

# Dynamical Systems: A Brief Introduction

## 1. Objects of Study

**Phase Space  $M$ :** A geometric object (e.g. sphere, tori, open set in  $R^n$ ).

**A System  $T$ :** A mapping  $M \rightarrow M$ .

**Orbits:**  $x \rightarrow T(x) \rightarrow T(T(x)) \rightarrow \dots$

**Ex1:**  $M = R^1$ ,  $T : R^1 \mapsto R^1$  defined by

$$T(x) = 1 - 2x^2.$$

$$x = 0.1, x_1 = 0.98, x_2 = -0.9208, \dots$$

**Ex2:**  $M = S^1$ ,  $T(\theta) = \theta + \pi$ .

$$\theta = 1, \theta_1 = 1 + \pi, \theta_2 = 1, \dots$$

**Ex3:** Example:  $M = R^2$ ,  $T : (x, y) \rightarrow (x_1, y_1)$

$$T : \begin{cases} x_1 & = & 2x \\ y_1 & = & \frac{1}{2}y \end{cases}$$

$$z = (1, 1), z_1 = (2, \frac{1}{2}), z_2 = (2^2, \frac{1}{2^2}), \dots$$

**Ex4:**  $M = \mathbb{R}^2$ ,  $T : (r, \theta) \rightarrow (r_1, \theta_1)$

$$T : \begin{cases} r_1 = r \\ \theta_1 = \theta + r \end{cases}$$

$$z = (1, 0), z_1 = (1, 1), \dots, z_n = (1, n \bmod (2\pi)), \dots$$

– For  $T : M \rightarrow M$ , and  $x_0 \in M$  given, the orbit started from  $x_0$  is denoted as  $\{x_n\}_{n=0}^{\infty}$ .

– If  $T^{-1}$  exists, we say that  $T$  is invertible. We then are able to talk about backward orbit from  $x_0$ :  $x_{-1} = T^{-1}x_0$  and so on.

– Ex1 in the above is not invertible. Ex2, Ex3 and Ex4 are invertible.

## 2. Fundamental Questions

**Q1:** Behave of individual orbits.

– Fixed points:  $T(x) = x$ .

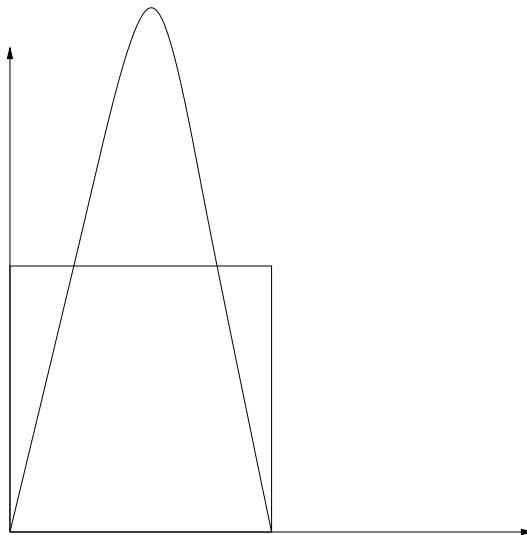
**Ex:**  $T(x) = \mu x(1 - x)$

$$x = \mu x(1 - x) \rightarrow x_1 = 0, \quad x_2 = 1 - \mu^{-1}.$$

– Periodic orbits:  $T^n(x) = x$ .

The smallest  $n$  is the period.

**Ex:**  $T(x) = 7x(1 - x)$



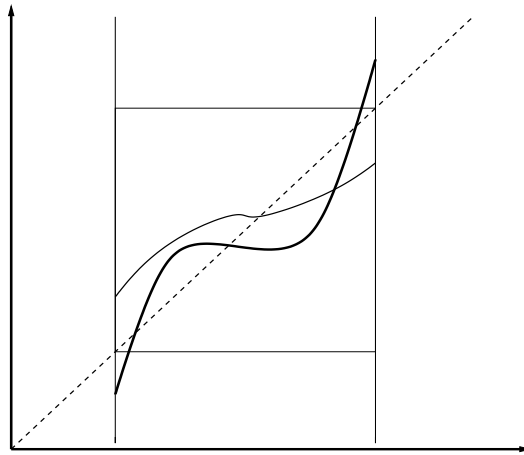
**Claim:** Periodic orbit of all period exists.

Proof: Let  $I_1 = [0, \frac{1}{2}]$ ,  $I_2 = [\frac{1}{2}, 1]$ .  $\forall n > 0$  given,  $\exists$  an interval  $I$ , such that

(i)  $T^i(I) \subset I_1, i = 0, 1, \dots, n - 2;$

(ii)  $T^{n-1}I \subset I_2, T^n I = I_1.$

*Fact:* Let  $T : I \rightarrow R$  be continuous such that  $T(I) \subset I$  (or  $T(I) \supset I$ ). The  $T$  has a fixed point in  $I$ .

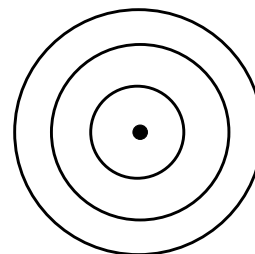
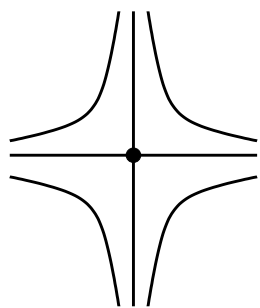


Note that the period of the orbit constructed is  $n$  by design.

## Q2: Organizations of Orbits.

– Phase portrait: (orbit structure)

$$\text{Example A: } T : \begin{cases} x_1 = 2x \\ y_1 = \frac{1}{2}y \end{cases}$$



Example B:  $T : \begin{cases} r_1 & = & r \\ \theta_1 & = & \theta + r \end{cases}$

– Local Stability:

Let  $x_0$  be a fixed point. An open neighborhood of  $x_0$  is denoted as  $U(x_0)$ ,  $V(x_0)$ , etc.

(a)  $x_0$  is **stable** if for every  $U(x_0)$  given, there exists an  $V(x_0)$  such that  $T^i V(x_0) \subset U(x_0)$  for all  $i \geq 1$ .

(b)  $x_0$  is **asymptotically stable** (a sink, or an attracting fixed point) if there exists  $U(x_0)$ , such that for all  $x \in U(x_0)$ ,  $T^i x \rightarrow x_0$  as  $i \rightarrow \infty$ .

$\{x \in M : T^i x \rightarrow x_0\}$  is the **attracting basin** for  $x_0$ .

(c) Let  $x_0 \in M$  be such that  $T^n x_0 = x_0$ , then  $x_0$  is a fixed point of  $T^n$ . (a) and (b) apply to define respectively **stable** and **asymptotically stable** periodic orbits.

Example B is Stable but not asymptotically stable.

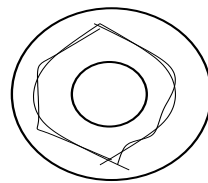
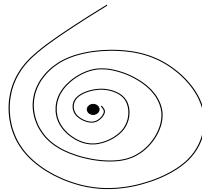
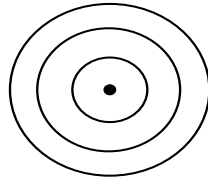
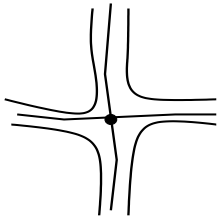
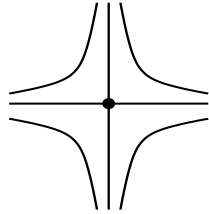
Example A is not stable.

No asymptotically stable fixed point for measure preserving maps.

$T(x) = \frac{1}{2}x$  has a unique fixed point  $x_0 = 0$  that is asymptotically stable. Its attracting basin is  $\mathbb{R}$ .

– Structure stability: Does the orbit structure of a given  $T$  change under small perturbation?

Hyperbolic structure and Elliptic structure



### 3. Source of Inspirations:

Differential Equation  $\Rightarrow$  Dynamical Systems

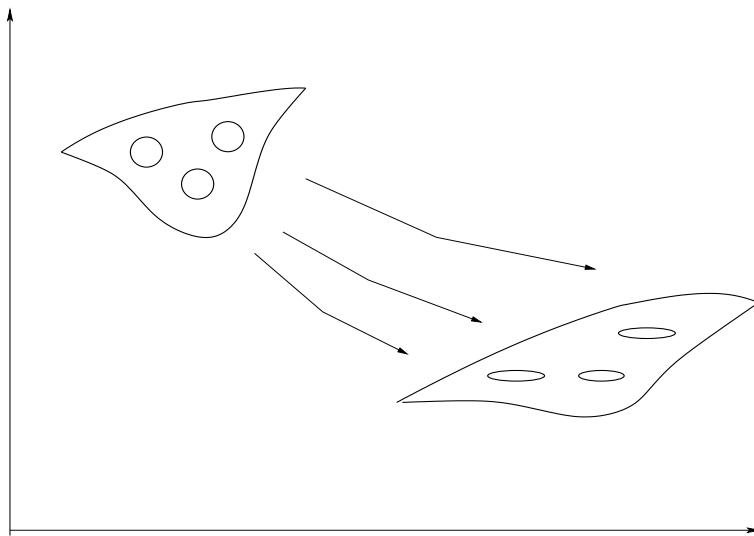
$$\frac{dx}{dt} = f(x, t) \quad x \in \mathbb{R}^n.$$

(a) Autonomous Equations:  $f(x, t) = f(x)$

Solution:  $x = x(t, x_0)$ ;

Map:  $T(x_0) = x(1, x_0)$ ;  $T^n(x_0) = x(n, x_0)$ .

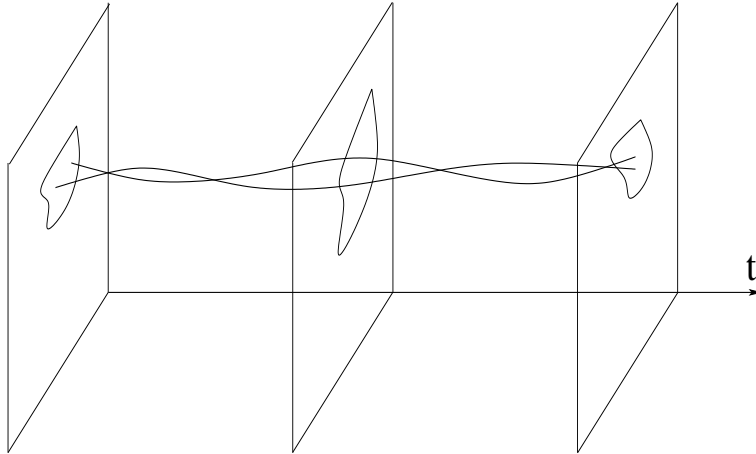
Arnold's cat:  $U \rightarrow TU$ .



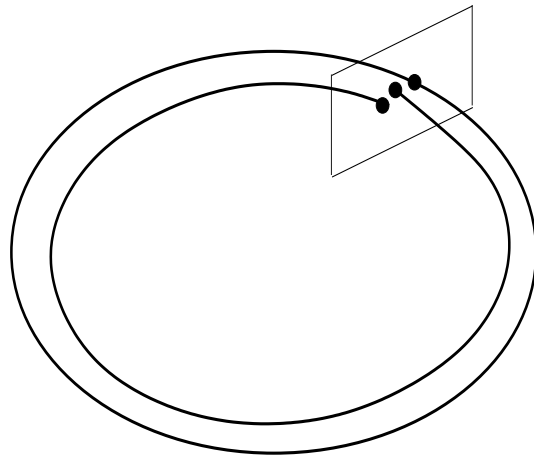
(b) Non-auto Equations:  $f(x, t) = f(x, t + T)$

Solution:  $x(t, x_0)$ ;

Map:  $T(x_0) = x(T, x_0)$ ;  $T^n(x_0, T) = x(nT, x_0)$ .



(c) Poincaré section.



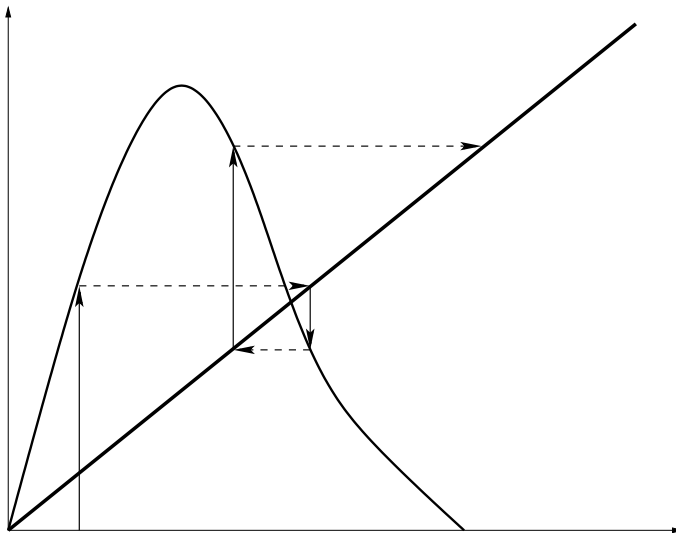
# 1D Dynamics: Periodic Orbits

**Maps of study:**  $T(x) : \mathbb{R} \rightarrow \mathbb{R}$ .

$T(x)$  is as smooth as we need along the way.

## 1. Graph of $T(x)$ and orbits

- Fixed points: Intersection of the graph  $y = T(x)$  and  $y = x$ .
- A given orbit: Trace the graph.



## 2. Stability of a fixed point

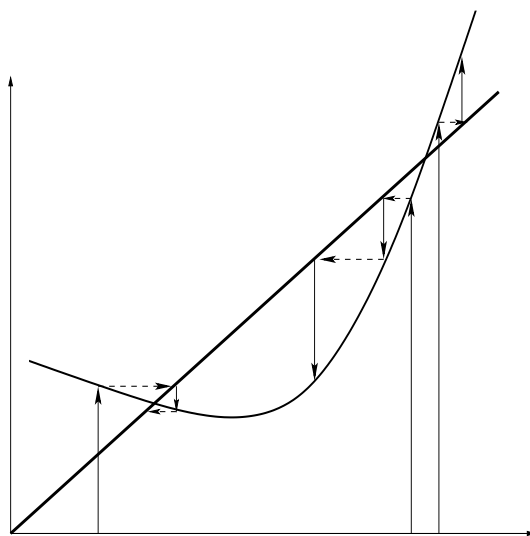
**Claim:** Let  $x_0$  be a fixed point.  $x_0$  is asymptotically stable if  $|T'(x_0)| < 1$ . It is unstable if  $|T'(x_0)| > 1$ .

**Proof:** If  $|T'(x_0)| < 1$ , then by continuity there exist  $I(x_0)$  (an interval contains  $x_0$ ), such that  $|T'(x)| < \lambda < 1$  for all  $x \in I(x_0)$ . By mean value theorem then,

$$|T(x) - x_0| = |T(x) - T(x_0)| < \lambda|x - x_0|$$

for all  $x \in I(x_0)$ . This implies  $|T^n(x) - x_0| < \lambda^n|x - x_0| \rightarrow 0$ . The other half is similar.

*A demonstration using graph*



### 3. $|T'(x_0)| = 1$ : Degenerate case

**Ex:**  $T(x) = x + x^3$ : unstable at  $x = 0$ .

**Proof:**  $T'(x) = 1 + 3x^2 > 1$  around  $x = 0$ . So  $|T(x) - 0| > |x - 0|$  for all  $x \neq 0$ .

**Ex:**  $T(x) = x - x^3$ : asymptotically stable at  $x = 0$ .

**Proof:**  $T'(x) = 1 - 3x^2 < 1$  around  $x = 0$ . So  $|T(x) - 0| < |x - 0|$ . Starting from, say,  $x \neq 0$ ,  $\{x_n\}$  is a decreasing sequence. So  $x_n \rightarrow \hat{x}$ .  $\hat{x}$  must be a fixed point. So  $\hat{x} = 0$  and  $x_n \rightarrow 0$ .

**Ex:**  $T(x) = x^3 \sin \frac{1}{x} + x$ : Stable but not asymptotically stable at  $x = 0$ .

**Proof:** Fixed points of this  $T(x)$  is defined by

$$x^3 \sin \frac{1}{x} = 0.$$

This is a case in which  $T(x)$  has infinitely many fixed points accumulating at  $x = 0$ .

## 4. Existence of periodic orbits

**Claim:** Let  $T : \mathbb{R} \rightarrow \mathbb{R}$ . If  $T$  has a periodic orbit of period three, then it has periodic orbit of all periods.

**Proof:** Assume  $a < b < c$  is such that  $f(a) = b$ ,  $f(b) = c$  and  $f(c) = a$ . Let  $I_0 = [a, b]$ ,  $I_1 = [b, c]$ , we have  $T(I_0) \supset I_1$  and  $f(I_1) \supset I_0 \cup I_1$ .

*Basic observation:* For any interval  $A$  such that  $T^i(A) \supset I_1$ , there are two sub-intervals  $A^0, A^1 \subset A$ , such that  $T^{i+1}(A^0) = I_0$ ,  $T^{i+1}(A^1) = I_1$ .

Let  $n$  be fix, we will be able to find an sub-interval  $A$  in  $I_1$  such that

(a)  $T^i(A) \subset I_1$  for all  $i < n - 1$ ;

(b)  $T^{n-1}(A) = I_0$ .

Since  $T^n(A) = T(I_0) = I_1 \supset A$ .  $T^n$  has a fixed point, which is a periodic orbit of  $T$  of period  $n$ . The period of this orbit can not be less than  $n$  by design.

## 5. Sarkovskii's Theorem

### Sarkovskii order

$$3 \triangleright 5 \triangleright 7 \triangleright \dots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \dots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \dots \\ \triangleright 2^m \cdot 3 \triangleright 2^m \cdot 5 \triangleright \dots \triangleright 2^n \triangleright 2^{n-1} \triangleright \dots \triangleright 2 \triangleright 1.$$

*Remark:* We can always write an integer  $n$  in the form  $n = p2^m$  where  $p \geq 1$  is odd and  $m \geq 0$  be positive.

**Theorem:** Assume that  $T : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. If  $T$  has a periodic orbit of period  $n$ , then for all  $n'$  such that  $n \triangleright n'$ ,  $T$  has a periodic orbit of period  $n'$ .

- The previous claim (period three implies all period) is a special case of this claim.
- If a 1D map has only finitely many periodic solutions, their period has to be multiples of 2.
- This claim is true only for interval maps (not even true for maps from  $S^1$  to  $S^1$ ).

## Homework

1. Find  $T^{-1}$  for Ex. 2-4 in the first two pages.
2. Let  $x_0$  be a periodic point of period  $n$  in  $[0, 1]$  for  $T(x) = 7x(1 - x)$ . Is  $x_0$  stable? Why?
3. Let  $x \in \mathbb{R}^n$ , and  $A = (a_{ij})_{n \times n}$  be a constant matrix. Find the time-1 map of the equation  $\frac{dx}{dt} = Ax$ .
4. For the given set of differential equations

$$\begin{aligned}\frac{dx}{dt} &= x + y - x(x^2 + y^2)^{\frac{1}{2}} \\ \frac{dy}{dt} &= -x + y - y(x^2 + y^2)^{\frac{1}{2}}\end{aligned}$$

- (a) Find all periodic solutions of this equation.
  - (b) Let  $S$  be the  $y$ -axis. Find the Poincare map induced by this equation around the indicated periodic solutions.
5. Discuss the stability of the fixed points of  $T(x) = \mu x(1 - x)$  for  $2 < \mu < 5$ .
  6. Let  $T(x) = x^3 - \lambda x$  for  $\lambda > 0$ .
    - (a) Find all periodic points and discuss their stabilities for  $0 < \lambda < 1$ .
    - (b) Prove that, if  $|x|$  is sufficiently large, then  $|f^n(x)| \rightarrow \infty$ .

7. Suppose  $A_0, A_1, \dots, A_n$  are closed intervals and  $T(A_i) \supset A_{i+1}$  for  $i = 0, \dots, n-1$ . Prove that there is a point  $x \in A_0$  such that  $T^i(x) \in A_i$  for all  $i \leq n$ .