RANK ONE ATTRACTORS IN A PERIODICALLY PERTURBED SECOND ORDER SYSTEM

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Abstract. In this paper we study a time dependent second order system in the form of

$$\frac{d^2q}{dt^2} + (a - bq^2)\frac{dq}{dt} - q + q^3 = \mu \sin \omega t$$

where $a > 0, b, \mu, \omega$ are parameters. We prove that this equation admits strange attractors with SRB measures for a positive measure set of parameters (a, b, μ, ω).

Periodically perturbed second order systems have been studied extensively in the history of the modern theory of chaos and dynamical systems [GH]. When a homoclinic solution is periodically perturbed, transversal intersections of stable and unstable manifolds occur within certain range of forcing parameters, generating homoclinic tangles and strange attractors. For concrete second order systems, a well-defined computational procedure (the Melnikov' method) has been developed in verifying the existence of the indicated homoclinic tangle. See, for an instance, Sects. 4.5 and 4.6 of [GH], for a detailed presentation on Melnikov's method and its applications to a list of periodically perturbed systems.

In [WO] we proved that, for certain periodically forced second order systems, there exists a form of chaos that is less complicated in a range of parameters where the perturbed stable and unstable manifolds *do not* intersect. We started with a *non-resonant* and *dissipative* saddle that admits a homoclinic solution. The perturbation we use is in the form of $(-\mu(\rho h(x, y) + \sin \omega t), \mu(\rho h(x, y) + \sin \omega t))$ where μ, ρ, ω are forcing parameters. The forced equations are written as an autonomous system in three dimensional space (x, y, θ) where x, y are phase variables and θ is an angular variable representing time. A 2D Poincaré section in the space of (x, y, θ) is constructed, and the return maps induced by the three-dimensional autonomous flow are explicitly computed. We proved that the return maps obtained are admissible families of rank one maps. The existence of strange attractors with SRB

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measures followed from a dynamics theory developed in recent years by Wang and Young ([WY1]-[WY3]) base on the theory of Benedicks and Carleson on strongly dissipative Hénon maps [BC2].

In this paper we apply the analysis of [WO] to the study of a time periodic second order system in the form of

$$\frac{d^2q}{dt^2} + (a - bq^2)\frac{dq}{dt} - q + q^3 = \mu \sin \omega t \tag{1}$$

where $a > 0, b, \mu, \omega$ are parameters. We prove that this equation admits strange attractors with SRB measures for a positive measure set of parameters (a, b, μ, ω). Observe that the unperturbed equation for (1) is the exercise 4.6.4 in Sect. 4.6 of [GH], a system that has been studied extensively in the past [HR].

Let us finish by noting that the analysis introduced in [WO] and in this paper can also be extended to the studies of many other periodically forced second order systems. We chose equation (1) because it fit most easily into the particular setup of [WO]. By properly modifying the setup of [WO], we could for example obtain similar results for periodically forced pendulum in the form of

$$\frac{d^2\theta}{dt^2} + \sin\theta - \varepsilon(\alpha - \delta v) = \mu \sin \omega t.$$

We will present the details of this result elsewhere.

1. Statement of results

We call a saddle fixed point of a two dimensional autonomous system a *dissipative* saddle if the magnitude of the associated negative eigenvalue is larger than that of the positive eigenvalue. We call a saddle point a *homoclinic* saddle if it is approached by one solution, which we call a *homoclinic* solution, from both the positive and the negative directions of time.

We start with the second order system

$$\frac{d^2q}{dt^2} - q + q^3 = 0.$$

Denote $p = \frac{dq}{dt}$. We write this equation as

$$\frac{dq}{dt} = p, \qquad \frac{dp}{dt} = q - q^3. \tag{2}$$

With a simple linear change of coordinates

$$x = \frac{1}{2}(q - p), \qquad y = \frac{1}{2}(q + p),$$

we write equation (2) in (x, y) as

$$\frac{dx}{dt} = -x + \frac{1}{2}(x+y)^3, \qquad \frac{dy}{dt} = y - \frac{1}{2}(x+y)^3. \tag{3}$$

Let

$$a(t) = \frac{2\sqrt{2}e^{3t}}{(1+e^{2t})^2}, \qquad b(t) = \frac{2\sqrt{2}e^{t}}{(1+e^{2t})^2}.$$

$(x, y) = (0, 0)$ is a homoclinic saddle and $(x, y) = (a(t), b(t))$ is a homoclinic solution initiated at $(\sqrt{2}, \sqrt{2})$. Denote

$$\ell = \{\ell(t) = (a(t), b(t)),\ t \in \mathbb{R}\}.$$

$(x, y) = (0, 0)$ is, however, not dissipative.

We construct systems with rank one attractors in two steps. First we add autonomous perturbations to the right of equation (3) to obtain

$$\begin{aligned} \frac{dx}{dt} &= -x + \frac{1}{2}(x+y)^3 + \gamma(y+x)^2(y-x) \\ \frac{dy}{dt} &= y - \lambda y - \frac{1}{2}(x+y)^3 - \gamma(y+x)^2(y-x). \end{aligned} \tag{4}$$

Proposition 1.1. *There exists $\lambda_0 > 0$ sufficiently small, such that for $\lambda \in (0, \lambda_0)$, there exists a $\gamma_\lambda, |\gamma_\lambda| < 10\lambda$ such that for $\gamma = \gamma_\lambda$;*

(i) equation (4) has a homoclinic solution for $(x, y) = (0, 0)$, which we denote as

$$\ell_\lambda = \{\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t)),\ t \in \mathbb{R}\};$$

(ii) for any given $L > 0$, there exists a $K(L)$ independent of λ, such that for all $t \in [-L, L]$,

$$|\ell_\lambda(t) - \ell_0(t)| < K(L)\lambda$$

where $\ell_0(t) = \ell(t) = (a(t), b(t))$.

Next we let $\gamma = \gamma_\lambda$ in equation (4) for $\lambda \in (0, \lambda_0)$. We add time dependent perturbations to the right of equation (4) to obtain

$$\begin{aligned} \frac{dx}{dt} &= -x + \frac{1}{2}(x+y)^3 + \gamma_\lambda(y+x)^2(y-x) \\ &\quad - \mu(\rho(y+x)^2(y-x) + \sin\omega t), \\ \frac{dy}{dt} &= (1-\lambda)y - \frac{1}{2}(x+y)^3 - \gamma_\lambda(y+x)^2(y-x) \\ &\quad + \mu(\rho(y+x)^2(y-x) + \sin\omega t). \end{aligned} \tag{5}$$

The following is the main theorem of this paper.

Theorem A *There exists a positive measure set $\mathbb{D} \subset (0, \lambda_0)$ for λ, an interval (ρ_1, ρ_2) for ρ, and an ω_0 sufficiently large for ω, such that* *for any given (λ, ρ, ω), $\lambda \in \mathbb{D}$, $\rho \in (\rho_1, \rho_2)$ and $\omega > \omega_0$, there exists a parameter set $\Delta_{\lambda,\rho,\omega}$ for μ with positive Lebesgue density at $\mu = 0$, such that for all $\mu \in \Delta_{\lambda,\rho,\omega}$, equation (5) admits a strange attractor with SRB measures.*

Let $q = x + y$. We have from (5),

$$\frac{d^2q}{dt^2} = \beta_1 q - \beta_2 q^3 - (a - bq^2)\frac{dq}{dt} + (1-\lambda)\mu \sin \omega t \tag{6}$$

where

$$\beta_1 = (1-\lambda), \quad \beta_2 = [1 - \frac{1}{2}\lambda - (\gamma_\lambda(1-\lambda) + \mu\rho(1-\lambda))\frac{\lambda}{2-\lambda}],$$
$$a = \lambda, \quad b = -(\gamma_\lambda(1-\lambda) + \mu\rho(1-\lambda))(1 - \frac{\lambda}{2-\lambda}).$$

Changing t and q by

$$\tilde{t} = \sqrt{\beta_1}t, \quad \tilde{q} = \sqrt{\frac{\beta_2}{\beta_1}}q,$$

we write equation (6) as

$$\frac{d^2\tilde{q}}{d\tilde{t}^2} - \tilde{q} + \tilde{q}^3 + (\tilde{a} - \tilde{b}\tilde{q}^2)\frac{d\tilde{q}}{d\tilde{t}} = \tilde{\mu}\sin\tilde{\omega}\tilde{t} \tag{7}$$

where

$$\tilde{a} = \frac{a}{\sqrt{\beta_1}}, \quad \tilde{b} = \frac{b\sqrt{\beta_1}}{\beta_2}, \quad \tilde{\mu} = \frac{(1-\lambda)}{\beta_1}\sqrt{\frac{\beta_2}{\beta_1}}\mu, \quad \tilde{\omega} = \frac{\omega}{\sqrt{\beta_1}}.$$

We have

Theorem B *There exists a positive measure set Δ of parameters $(\tilde{a}, \tilde{b}, \tilde{\mu}, \tilde{\omega})$ so that the corresponding equation (7) admits a strange attractor with SRB measures.*

Theorem B follows directly from Theorem A.

2. Existence of a dissipative homoclinic saddle

In Sect. 2.1 we study the unperturbed equation (3), evaluating a few important definite integrals. In Sects. 2.2-2.4 we prove Proposition 1.1 for the perturbed autonomous equation (4).

2.1. **On the unperturbed system.** Let us recall that $\ell(t) = (a(t), b(t))$ where

$$a(t) = \frac{2\sqrt{2}e^{3t}}{(1+e^{2t})^2}, \qquad b(t) = \frac{2\sqrt{2}e^{t}}{(1+e^{2t})^2} \tag{8}$$

is a homoclinic solution of equation (3). It follows that

$$\begin{aligned} u(t) &= \frac{-(e^{2t}-3)}{\sqrt{(e^{2t}-3)^2 + (e^{-2t}-3)^2}} \\ v(t) &= \frac{e^{-2t}-3}{\sqrt{(e^{2t}-3)^2 + (e^{-2t}-3)^2}}, \end{aligned} \tag{9}$$

where

$$(u(t), v(t)) = \left| \frac{d}{dt}\ell(t) \right|^{-1} \frac{d}{dt}\ell(t)$$

is the unit tangent vector of ℓ at $\ell(t)$.

Let us re-write equation (3) as

$$\frac{dx}{dt} = -x + f(x,y) \qquad \frac{dy}{dt} = y + g(x,y) \tag{10}$$

where

$$f(x,y) = \frac{1}{2}(x+y)^3, \qquad g(x,y) = -\frac{1}{2}(x+y)^3.$$

Let

$$\begin{aligned} E(t) = {} & v^2(t)(-1 + \partial_x f(a(t), b(t))) + u^2(t)(1 + \partial_y g(a(t), b(t))) \\ & - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))). \end{aligned}$$

$E(t)$ measures the rate of expansion of the solutions of equation (10) in the direction normal to ℓ at $\ell(t)$. For the homoclinic solution $\ell(t) = (a(t), b(t))$,

$$E(t) = -\frac{(e^{-2t}-3)^2 - (e^{2t}-3)^2}{(e^{2t}-3)^2 + (e^{-2t}-3)^2} \left(1 - \frac{12e^{2t}}{(1+e^{2t})^2} \right). \tag{11}$$

Let

$$K(s) = -\int_0^s E(s)ds. \tag{12}$$

Lemma 2.1. *For $s \in (-\infty, \infty)$,*

$$K(s) = \frac{1}{2} \ln \frac{8e^{2s}((1-3e^{2s})^2 + e^{4s}(e^{2s}-3)^2)}{(e^{2s}+1)^6}.$$

Proof: Let $x = e^{2t}$. We have[1]

$$K(s) = \frac{1}{2}\int_1^{e^{2s}} \frac{1}{x}\frac{(1-3x)^2 - x^2(x-3)^2}{(1-3x)^2 + x^2(x-3)^2}\left(1 - \frac{12x}{(1+x)^2}\right)dx.$$

Observe that

$$(1-3x)^2 + x^2(x-3)^2 = ((x-a)^2 + a^2)((x-b)^2 + b^2) \tag{13}$$

where $a, b > 0$ satisfying $a + b = 3,\ ab = \frac{1}{2}$. We have

$$\begin{aligned}
K(s) &= \frac{1}{2}\int_1^{e^{2s}} \frac{(-x^4 + 6x^3 - 6x + 1)(x^2 - 10x + 1)}{x(x+1)^2((x-a)^2 + a^2)((x-b)^2 + b^2)}dx \\
&= \frac{1}{2}\int_1^{e^{2s}} \left(\frac{1}{x} - \frac{6}{1+x} + \frac{4x^3 - 18x^2 + 36x - 6}{((x-a)^2 + a^2)((x-b)^2 + b^2)}\right)dx \\
&= \frac{1}{2}\int_1^{e^{2s}} \left(\frac{1}{x} - \frac{6}{1+x} + \frac{2(x-a)}{(x-a)^2 + a^2} + \frac{2(x-b)}{(x-b)^2 + b^2}\right)dx.
\end{aligned}$$

We then have

$$\begin{aligned}
K(s) &= \frac{1}{2}\ln\frac{8e^{2s}((e^{2s} - a)^2 + a^2)((e^{2s} - b)^2 + b^2)}{(e^{2s} + 1)^6} \\
&= \frac{1}{2}\ln\frac{8e^{2s}((1 - 3e^{2s})^2 + e^{4s}(e^{2s} - 3)^2)}{(e^{2s} + 1)^6}.
\end{aligned}$$

□

Through direct evaluation using Lemma 2.1, we obtain the values of the following definite integrals as

$$\begin{aligned}
&\int_{-\infty}^{\infty} (u(s) + v(s))(b(s) + (a(s))^2(b(s) - a(s))e^{-\int_0^s E(t)dt}ds = \frac{16}{15}; \\
&\int_{-\infty}^{\infty} u(s)b(s)e^{-\int_0^s E(t)dt}ds = -\int_{-\infty}^{\infty} v(s)a(s)e^{-\int_0^s E(t)dt}ds = \frac{2}{3}; \\
&\int_{-\infty}^{\infty} v(s)b(s)e^{-\int_0^s E(t)dt}ds = \int_{-\infty}^{\infty} u(s)a(s)e^{-\int_0^s E(t)dt}ds = 0.
\end{aligned} \tag{14}$$

We have two more integrals to evaluate, and they are

$$\begin{aligned}
C(\omega) &= \int_{-\infty}^{\infty} u(s)\cos(\omega s)e^{-\int_0^s E(t)dt}ds, \\
S(\omega) &= \int_{-\infty}^{\infty} u(s)\sin(\omega s)e^{-\int_0^s E(t)dt}ds.
\end{aligned} \tag{15}$$

[1] The author would like to thank Ali Oksasoglu for making this computation.

Lemma 2.2. *We have*

$$C(\omega) = \frac{\sqrt{2}\pi\omega^2}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}, \qquad S(\omega) = -\frac{\sqrt{2}\pi\omega}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}.$$

Proof: Using Lemma 2.1, we have

$$C(\omega) = -2\sqrt{2}I_c, \qquad S(\omega) = -2\sqrt{2}I_s$$

where

$$I_c = \int_{-\infty}^{\infty} \frac{e^{3s}(e^{2s}-3)}{(e^{2s}+1)^3} \cdot \cos(\omega s) \cdot ds$$

$$I_s = \int_{-\infty}^{\infty} \frac{e^{3s}(e^{2s}-3)}{(e^{2s}+1)^3} \cdot \sin(\omega s) \cdot ds.$$

Let $I = I_c + iI_s$.

$$I = \int_{-\infty}^{\infty} \frac{e^{3s}(e^{2s}-3)}{(e^{2s}+1)^3} \cdot e^{i\omega s} \cdot ds.$$

We evaluate I as follows. On the $z = x + iy$ plan, let

$$\ell_1 = \{x, \quad x : -\infty \to \infty\}, \qquad \ell_2 = \{x + \pi i, \quad x : \infty \to -\infty\}$$

and

$$f(z) = \frac{e^{3z}(e^{2z}-3)}{(e^{2z}+1)^3} \cdot e^{i\omega z}.$$

We have

$$\int_{\ell_1} f(z)dz = I, \qquad \int_{\ell_2} f(z)dz = e^{-\omega\pi}I.$$

So by the residue theorem,

$$(1 + e^{-\omega\pi})I = 2\pi i Res(f(z))_{z=\frac{\pi i}{2}}.$$

Let $t = z - \frac{\pi i}{2}$, we write $f(z)$ as

$$\begin{aligned} f(z) &= -ie^{-\frac{1}{2}\omega\pi} \frac{e^{3t}(e^{2t}+3)}{(e^{2t}-1)^3} \cdot e^{i\omega t} \\ &= -\frac{i}{8} e^{-\frac{1}{2}\omega\pi} \frac{e^{(5+\omega i)t} + 3e^{(3+\omega i)t}}{t^3(1+t+\frac{2}{3}t^2+\mathcal{O}(t^3))^3}. \end{aligned}$$

We have

$$\begin{aligned} e^{(5+i\omega)t} + 3e^{(3+i\omega)t} &= 4 + (14 + 4i\omega)t \\ &\quad + (26 + 14i\omega - 2\omega^2)t^2 + \mathcal{O}(t^3). \end{aligned}$$

We also have

$$(1 + t + \frac{2}{3}t^2)^{-3} = 1 - 3t + 4t^2 + \mathcal{O}(t^3).$$

Consequently,

$$Res(f(z)) = -\frac{i}{4}e^{-\frac{1}{2}\omega\pi}(i\omega - \omega^2),$$

from which it follows that

$$I = \frac{1}{2}\frac{\pi e^{-\frac{1}{2}\omega\pi}}{1 + e^{-\omega\pi}}(i\omega - \omega^2).$$

In conclusion, we have

$$C(\omega) = \frac{\sqrt{2}\pi\omega^2}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}, \qquad S(\omega) = -\frac{\sqrt{2}\pi\omega}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}.$$

□

The rest of this section is devoted to the proof of Proposition 1.1. There are two small scales of different magnitude in the proof of this proposition, which we represent using $\lambda << \varepsilon << 1$. ε represents the size of a small neighborhood of $(x, y) = (0, 0)$ that allows the conclusions of typical local analysis, such as the computations of the local stable and unstable manifolds of equations (3) and (4), to be valid. In principle it is exclusively determined by equation (3). Parasites to ε are the small neighborhood

$$U_\varepsilon = \{(x, y) : x^2 + y^2 < 4\varepsilon^2\},$$

and two parameters $L^\pm$, the times in the positive and the negative direction respectively for the homoclinic solution of equation (3) to hit $U_{\frac{1}{2}\varepsilon}$. By symmetry we have $L^+ = L^- = L$. It is important that L is *not* an independent parameter. It is a quantity determined completely by ε and the homoclinic solution $\ell(t) = (a(t), b(t))$.

The second small scale is represented by the parameter λ in equation (4). To control the size of the parameter γ through λ, we write $\gamma = \kappa\lambda$, and assume in the rest of this section that

$$|\kappa| < 10. \tag{16}$$

The conventional way of representing the magnitude of a term with $\mathcal{O}(\cdot)$ is used in conjunction with the two different scales as follows:

– $\mathcal{O}(\varepsilon)$ represents a quantity, the magnitude of which is bounded by $K\varepsilon$ where K is a constant independent of ε and λ (provided that both are sufficiently small, of course). By adjusting the size of ε, these terms can be set as small as we wish.

– $\mathcal{O}(\lambda)$ represents a term, the magnitude of which is bounded from above by $K(\varepsilon)\lambda$ where $K(\varepsilon)$ is a constant depending on ε (and L). By adjusting the size of λ, regarding ε as being fixed, we can set these terms, again, arbitrarily small. As a function of ε, however, $K(\varepsilon)$ could go to infinite in size as $\varepsilon \to 0$.

2.2. **Around the hyperbolic fixed point.** In this subsection we compute the local stable and unstable manifolds of $(0,0)$ for equation (4). All coordinate changes are define on U_ε where the size of ε is determined by the unperturbed equation (3).

A. First coordinate change: $(x, y) \to (\xi, \eta)$

Denote the local stable manifold of equation (10) at $(x, y) = (0, 0)$ as $y = S(x)$ and the local unstable manifold as $x = U(y)$. We have

$$\begin{aligned} -U(y) + f(U(y), y) &= U'(y)(y + g(U(y), y)) \\ S(x) + g(x, S(x)) &= S'(x)(-x + f(x, S(x))). \end{aligned}$$

where $U'(y)$ is the derivative of $U(y)$ with respect to y and $S'(s)$ is the derivative of $S(x)$ with respect to x. We introduce

$$\xi = x - U(y), \qquad \eta = y - S(x) \tag{17}$$

to transform equation (10) into

$$\begin{aligned} \frac{d\xi}{dt} &= (-1 + \hat{f}(\xi, \eta))\xi, \\ \frac{d\eta}{dt} &= (1 + \hat{g}(\xi, \eta))\eta \end{aligned}$$

where

$$\begin{aligned} \hat{f}(\xi, \eta)\xi &= f(\xi + U(y), y) - f(U(y), y) \\ &\quad - U'(y)(g(\xi + U(y), y) - g(U(y), y)) \\ \hat{g}(\xi, \eta)\eta &= g(x, \eta + S(x)) - g(x, S(x)) \\ &\quad - S'(x)(f(x, \eta + S(x)) - f(x, S(x))). \end{aligned}$$

Observe that because $f(x, y), g(x, y)$ are degree three in (x, y), $\hat{f}(\xi, \eta), \hat{g}(\xi, \eta)$ start with the second degree terms in (ξ, η). We now study the perturbed equation (4), which we write as

$$\begin{aligned} \frac{dx}{dt} &= -x + f(x, y) + \gamma h(x, y) \\ \frac{dy}{dt} &= (1 - \lambda)y + g(x, y) - \gamma h(x, y) \end{aligned} \tag{18}$$

where

$$h(x, y) = (y + x)^2(y - x).$$

Let the relations between x, y and ξ, η be defined through (17). We derive the corresponding equations for (18) in ξ, η. We have

$$\begin{aligned}\frac{d\xi}{dt} &= -x + f(x,y) - \gamma h(x,y) \\ &\quad -U'(y)((1-\lambda)y + g(x,y) + \gamma h(x,y)) \\ &= (-1 + \hat{f}(\xi,\eta))\xi + \lambda y U'(y) - \gamma h(x,y)(1 + U'(y)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{d\eta}{dt} &= (1-\lambda)y + g(x,y) + \gamma h(x,y) \\ &\quad -S'(x)(-x + f(x,y) - \gamma h(x,y)) \\ &= (1 + \hat{g}(\xi,\eta))\eta - \lambda y + \gamma h(x,y)(1 + S'(x)).\end{aligned}$$

We obtain the new equations in ξ, η for (18) as

$$\begin{aligned}\frac{d\xi}{dt} &= (-1 + \hat{f}(\xi,\eta))\xi + \lambda w_1(\xi,\eta) \\ \frac{d\eta}{dt} &= (1 - \lambda + \hat{g}(\xi,\eta))\eta + \lambda w_2(\xi,\eta)\end{aligned} \tag{19}$$

where

$$\begin{aligned}w_1(\xi,\eta) &= yU'(y) - \kappa h(x,y)(1 + U'(y)) \\ w_2(\xi,\eta) &= -S(x) + \kappa h(x,y)(1 + S'(x)).\end{aligned}$$

Recall that $\kappa = \gamma\lambda^{-1}$, and we assume that $|\kappa| < 10$. $w_1(\xi,\eta)$ and $w_2(\xi,\eta)$ are terms of order at least three in ξ, η.

B. The second coordinate change: $(\xi,\eta) \to (X,Y)$

We denote the local unstable manifold of $(0,0)$ for equation (19) as

$$\xi = \lambda W^u(\eta,\lambda)$$

and the local stable manifold as

$$\eta = \lambda W^s(\xi,\lambda).$$

We have

Lemma 2.3. *There exists an $\varepsilon > 0$, independent of λ, and a $\lambda_0 > 0$ sufficiently small, so that $W^u(\eta,\lambda), W^s(\xi,\lambda)$ are analytically defined on*

$$(-\varepsilon,\varepsilon) \times [0,\lambda_0).$$

In addition, there exists K independent of ε and λ so that

$$|W^u(\eta,\lambda)|, \quad |W^s(\xi,\lambda)| < K\varepsilon^2.$$

Proof: Let

$$X = \xi - \lambda \sum_{k=2}^{\infty} u_k \eta^k.$$

We have

$$\begin{aligned}
\frac{dX}{dt} &= (-1 + \hat{f}(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta))X + \lambda(-1 + \hat{f}(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta)) \sum_{k=2}^{\infty} u_k \eta^k \\
&\quad + \lambda w_1(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) - \lambda \sum_{k=2}^{\infty} k u_k \eta^k (1 - \lambda + \hat{g}(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta)) \\
&\quad - \lambda^2 \sum_{k=2}^{\infty} k u_k \eta^{k-1} w_2(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta).
\end{aligned}$$

Let us set the new equation for X as

$$\begin{aligned}
\frac{dX}{dt} &= (-1 + \hat{f}(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta))X \\
&+ \lambda \sum_{k=2}^{\infty} u_k \eta^k (\hat{f}(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) - \hat{f}(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta)) \\
&+ \lambda \left(w_1(X + \sum_{k=1}^{\infty} \lambda u_k \eta^k, \eta) - w_1(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) \right) \\
&- \lambda \sum_{k=2}^{\infty} k u_k \eta^k (\hat{g}(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) - \hat{g}(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta)) \\
&- \lambda^2 \sum_{k=2}^{\infty} k u_k \eta^{k-1} \left(w_2(X + \sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) - w_2(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) \right)
\end{aligned}$$

which we can re-write as

$$\frac{dX}{dt} = (-1 + \hat{f}(X, \eta) + \mathcal{O}(\lambda))X. \tag{20}$$

To obtain (20) we should set

$$\begin{aligned}
&- \sum_{k=2}^{\infty} u_k \eta^k + \sum_{k=2}^{\infty} u_k \eta^k \hat{f}(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) + w_1(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta) \\
&- \sum_{k=2}^{\infty} k u_k \eta^k (1 - \lambda + \hat{g}(\sum_{k=2}^{\infty} \lambda u_k \eta^k, \eta)) \\
&- \lambda \sum_{k=2}^{\infty} k u_k \eta^{k-1} w_2(\lambda \sum_{k=2}^{\infty} u_k \eta^k, \eta) = 0
\end{aligned} \tag{21}$$

to solve for $u_k, k \geq 2$. Let us observe that (i) $u_k, k \geq 2$ are inductively determined by (21), and (ii) since there is no resonance issue here, the convergence of the formal series $\sum_{k=2}^{\infty} u_k \eta^k$ can be proved by the classical majorant method. We are not having a resonance issue here because our objective is to flat stable manifold, not to conjugate the equations with its linear part.

Observe that the local unstable stable manifold is defined by $X = 0$. We have

$$W^u(\eta, \lambda) = \sum_{k=2}^{\infty} u_k \eta^k.$$

This proves the $W^u(\eta, \lambda)$ part. Proof for $W^s(\xi, \lambda)$ is similar.

Note that it is important for us to assume that $\kappa < K$ with a K that is independent of *both* ε and λ (In the current case $K = 10$). This is because the upper bound of u_k and v_k is, in general, a dependent of K, and we would not be able to get the estimates of this lemma if K depends on either ε or λ. □

Corollary 2.1. *On the local stable manifold of equation (18) for* $(\xi, \eta) = (0, 0)$, $\eta = \lambda \mathcal{O}(\varepsilon^2)$*; and on the local unstable manifold,* $\xi = \lambda \mathcal{O}(\varepsilon^2)$.

Proof: Follows directly from Lemma 2.3. □

2.3. **Around the homoclinic loop.** Let $\ell(s) = (a(s), b(s))$ be the homoclinic loop for the unperturbed equation (10). We have

$$\frac{da(s)}{ds} = -a(s) + f(a(s), b(s)), \qquad \frac{db(s)}{ds} = b(s) + g(a(s), b(s)). \tag{22}$$

Denote

$$u(s) = \frac{-a(s) + f(a(s), b(s))}{\sqrt{(-a(s) + f(a(s), b(s)))^2 + (b(s) + g(a(s), b(s)))^2}},$$
$$v(s) = \frac{b(s) + g(a(s), b(s))}{\sqrt{(-a(s) + f(a(s), b(s)))^2 + (b(s) + g(a(s), b(s)))^2}}$$

and let

$$\mathbf{e}(s) = (v(s), -u(s)).$$

Let (s, z) be such that

$$(x, y) = \ell(s) + z\mathbf{e}(s).$$

We have

$$x = x(s, z) := a(s) + v(s)z, \quad y = y(s, z) := b(s) - u(s)z. \tag{23}$$

Differentiating (23), we obtain

$$\begin{aligned}
\frac{dx}{dt} &= (-a(s) + f(a(s), b(s)) + v'(s)z)\frac{ds}{dt} + v(s)\frac{dz}{dt} \\
\frac{dy}{dt} &= (b(s) + g(a(s), b(s)) - u'(s)z)\frac{ds}{dt} - u(s)\frac{dz}{dt}
\end{aligned} \tag{24}$$

where $u'(s) = \frac{du(s)}{ds}$, $v'(s) = \frac{dv(s)}{ds}$. Let us denote

$$\begin{aligned}
F_\lambda(s, z) &= -(a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)) \\
&\quad + \gamma h(a(s) + zv(s), b(s) - zu(s)), \\
G_\lambda(s, z) &= (b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)) \\
&\quad - \lambda(b(s) - zu(s)) - \gamma h(a(s) + zv(s), b(s) - zu(s)).
\end{aligned}$$

We obtain from equation (24) the new equations for (18) in s, z as

$$\begin{aligned}
\frac{ds}{dt} &= \frac{u(s)F_\lambda(s, z) + v(s)G_\lambda(s, z)}{\sqrt{F_0(s, 0)^2 + G_0(s, 0)^2} + z(u(s)v'(s) - v(s)u'(s))} \\
\frac{dz}{dt} &= v(s)F_\lambda(s, z) - u(s)G_\lambda(s, z).
\end{aligned}$$

We now re-write these equations as

$$\begin{aligned}
\frac{ds}{dt} &= 1 + \mathcal{O}(z) + \lambda B(s) \\
\frac{dz}{dt} &= (E(s) + \mathcal{O}(\lambda))z + \mathcal{O}(z^2) \\
&\quad + \lambda(b(s)u(s) + \kappa(u(s) + v(s))h(a(s), b(s)))
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
B(s) &= \frac{-v(s)b(s) + \kappa(u(s) - v(s))h(a(s), b(s))}{\sqrt{F_0(s, 0)^2 + G_0(s, 0)^2}}; \\
E(s) &= v^2(s)(-1 + \partial_x f(a(s), b(s)) + u^2(s)(1 + \partial_y g(a(s), b(s))) \\
&\quad - u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s))).
\end{aligned}$$

We caution that, in using $\mathcal{O}(z)$, we represent a term that is $< K(L)|z|$ where $K(L)$ is a dependent of L. This is because we have

$$\sqrt{F_0(s, 0)^2 + G_0(s, 0)^2}$$

in the denominator. For $s \in [-2L, 2L]$, this is a small number depending on L.

Let us now re-scale the variable z by letting

$$Z = \lambda^{-1} z$$

we arrive at the following equations for the new variables (s, Z)

$$(26)\quad \begin{aligned} \frac{ds}{dt} &= 1 + \mathcal{O}(\lambda) \\ \frac{dZ}{dt} &= E(s)Z + \mathcal{O}(\lambda) + b(s)u(s) + \kappa(u(s) + v(s))h(a(s), b(s)). \end{aligned}$$

We caution that equation (26) is valid only on a prefixed domain

$$\mathcal{D} = \{(s, Z), \quad s \in [-2L, 2L], \ |Z| < \hat{K}\}$$

where $L, \hat{K} > 0$ are independent of λ.

Let $(s(t), Z(t))$ be the solution of equation (26) initiated at $(-L, Z_0)$. We obtain at $s = L$,

$$(27)\quad Z(L) = P_L(Z_0 + \Phi_L + \mathcal{O}(\lambda))$$

where

$$P_L = e^{\int_{-L}^{L} E(s)ds};$$

$$\Phi_L = \int_{-L}^{L} (u(s)b(s) + \kappa(u(s) + v(s))h(a(s), b(s)))e^{-\int_{-L}^{s} E(\tau)d\tau} ds.$$

(27) is what we will use in proving Proposition 1.1.

2.4. **Existence of a homoclinic loop.** In this subsection we prove Proposition 1.1. For a given $\varepsilon > 0$ sufficiently small, there exists $L > 0$ so that the homoclinic loop $\ell(t) = (a(t), b(t)), t \geq L$ for the unperturbed system is such that

$$(28)\quad \begin{aligned} \xi(-t) = a(-t) - U(b(-t)) = 0, \\ \eta(t) = b(t) - S(a(t)) = 0 \end{aligned}$$

where ξ, η are the variables defined through (17).

Let K be a constant independent of λ, and

$$(29)\quad \begin{aligned} \sigma_K^- &= \{(x, y) : x = x(-L, z), \ y = y(-L, z), \ |z| \leq \lambda K\} \\ \sigma_K^+ &= \{(x, y) : x = x(L, z), \ y = y(L, z), |z| < \lambda K\}. \end{aligned}$$

Let us also use the scaled variable $Z = \lambda^{-1}z$ and denote

$$\sigma_K^{\pm} = \{(s, Z) : \quad s = \pm L, \ |Z| < K\}.$$

Let $\mathbf{W}^s, \mathbf{W}^u$ be the local stable and unstable manifold of the perturbed equation (18) for $(0, 0)$, and

$$p^- = \mathbf{W}^u \cap \sigma^-, \quad p^+ = \mathbf{W}^s \cap \sigma^+.$$

Denote the Z-coordinate for p^- as Z_- and the Z-coordinate for p^+ as Z_+.

Lemma 2.4. *We have*

$$Z_- = \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda), \qquad Z_+ = \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda).$$

Proof: First we compute Z_-. Let $\ell(-L) = (a(-L), b(-L))$ be the intersection point of ℓ with σ_K^-. We have $\xi = 0, \eta = \varepsilon$ for $\ell(-L)$.

We compute the corresponding values of ξ for $p = (-L, z) \in \sigma^-$. By definition,

$$\begin{aligned}
\xi &= a(-L) + v(-L)z - U(b(-L) - u(-L)z) \\
&= v(-L)z - U(b(-L) - u(-L)z) + U(b(-L)) \\
&= (v(-L) + U'(b(-L))u(-L) + \mathcal{O}(z))z
\end{aligned}$$

where the first line of (28) is used for the second equality. By using the first line of (28), we also have

$$u(-L) - U'(b(-L))v(-L) = 0.$$

It then follows that

$$\xi = \left(\frac{1}{v(-L)} + \mathcal{O}(z) \right) z.$$

On the other hand, since p^- is on the unstable manifold of equation (18), by Corollary 2.1 we have for p^-,

$$\xi = \lambda \mathcal{O}(\varepsilon^2).$$

We put the last two ξ together to obtain,

$$Z_- = \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda).$$

Proof for Z_+ is similar. □

Proof of Proposition 1.1: Let $(s(t), Z(t))$ be the solution of equation (26) initiated at $p^- = (-L, Z_-)$. Observe that in (27), $P_L = 1$ from Lemma 2.1. So we have

$$Z(L) = Z_- + \Phi_L + \mathcal{O}(\lambda).$$

In order for the perturbed equation (4) to have a homoclinic loop, it suffices for us to have $Z_+ = Z(L)$. This is to say that

$$Z_+ = Z_- + \Phi_L + \mathcal{O}(\lambda). \tag{30}$$

We also have from Lemma 2.4 that

$$Z_- = \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda), \quad Z_+ = \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda).$$

So from (30) we obtain

$$\kappa = \frac{\mathcal{O}(\varepsilon) + \mathcal{O}(\lambda) - \int_{-L}^{L} u(s)b(s)e^{-\int_{-L}^{s} E_0(\tau)d\tau} ds}{\int_{-L}^{L} (u(s) + v(s))h(a(s), b(s))e^{-\int_{-L}^{s} E(\tau)d\tau} ds}.$$

which we re-write as

$$\kappa = \frac{e^{\int_{-L}^{0} E(t)dt} \cdot (\mathcal{O}(\varepsilon) + \mathcal{O}(\lambda)) - \int_{-L}^{L} u(s)b(s)e^{-\int_0^s E(\tau)d\tau}ds}{\int_{-L}^{L}(u(s)+v(s))h(a(s),b(s))e^{-\int_0^s E(\tau)d\tau}ds}. \tag{31}$$

We caution that, because all the derivations in this subsection has be subjected to the assumption that $\kappa < 10$. This solution is only valid if it satisfies

$$\kappa < 10. \tag{32}$$

From Lemma 2.1, we have

$$e^{\int_{-L}^{0} E(t)dt} = K(-L) = K(L) << 1.$$

(32) now follows from (14). This proves Proposition 1.1(i).

To indicate explicitly the dependency on λ, let us denote the intersection point of the local unstable manifold with σ^- as p_λ^-. We make $\ell_\lambda(t)$ uniquely determined by letting $\ell_\lambda(-L) = p_\lambda^-$. Denote this homoclinic solution written in (x, y)-space as $\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t))$. In particular, $\ell_0(t) = (a_0(t), b_0(t))$ is the homoclinic solution for the unperturbed equation (10). Proposition 1.1(ii) follows from

$$|\ell_\lambda(-L) - \ell_0(-L)| = \mathcal{O}(\lambda) \tag{33}$$

and the continuous dependency of solutions with respect to vector fields and initial conditions. □

3. Proof of Theorem A

Our proof of Theorem A applies the main theorem of [WO], the setups of which we present in details in Sect. 3.1. Theorem A is proved in Sect. 3.2.

3.1. **The main theorem of [WO].**

The setups and the main theorem of [WO] are as follows. Let $(x, y) \in \mathbb{R}^2$ be the phase variables and t be the time. We start with an autonomous system

$$\frac{dx}{dt} = -\alpha x + f(x, y), \qquad \frac{dy}{dt} = \beta y + g(x, y) \tag{34}$$

where $f(x, y), g(x, y)$ are real analytic at $(x, y) = (0, 0)$, and $f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0$. We assume that α, β satisfy certain Diophantine non-resonance condition and $(x, y) = (0, 0)$ is a dissipative saddle point. Namely,

(H1) *(i) there exists $d_1, d_2 > 0$ so that for all $n, m \in \mathbb{Z}^+$,*

$$|n\alpha - m\beta| > d_1(|n| + |m|)^{-d_2};$$

(ii) $0 < \beta < \alpha$.

Let us also assume that the positive x-side of the local stable manifold and the positive y-side of the local unstable manifold of $(0,0)$ are included as part of a homoclinic solution, which we denote as $x = a(t), y = b(t)$. Let

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$

We further assume that $f(x,y), g(x,y)$ are C^4 in a sufficiently small neighborhood surrounding ℓ.

To the right of equation (34) we add a time-periodic term to form a non-autonomous system

$$\begin{aligned} \frac{dx}{dt} &= -\alpha x + f(x,y) - \mu(\rho h(x,y) + \sin \omega t), \\ \frac{dy}{dt} &= \beta y + g(x,y) + \mu(\rho h(x,y) + \sin \omega t) \end{aligned} \tag{35}$$

where μ, ρ and ω are parameters. $0 < \mu << 1$ controls the magnitude of the forcing term and ρ, ω are much larger parameters, the ranges of which we will make explicit momentarily. To study (35), we introduce an angular variable $\theta \in S^1$ to write it as

$$\begin{aligned} \frac{dx}{dt} &= -\alpha x + f(x,y) - \mu(\rho h(x,y) + \sin \theta) \\ \frac{dy}{dt} &= \beta y + g(x,y) + \mu(\rho h(x,y) + \sin \theta) \\ \frac{d\theta}{dt} &= \omega. \end{aligned} \tag{36}$$

We denote

$$(u(t), v(t)) = \left| \frac{d}{dt}\ell(t) \right|^{-1} \frac{d}{dt}\ell(t)$$

where $\ell(t) = (a(t), b(t))$ is the homoclinic loop of equation (34). $(u(t), v(t))$ is a unit vector tangent to ℓ at $\ell(t)$. In what follows we let

$$\begin{aligned} E(t) = v^2(t)(-\alpha + \partial_x f(a(t), b(t))) + u^2(t)(\beta + \partial_y g(a(t), b(t))) \\ - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))). \end{aligned} \tag{37}$$

$E(t)$ measures the rate of expansion of the solutions of equation (34) in the direction normal to ℓ at $\ell(t)$.

In what follows we let

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} (u(s) + v(s)) h(a(s), b(s)) e^{-\int_0^s E(\tau)d\tau} ds \\
C &= \int_{-\infty}^{\infty} (u(s) + v(s)) \cos(\omega s) e^{-\int_0^s E(\tau)d\tau} ds \qquad (38) \\
S &= \int_{-\infty}^{\infty} (u(s) + v(s)) \sin(\omega s) e^{-\int_0^s E(\tau)d\tau} ds.
\end{aligned}
$$

A, C and S are all well-defined. We assume that

(H2) (i) $A \neq 0$ and (ii) $C^2 + S^2 \neq 0$.

For a given equation (35) satisfying (H1) and (H2), we let

$$
\rho_1 = -\frac{200}{101}\frac{\sqrt{C^2+S^2}}{A}, \quad \rho_2 = -\frac{400}{99}\frac{\sqrt{C^2+S^2}}{A}. \qquad (39)
$$

We also let

$$
I = \{z \in \mathbb{R}, \ |z| < K\mu\}
$$

for some $K > 1$ sufficiently large independent of μ; and

$$
\Sigma_I = \{\ell(0) + (v(0), -u(0))z \in \mathbb{R}^2, z \in I\} \times S^1.
$$

We have

Theorem C (Main Theorem of [WO]) *Assume (H1) and (H2) for (35). Then there exists $\omega_0 > 0$ such that for every ω satisfying $|\omega| > \omega_0$ and every $\rho \in [\rho_1, \rho_2]$, there exists a parameter set $\Delta_{\omega,\rho}$ for μ with positive Lebesgue density at $\mu = 0$, such that for every $\mu \in \Delta_{\omega,\rho}$, (i) equation (36) induces a well-defined family of return maps $\mathcal{F} = \mathcal{F}_\mu : \Sigma_I \to \Sigma_I$ such that $\mathcal{F}(\Sigma_I) \subset \Sigma_I$; and (ii) $\mathcal{F}$ admits an ergodic SRB measures on Σ_I.*

We recall that an $\mathcal{F}$-invariant Borel probability measure m on Σ is an *SRB measure* if (i) $\mathcal{F}$ has a positive Lyapunov exponent m-a.e.; (ii) the conditional measures of m on unstable manifolds are absolutely continuous with respect to the Riemannian measures on these unstable leaves. SRB measures represent *visible statistical law* in chaotic systems.

The proof of Theorem C in [WO] is based on a theory of rank one maps developed by Wang and Young in recent years ([WY1], [WY2]). It is proved in [WO] that, assuming (H1) and (H2), the flow induced map $\mathcal{F} : \Sigma_I \to \Sigma_I$, for μ sufficiently small, is well defined and is an admissible family of rank one maps. Consequently, all results of [WY1] for maps of good parameters apply to $\mathcal{F}_\mu$. In particular, we have the existence of SRB measures as stated in Theorem C(ii).

3.2. **Proof of Theorem A.** Let λ_0 be sufficiently small satisfying Proposition 1.1, and $\Delta \subset (0, \lambda_0)$ be such that, for $\lambda \in \Delta$,

$$|n-(1-\lambda)m| > \lambda_0^2(n+m)^{-2} \tag{40}$$

for all $n, m \in \mathbb{Z}^+$. Let $\lambda \in \mathbb{D}$ and $\gamma = \gamma_\lambda$ be fixed in equation (4). We re-write that equation as

$$\begin{aligned} \frac{dx}{dt} &= -x + f_\lambda(x,y) \\ \frac{dy}{dt} &= (1-\lambda)y + g_\lambda(x,y) \end{aligned} \tag{41}$$

where

$$f_\lambda(x,y) = -g_\lambda(x,y) = \frac{1}{2}(x+y)^3 + \gamma_\lambda(y+x)^2(y-x).$$

Equation (41) is naturally in the form assumed for the unperturbed equation (34) in Sect. 3.1 with

$$\alpha = 1, \quad \beta = 1-\lambda.$$

$(x,y) = (0,0)$ is a *dissipative homoclinic saddle* for equation (41) and the homoclinic solution is $\ell_\lambda = (a_\lambda(t), b_\lambda(t))$ from Proposition 1.1(i). Equation (5), which we write as

$$\begin{aligned} \frac{dx}{dt} &= -x + f_\lambda(x,y) - \mu(\rho(x+y)^2(x-y) + \sin \omega t), \\ \frac{dy}{dt} &= (1-\lambda)y + g_\lambda(x,y) + \mu(\rho(x+y)^2(x-y) + \sin \omega t), \end{aligned} \tag{42}$$

assumes the form of equation (35) with

$$h(x,y) = (x+y)^2(x-y).$$

We regard (41) as the unperturbed system, and (42) as the forced equation. Let

$$(u_\lambda(t), v_\lambda(t)) = \left|\frac{d}{dt}\ell_\lambda(t)\right|^{-1} \frac{d}{dt}\ell_\lambda(t).$$

Write the correspondences of $E(t)$ of (37), A, C, S of (38) as $E_\lambda(t)$, $A_\lambda, C_\lambda, S_\lambda$ respectively, we have

(43)

$$\begin{aligned} E_\lambda(t) = v_\lambda^2(t)(-1 + \partial_x f_\lambda(a_\lambda(t), b_\lambda(t))) + u_\lambda^2(t)(1-\lambda+\partial_y g_\lambda(a_\lambda(t), b_\lambda(t))) \\ - u_\lambda(t)v_\lambda(t)(\partial_y f_\lambda(a_\lambda(t), b_\lambda(t)) + \partial_x g_\lambda(a_\lambda(t), b_\lambda(t))); \end{aligned}$$

$$(44)\qquad \begin{aligned} A_\lambda &= \int_{-\infty}^{\infty} (u_\lambda(s) + v_\lambda(s)) h(a_\lambda(s), b_\lambda(s)) e^{-\int_0^s E_\lambda(\tau) d\tau} ds \\ C_\lambda &= \int_{-\infty}^{\infty} (u_\lambda(s) + v_\lambda(s)) \cos(\omega s) e^{-\int_0^s E_\lambda(\tau) d\tau} ds \\ S_\lambda &= \int_{-\infty}^{\infty} (u_\lambda(s) + v_\lambda(s)) \sin(\omega s) e^{-\int_0^s E_\lambda(\tau) d\tau} ds. \end{aligned}$$

We also let $A_{\lambda,L}, C_{\lambda,L}, S_{\lambda,L}$ be the ones obtained by changing the integral bounds from $\pm\infty$ to $\pm L$ respectively in (44).

Our next two lemmas make connection between the proof of Theorem A and the computations of Sect. 2.1.

Lemma 3.1. *There exist λ_0 sufficiently small and L_0 sufficiently large, such that for any given $L > L_0$,*

$$|A_\lambda - A_{\lambda,L}|, \quad |C_\lambda - C_{\lambda,L}|, \quad |S_\lambda - S_{\lambda,L}| < 10^3 e^{-\frac{1}{2}L}$$

for all $\lambda \in [0, \lambda_0)$.

Proof: Observe that, for $t \in (-\infty, -L)$, $E_\lambda(t) < -\frac{1}{2}$ and for $t \in (L, \infty)$, $E_\lambda(t) > \frac{1}{2}$ provided that L is sufficiently large and λ sufficiently small. □

We also have

Lemma 3.2. *For any given $L > 0$, there exists $K(L)$ independent of λ so that*

$$|A_{\lambda,L} - A_{0,L}|, \quad |C_{\lambda,L} - C_{0,L}|, \quad |S_{\lambda,L} - S_{0,L}| < K(L)\lambda.$$

Proof: With L been fixed independent of λ, this lemma follows from Proposition 1.1(ii). □

Proof of Theorem A: We apply Theorem C to equation (42). Let $\lambda \in \mathbb{D}$ be fixed. (40) is (H1)(i) and (H1)(ii) holds because $\alpha = 1$, $\beta = 1 - \lambda$. For (H2) we use Lemmas 3.1, 3.2, (14) and Lemma 2.2 for A, C, S. To prove $A_\lambda \neq 0$, we first chose $L > L_0$ sufficiently large so that

$$10^3 e^{-\frac{1}{2}L} < 10^{-3} A.$$

Note that $A = \frac{16}{15}$ from (14). It then follows from Lemma 3.1 that

$$|A_\lambda - A_{\lambda,L}|, \quad |A_0 - A_{0,L}| < 10^{-3} A.$$

Next we let λ_0 be sufficiently small so that

$$|A_{\lambda,L} - A_{0,L}| < K(L)\lambda_0 < 10^{-3} A$$

where $K(L)$ is as in Lemma 3.2. We obtain

$$A_\lambda \geq A_0 - |A_0 - A_{0,L}| - |A_{\lambda,L} - A_{0,L}| - |A_\lambda - A_{\lambda,L}| > 1.$$

Proof for (H2)(ii) is similar. □

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