

The eigenvector for λ_1 is $(1, (1 + \sqrt{5})/2)^T$. The eigenvalue for λ_2 is $(1, (1 - \sqrt{5})/2)^T$. Let

$$C = \begin{pmatrix} 1 & 1 \\ (1 + \sqrt{5})/2 & (1 - \sqrt{5})/2 \end{pmatrix}.$$

We have

$$C^{-1} = \begin{pmatrix} (1 - \sqrt{5})/2 & -1 \\ -(1 + \sqrt{5})/2 & 1 \end{pmatrix}$$

and

$$C^{-1}AC = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}.$$

We now introduce Y such that $X = CY$ to derive equation for Y as

$$\frac{dY}{dt} = C^{-1}ACY = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix} Y.$$

It then follows that

$$Y(t) = \begin{pmatrix} e^{\frac{3+\sqrt{5}}{2}t} & 0 \\ 0 & e^{\frac{3-\sqrt{5}}{2}t} \end{pmatrix} Y_0.$$

We finally have

$$X(t) = C \begin{pmatrix} e^{\frac{3+\sqrt{5}}{2}t} & 0 \\ 0 & e^{\frac{3-\sqrt{5}}{2}t} \end{pmatrix} C^{-1} X_0$$

The time map is the map from $X_0 \to X(1)$. It is the 2D map defined by the matrix

$$C \begin{pmatrix} e^{\frac{3+\sqrt{5}}{2}} & 0 \\ 0 & e^{\frac{3-\sqrt{5}}{2}} \end{pmatrix} C^{-1}.$$

Solution on (9)(b): If the periodic orbits are not dense in $[0, 1]$, there exists an interval $I_0 = [a, b]$ so that no periodic orbit is inside of I_0. Without loss of generality we assume I_0 is completely inside of either $[0.1/2]$ or $[1/2, 1]$. For the tent map T the slopes for both segments are 2. This implies that the size of the image of I_0 doubles that of I_0: $|T(I_0)| = 2|I_0|$. Let $I_i = T^i(I_0)$, and i_0 be the smallest i so that I_i crosses $x = 1/2$. This is bound to happen because we can not have $|I_i| = 2^i|I_0| < 1/2$ for all i. Now I_{i_0} is divided into two, both hold $x = 1/2$ as one end. We notice that this end will be held at $x = 0$ forever in future iterations of T. Since the images will double in size by 2 for each iteration of T. It will eventually cover I_0. But a map from a subset of I_0 coming back to cover I_0 must have a fixed point, which is a periodic orbit inside of I_0, a contradiction.