Math 485, HW5

(1) Starting with the interval $[0,1]$, we first take away the middle third to obtain two small intervals. The one that is taken away is $(1/3., 2/3)$ and the two remaining intervals are $[0,1/3]$ and $[2/3,1]$. Next we take away the middle third for each of the two remaining intervals to produce four even smaller intervals. Keep taking the middle third for each of the remaining intervals, and so on forever, we will end up with a set of point in $[0,1]$, which we name as the 1/3-Cantor set and denote as $\Lambda_{1/3}$. We let $m(\Lambda_{1/3})$ be 1 minus the total length of intervals that was taken away in this infinite process, and call $m(\Lambda_{1/3})$ the measure of $\Lambda_{1/3}$.

(a) Calculate $m(\Lambda_{1/3})$.

(b) We obtain $\Lambda_{1/4}$ by doing the same as in the above, but only taking away middle forth of each remaining interval. Calculate $m(\Lambda_{1/4})$.

(c) How about $\Lambda_{1/1000}$?

(2) Do the same construction as in the previous problem, with the adjustment that in first step, we take away middle 1/3, but in the second step we take away middle $(1/3)^2$, and in the n-th step we take away the middle $(1/3)^n$. We denote the resulted Cantor set as $K_{1/3}$.

(a) Calculate $m(K_{1/3})$.

(b) Calculate $m(K_{1/4})$.

(3) Let $T : \mathbb{R}^1 \to \mathbb{R}^1$ be defined by $T(x) = 7x(1-x)$. For any given $x \in [0,1]$, we let $A(x) = 1$ if $x \in [0,1/2)$, and $A(x) = 2$ if $x \in [1/2,1]$. We call $A(x)$ the *address* of x. For a given orbit $x_0, x_1, \cdots x_n, \cdots$ inside of $[0,1]$, we obviously have a sequence of $\{1,2\}$ as $A(x_0), A(x_1), \cdots, A(x_n), \cdots$.

Now prove the reverse, that is, for any given sequence of $\{1,2\}$, there exists orbit $x_0, x_1, \cdots, x_n \cdots$, such that $A(x_0), A(x_1), \cdots, A(x_n), \cdots$ is the given sequence.

(4) Let $y = 3x$ be the straight line in the (x,y)-plane. Sketch this curve on $[0,1] \times [0,1]$ representing the 2D torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. How about $y = 4x$? How about $y = 2x/3$?

(5) Prove that for Anosov map induce on the $2D$-torus $\mathbb{T}^2$ by using

$$x_1 = 2x + y, \quad y_1 = x + y,$$

all (x,y) of coordinates of rational numbers are on periodic orbits.