On the Dynamics of Homoclinic Tangles

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Part I

Dynamics of Homoclinic Tangles
Unperturbed equation

\[ \frac{dx}{dt} = -\alpha x + f(x, y), \]
\[ \frac{dy}{dt} = \beta y + g(x, y) \]

- Dissipative saddle: \( 0 < \beta < \alpha \).

- Homoclinic solution: (0, 0) is with a homoclinic solution

\[ \ell = \{ \ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \ t \in \mathbb{R} \}. \]
Periodically perturbed equation

\[
\frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y, t), \\
\frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y, t).
\]

Two Scenarios:

Scenario (a) Scenario (b)

— Long history dates back to Poincaré.

— Chaos theory has been focused on (a).
Smale’s Horseshoe (1960)

- The horseshoe map

- Horseshoe implies complicated dynamics

The dynamics of a horseshoe map conjugates to the full shift of two symbols.

- Scenario (a) implies a horseshoe map
Analysis of a Given Equation

Part I  Theoretic Analysis

• Compute the Melnikov function to verify that we are in scenario (a);

• Use Smale’s construction to confirm the existence of complicated dynamics.

part II  Numerical Evidence

• To find a few values of $\mu$ that offer numerical plots of chaos
A Gap Between Theory and the Plots:

Plots are those of physical measures, not that of horseshoes.

Physical measures:

Invariant distribution with an attractive basin of positive Lebesgue measure.

- Sinks represent stable dynamics;
- SRB measures representing chaos.
- Horseshoe is always with an attractive basin of measure zero.
A Recent Dynamical Theory

There are THREE major dynamical scenarios:

(I) Transient tangles Homoclinic tangles admits no physical measures.

(II) Tangles dominated by sinks The only physical measures admitted are sinks.

(III) Hénon-like attractors SRB measures associated to Newhouse tangency representing chaos.

As \( \mu \to 0 \), there is a invariant pattern of dynamical behavior that would be repeat in an accelerated fashion as \( \mu \to 0 \).

Multiplicative Period: \( \mu_1 = e^{\beta T} \mu_0 \).
Part II

Numerical Investigation
Equation of Study

Step 1: We start with

\[ \frac{d^2q}{dt^2} - q + q^2 = 0. \]

Step 2: Add non-linear damping terms

\[ \frac{d^2q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^2 = 0. \]

For \( \lambda > 0 \) small, there exists \( \gamma_\lambda \) so that it has a homoclinic loops to a dissipative saddle.

Step 3: Add periodic perturbations

\[ \frac{d^2q}{dt^2} + (\lambda - \gamma_\lambda q^2) \frac{dq}{dt} - q + q^2 = \mu \sin 2\pi t \]
Numerical Simulation

- Numerical scheme
  - Fourth-order Runge-Kutta routine

- Values of $\lambda$ and $\gamma$
  \[
  \lambda = 0.5, \quad \gamma_{\lambda} = 0.5770285901
  \]

- Initial phase position
  \[
  (q_0, p_0) = (0.01, 0.0).
  \]

- Parameters varied
  - $\mu$ ranged from $10^{-3}$ to $10^{-7}$.
  - $t_0$ ranged in $[0, 1)$.
  - Compute the solution for each fixed combination of $\mu$ and $t_0$. 
Simulation Results

- Transient Tangle
  - Let $t_0$ run over $[0, 1)$ with an increment of 0.001.
  - all 1,000 solutions leave the neighborhood of the homoclinic solution.
  - Homoclinic tangle contains no object directly observable.

- Periodic Sink
- Hénon-Like Attractor

These are plots of a chaotic attractor close to a periodic solution.

They represent a Hénon-like attractor associated to Newhouse tangency (Mora-Viana)
Theory of Newhouse Tangency (1970’s)

- Return map on $B_n$ is essentially a Hénon family after re-normalization.

  — Newhouse’s infinitely many sinks.
  

- Persistency of tangency.

  — Tangency is structured as intersection of two Cantor sets.

  — Small change of parameters can not do away the tangency.
### Periodicity of Dynamical Behavior

Theoretical Multiplicity $= e^{\beta T} = 2.1831$

<table>
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<tr>
<th>$\mu$</th>
<th>Dynamics</th>
<th>Actual Ratio</th>
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<td>$7.041 \cdot 10^{-5}$</td>
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Summary

(I) A Theory on Homoclinic Tangles

We provided a comprehensive description on the overall dynamical structure of homoclinic tangles from a periodically perturbed homoclinic solution.

(II) Theories on maps come together

Horseshoes, Newhouse sinks and Hénon-like attractors all fall into their places as part of a larger picture.

(III) Applications

Our results can be applied to the analysis of given equations, such as Duffing equation.

(IV) A systematic numerical investigation

The behavior of numerical solutions are entirely predictable and are consistent with the predictions of our new theory.