On the Homoclinic Tangles of Henri Poincaré

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(Joint with Ali Osasaoglu)
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Part I

Snapshots of History
King Oscar II’s Prize Problem

• A prize for solving the N-body problem

Given a system of arbitrarily many mass points that attract each other according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

– Acta Mathematica, vol. 7, of 1885-1886

• Prize was awarded to Henri Poincaré for his work on the restricted 3-body problem

In this particular case, I have found a rigorous proof of stability and a method of placing precise limits on the elements of the third body... I now hope that I will be able to attack the general case and ... if not completely resolve the problem (of this I have little hope), then at least found sufficiently complete results to send into the competition.

– Henri Poincaré, 1887
A Prize Mess

• Hugo Gyldén informed the prize committee that the prize was wrongly bestowed.
  — He published a paper two years earlier, in which Poincaré stability claim was proved.
  — Poincaré said Gyldén’ paper is not readable and is inconclusive.

• Poincaré then found a fatal mistake in his own essay.
  — The stability conclusion he claimed was wrong.
  — Acta Mathematica with the prize essay was recalled and all destroyed.
  — Poincaré completely re-wrote his essay and kept the prize.
  — Poincaré paid double the amount of the prize money to cover the cost of the recall.

• Power series solution constructed much later.
  By Sundman for the 3-body problem (1912) and by myself for all \( N \) (1985).
Homoclinic tangles of Henri Poincare

\[ \frac{dx}{dt} = f(x) + \varepsilon g(x, t) \]

- The mistake Poincaré made

- The homoclinic tangle of Henri Poincaré

- Appeared to be an incompressible mess.
Smale’s Horseshoe (1960)

- The horseshoe map

- Horseshoe embedded in homoclinic tangle

- Melnikov method

A standard computational procedure in verifying chaos in differential equations
Hénon Attractors

\[ H_{a,b} : \quad x_1 = 1 - ax^2 + y, \quad y_1 = bx \]

- Numerical evidence of strange attractors

- Theory on Hénon maps

**Theorem**  For every \( b \) sufficiently small, there exists a positive measure set \( \Delta_b \subset (1, 2) \), such that for \( a \in \Delta_b \), \( H_{a,b} \) has a positive Lyapunov exponent Lebesgue almost everywhere on \( (x, y) \in (0, 2) \times (-1, 1) \)

Remark: This theorem is mainly due to Benedicks and Carleson. Proof is long and hard.
The Theory of Newhouse Tangency

- Return map on $B_n$ is essentially a Hénon family after re-normalization.

  — Newhouse’s infinitely many sinks.

  — Hénon-like attractors (Mora-Viana).

- Persistency of tangency.

  — Tangency is structured as intersection of two Cantor sets.

  — Small change of parameters can not do away the tangency.
Part II

Structure of Homoclinic Tangles
The Question of Homoclinic Tangles

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha x + f(x, y) + \mu P(x, y, t), \\
\frac{dy}{dt} &= \beta y + g(x, y) + \mu Q(x, y, t).
\end{align*}
\]

- **Fundamental object:** \( \Lambda = \) the maximum set of solutions staying around the homoclinic loop.

- **Fundamental question:** The geometric and dynamical structure of \( \Lambda \)?

  — Horseshoe is only a participating part.
The separatrix map $\mathcal{R}$

$\mathcal{R}$ is an annulus map proposed by Afraimovich and Shilnikov.

It is only partially defined on the annulus.

We rigorously derived a formula for $\mathcal{R}$ from the equation.
The structure of homoclinic tangle

- $\mathcal{R}$ is defined on vertical strips.

- **The action of $\mathcal{R}$ on one strip**
  - Compressing in vertical direction
  - Stretching in horizontal direction, to infinity in length towards both end
  - Folded, and put back into $\mathcal{A}$.

- **The image moves horizontally in a constant speed with respect to** $a \sim \ln \mu^{-1}$ as $\mu \to 0$. 
Dynamical Consequences

(I) There exists an infinitely many disjoint open intervals for $\mu$ as $\mu \to 0$, such that $\Lambda_\mu$ is conjugate to a horseshoe of infinitely many branches.

(II) There exists an infinitely many disjoint open intervals for $\mu$ as $\mu \to 0$, such that $\Lambda_\mu$ is the union of an periodic sink and a horseshoe of infinitely many branches.

(III) There are infinitely many open intervals of $\mu$ for $\mu \to 0$, such that $\Lambda_\mu$ admits Newhouse tangency.

--- Newhouse sinks;

--- Hénon attractors

(IV) As $\mu \to 0$, there is a periodic pattern of dynamical behavior $\Lambda_\mu$ would repeat in an accelerated fashion as $\mu \to 0$. 
Part III

Numerical Investigation
Equation of Study

Step 1: We start with the Duffing equation
\[
\frac{d^2 q}{dt^2} - q + q^2 = 0.
\]

Step 2: Add non-linear dumping terms
\[
\frac{d^2 q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^2 = 0.
\]

For \( \lambda > 0 \) small, there exists \( \gamma_\lambda \) so that it has a homoclinic loops to a dissipative saddle.

Step 3: Add periodic perturbations
\[
\frac{d^2 q}{dt^2} + (\lambda - \gamma_\lambda q^2) \frac{dq}{dt} - q + q^2 = \mu \sin 2\pi t
\]
Numerical Simulation

- Numerical scheme
  - Fourth-order Runge-Kutta routine

- Values of $\lambda$ and $\gamma$
  \[
  \lambda = 0.5, \quad \gamma_\lambda = 0.5770285901
  \]

- Initial phase position
  \[
  (q_0, p_0) = (0.01, 0.0)
  \]

- Parameters varied
  - $\mu$ ranged from $10^{-3}$ to $10^{-7}$.
  - $t_0$ ranged in [0, 1).
  - Compute the solution for each fixed combination of $\mu$ and $t_0$. 
Simulation Results

- **Transient Tangle**
  
  - Let $t_0$ run over [0, 1) with an increment of 0.001.
  
  - All 1,000 solutions leave the neighborhood of the homoclinic solution.
  
  - Homoclinic tangle contains no object *directly observable*.

- **Periodic Sink**

![Graphs](image.png)
• Hénon-Like Attractor

— These are a plot of a strange attractor associated to a Newhouse tangency.

— This is the kind of Chaos predicted by Mora-Viana based on Benedick-Carleson Theory on Hénon Maps.
**Periodicity of Dynamical Behavior**

Theoretical Multiplicity = $e^{\beta T} = 2.1831$

<table>
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<tr>
<th>$\mu$</th>
<th>Dynamics</th>
<th>Actual Ratio</th>
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<tr>
<td>$7.041 \cdot 10^{-5}$</td>
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<td>Chaotic</td>
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Summary

(I) A Theory on Homoclinic Tangles

We provided a comprehensive description on the overall dynamical structure of homoclinic tangles from a periodically perturbed homoclinic solution.

(II) Theories on maps come together

Horseshoes, Newhouse sinks and Hénon-like attractors all fall into their places as part of a larger picture.

(III) Applications

Our results can be applied to the analysis of given equations, such as Duffing equation.

(IV) A systematic numerical investigation

We know fully what to expect in numerical simulations around a periodically perturbed homoclinic solution.