Dynamical Profile of a Class of Rank One Attractors

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This paper is about dynamical properties of a class of rank one attractors. Roughly speaking, a rank one attractor is an attractor on which there is exactly one neutral or unstable direction and all other directions are contracted strongly. In [WY2], we identified a class of well behaved rank one maps (called $\mathcal{G}$ in that paper). These maps are characterized by hyperbolic behavior away from “critical structures”, which are localized sources of nonhyperbolicity. The present paper contains an in-depth study of the geometric and ergodic theories of $T \in \mathcal{G}$. We examine these maps from several different angles: Lyapunov exponents, SRB measures, basins of attraction, statistics of time series, global geometric and combinatorial structures, symbolic coding and periodic points. In short, we seek to build a dynamical profile for the class of maps $T \in \mathcal{G}$, to connect them to the existing literature on hyperbolic dynamics.

Since the statement of our results in Section 1 are quite straightforward, we would like to use the rest of this introduction to better acquaint the reader with the class of maps $\mathcal{G}$: Where did it come from? Why is it interesting? Do maps of the type in $\mathcal{G}$ appear naturally? Are our results here relevant in applications?

The class $\mathcal{G}$ has its origin in the Hénon family with $0 < b \ll 1$, studied in the groundbreaking work [BC2] and subsequent papers, e.g. [BY1, BY2, BV]. These ideas were taken to a larger context for the first time in [WY1], where the formulas of the Hénon maps were replaced by generic geometric conditions. The aim of [WY1] was to make contact with more general rank one phenomena in dynamical systems (see below). The systems treated in [WY1] are 2D; their generalization to arbitrary dimensions is the class $\mathcal{G}$ studied in [WY2]. We believe this is the context to which the body of ideas begun in [BC2] truly belongs.

A justification for studying $\mathcal{G}$ is that it provides a unique window into the workings of nonuniform hyperbolicity. While abstract nonuniform hyperbolic theory (as in e.g. [P, R, LY]) is fairly well developed, there are few concrete examples, the majority of which (e.g. billiards and the Lorenz attractor) have \textit{a priori} invariant cones or separation of stable and unstable directions. The same is true for some works on partially hyperbolic systems. The maps in $\mathcal{G}$ admit no continuous families of invariant cones, and derivative growth occurs in genuinely nonuniform ways. But the mechanism that leads to the loss of hyperbolicity is known, and as we will show, conditions imposed on critical orbits (required for $T$ to be in $\mathcal{G}$) translate into a fair amount of control on the hyperbolic properties of other orbits in the system. It is a model of \textit{controlled nonuniform hyperbolicity}.

Equally important to us is that the maps in $\mathcal{G}$ arise naturally in applications, in differential equations modeling commonly occurring phenomena. For example, strange attractors with one direction of instability have been shown to appear shortly after the breakdown

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of stability in situations involving periodically kicked oscillators and Hopf bifurcations e.g. [WY3, WY4, LWY]. They are also known to be associated with homoclinic behavior and slow-fast systems e.g. [MV, V, WO, GWY]. The class $\mathcal{G}$ is therefore relevant to both theory and applications of dynamical systems, and our goal here is to provide a comprehensive profile for this class of maps.

The organization of this paper is as follows: results are stated in Section 1, and proofs are given in Sections 3–6. In terms of exposition, our biggest challenge was how to recall properly the many facts needed from [WY2]. To help make the presentation more accessible, we have included in Section 2 a 10-page review of the geometry and dynamics of $T \in \mathcal{G}$. This is not an abridged version of [WY2]: we have extracted the picture at the end of the inductive construction (which is all that is needed here), setting aside the induction itself and parameter issues. We hope that this partial synopsis, which is considerably simpler than the full treatment, will be useful not only for the present paper but also as an aid to reading [WY2].

1 Statement of Results

As explained in the introduction, we seek in this paper to build a dynamical profile for the class of maps $\mathcal{G}$ identified in [WY2], and to relate it to the general theory of (nonuniformly) hyperbolic systems. To provide context, we begin with a very brief introduction to these maps; a detailed review is postponed to Section 2. The results which taken together comprise the dynamical profile we put forward, namely Theorems 1–8 and Corollaries 1–3, are stated in Sects. 1.2–1.4.

There is a body of work on Hénon and 2D rank-one attractors of which the results presented here are generalizations. References to these papers are collected at the end of the section.

1.1 A brief introduction to the class of maps $\mathcal{G}$

The class $\mathcal{G}$ is arrived at as follows.

- **A class of 1D maps called $\mathcal{M}$**. These are slight generalization of the maps introduced by Misiuriewicz in [M]. Let $I = S^1$ or $[0,1]$, and let $f : I \rightarrow I$ be a $C^2$ map. Let $C = C(f) = \{ x \in I : f'(x) = 0 \}$ be the set of critical points, and let $C_\delta$ be the $\delta$-neighborhood of $C$. We assume that all $x \in C$ are non-degenerate, and say $f \in \mathcal{M}$ if there exists a $\delta_0 > 0$ so that (a) $|(f^n)'|$ grows exponentially along orbit segments outside of $C_{\delta_0}$; (b) for $x \in C$, $f^n(x) \not\in C_{\delta_0}$ for all $n > 0$, and for $x \in C_{\delta_0} \setminus C$, the orbit of $x$ stays outside of $C_{\delta_0}$ until the loss of derivative at $x$ is sufficiently compensated by the exponential growth in (a). The precise definition of $\mathcal{M}$ is given in Sect. 1.1 of [WY2]. In what follows, $\lambda_0$ is a lower bound for the rate of exponential growth in (a), and $\lambda$ is a number with $0 < \lambda < \frac{1}{2} \lambda_0$.

- **Rank one maps of higher dimensions from unfolding of 1D maps**. Let $m \geq 2$ be an integer and $X = \{ (x,y) \in I \times \mathbb{R}^{m-1} : |y| \leq 1 \}$. For $f_0 \in \mathcal{M}$, $K_0 > 1$, and $a,b$ such that $0 << b << a << K_0^{-1}$, we introduce an open set of $C^3$ embeddings of $X$ into itself denoted by $\mathcal{G}_0 = \mathcal{G}_0(f_0,K_0,a,b)$: For $T \in \mathcal{G}_0$, let $T = (\hat{T}^1, \cdots, \hat{T}^m)$ denote its component functions. Then
We will refer to $z$ be the global stable manifold of $W_e$.

We also say $g$ (see e.g. [R, Y3]) that there is a that the following hold at every $z$:

(i) $\|\hat{T}^1\|_{C^3} < K_0$, and $\|\hat{T}^j\|_{C^3} < b$ for $j = 2, \cdots, m$;

(ii) $\|f - f_0\|_{C^2} < a$ where $f = \hat{T}^1|_{f^{-1}(I)} : I \to I$;

(iii) $\exists v$, a unit m-vector with zero $x$-component such that $|D\hat{T}^1_{(x,0)}v| > K_0^{-1}$ for all $x \in C(f_0)$.

• The class $G$ as a subset of $G_0$ with controlled critical behavior. For $f_0$ and $G_0$ as above, a subset $G \subset G_0$ is identified in [WY2] via an inductive selection procedure. These maps are chosen to have a well defined critical set $C$ with certain properties. Specifically, if $C(f_0) = \{x_1, \cdots, x_q\}$, then $C = \bigcup_{i=1}^q C_i$ where $C_i$ is a Cantor set in $X$ located near $(x_i,0)$, and all critical orbits, i.e., orbits starting from $z \in C$, are assumed to satisfy the following conditions: There are two exponential rates $\alpha << \lambda$ such that in forward time, the orbit of $z$ is not allowed to approach $C$ faster than rate $\alpha$, and $|DT_z^s v|$ grows exponentially at a rate $> \lambda$. In [WY2], we proved that $G$ is “large” as a subset of $G_0$ in the sense that it occupies positive measure sets of parameters in certain generic 1-parameter families.

Standing Hypotheses: All the theorems in this paper pertain to $T \in G$. For some of these theorems, we impose the additional condition

(*) For all $z \in X$, $(K_0^{-1}b)^{m-1} < |\det(DT(z))| < (K_0b)^{m-1}$.

This condition will be mentioned explicitly where needed.

1.2 Ergodic properties of SRB measures

Theorem 1 below is proved in Section 9 of [WY2]. We include its statement as it is the starting point for our other results, all of which are new.

A $T$-invariant Borel probability measure $\mu$ is called an SRB measure if (i) $T$ has a positive Lyapunov exponent $\mu$-a.e.; (ii) the conditional measures of $\mu$ on unstable manifolds are absolutely continuous with respect to the Riemannian measures on these leaves.

**Theorem 1 (Existence of SRB measures)** $T$ admits an SRB measure.

Before proceeding to our other results on the SRB measures of $T \in G$, we first recall some facts from general nonuniform hyperbolic theory. Let $g : M \to M$ be a diffeomorphism of a compact manifold $M$, and let $\nu$ be an ergodic Borel probability measure for $g$. Denote the Lyapunov exponents of $(g, \nu)$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. It follows from standard theory (see e.g. [R, Y3]) that there is a $g$-invariant Borel measurable set $\Gamma_\nu$ with $\nu(\Gamma_\nu) = 1$ such that the following hold at every $z \in \Gamma_\nu$:

1. for every continuous function $\varphi : M \to \mathbb{R}$, $\frac{1}{n} \sum_{i=0}^{n-1} \varphi(g^i(z)) \to \int \varphi d\nu$;

2. Lyapunov exponents are well defined at $z$ and are equal to $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$;

3. local stable manifolds $W^s_{loc}(z)$ are defined at $z$, with $g(W^s_{loc}(z)) \subset W^s_{loc}(gz)$.

We will refer to $z \in \Gamma_\nu$ as a $\nu$-typical point. For $z \in \Gamma_\nu$, we let $W^s(z) = \bigcup_{n \geq 0} g^{-n}W^s_{loc}(g^n z)$ be the global stable manifold of $z$, and define

$$W^s(\nu) = \bigcup_{z \in \Gamma_\nu} W^s(z).$$

We also say $\xi \in M$ is future generic with respect to $\nu$ if (1) holds for $\xi$. 

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Theorem 2 (Number of ergodic components) Assume (*). Then $T$ has at most $q$ ergodic SRB measures where $q$ is the number of critical points of $f_0$.

Theorem 3 (Basin of attraction) Assume (*), and let $\nu_1, \nu_2, \ldots, \nu_r$ be the ergodic SRB measures of $T$. Then Leb-a.e. $\xi \in X$ lies in $\bigcup_{i=1}^r W^s(\nu_i)$. Consequently, we have

(a) Leb-a.e. $\xi \in X$ is future generic with respect to an ergodic SRB measure.
(b) $T$ has a (well defined) positive Lyapunov exponent at Leb-a.e. $\xi \in X$.

Assertion (b) describes a form of instability seen from the basin.

General nonuniform hyperbolic theory tells us that for any ergodic SRB measure $\mu$ with no zero Lyapunov exponents, $W^s(\mu)$ has positive Lebesgue measure by the absolute continuity of the stable foliation (see [L], [PS]). It follows immediately that such a measure can have at most a countable number of ergodic components. In general,

(i) it is entirely possible for a map to have an infinite number of ergodic SRB measures;
(ii) $\bigcup_{\mu} W^s(\mu)$ where the union is taken over all ergodic SRB measures $\mu$ need not have full Lebesgue measure on any neighborhood of the attractor.

Theorems 2 and 3 above assert that, under the additional condition (*), neither (i) or (ii) occurs for $T \in G$.

The next result, which is useful in applications, gives easy-to-check conditions parallel to those for finite-state Markov chains for the ergodicity and mixing of SRB measures. Recall that $|DT^n v|$ grow exponentially at rates $> \lambda$ along all critical orbits of $T$. Let $J_1, \ldots, J_q$ (or $J_{q+1}$ in the case where $I$ is an interval) be the intervals of monotonicity of $f_0$, and let $P = (p_{i,j})$ be the matrix defined by $p_{i,j} = 1$ if $f_0(J_i) \supset J_j$, $p_{i,j} = 0$ otherwise. Powers of $P$ are denoted as $P^N = (p_{i,j}^N)$.

Theorem 4 (Conditions for ergodicity and mixing) Assume (*) and $e^\lambda > 2$.

(a) If for every $(i,j)$, $\exists N_1 = N_1(i,j) > 0$ such that $p_{i,j}^{N_1} > 0$, then $T$ admits a unique (ergodic) SRB measure $\nu$.
(b) If $\exists N_2 > 0$ such that $p_{i,j}^{N_2} > 0$ for all $(i,j)$, then $(T,\nu)$ is mixing.

1.3 Statistics of dynamically generated observations

In this subsection, we let $\varphi : X \to \mathbb{R}$ be an observable on $X$, and consider the distribution and limit laws of the sequence of random variables

$$\varphi, \varphi \circ T, \varphi \circ T^2, \ldots, \varphi \circ T^n, \ldots,$$

with an SRB measure $\nu$ as the underlying probability on $X$. We will use previously established ideas on Markov tower extensions, which we briefly recall, referring the reader to [Y1] for the precise formulation:

In [Y1], the idea of a positive measure horseshoe $g^R : \Lambda \to \Lambda$ with infinitely many branches and variable return times for a general diffeomorphism $g$ is introduced. Roughly speaking, $\Lambda$ is a hyperbolic product set, i.e. it is the intersection of two transversal families of local stable and unstable leaves $\Gamma^s$ and $\Gamma^u$, $R : \Lambda \to \mathbb{Z}^+$ is a return time function (usually not the first return), $g^R$ maps $\Lambda$ into itself and has the structure of a horseshoe with infinitely many branches. Moreover, letting $m_\gamma$ denote the induced Riemannian measure
on a submanifold $\gamma$, we have $m_{\gamma u}(\Lambda \cap \gamma^u) > 0$ for every $\gamma^u \in \Gamma^u$. Obviously, not every $g$ admits such a horseshoe.

The following known facts are relevant:

**Fact 1.** It is proved in [Y1] that if $g$ admits a positive measure horseshoe as above, and if additionally

(i) $g^R: \Lambda \to \Lambda$ satisfies hyperbolic conditions (P1)–(P5) in [Y1], and

(ii) $R$ has an exponential tail, i.e.

$$m_{\gamma^u}\{R > n\} < C_0e^{-cn} \quad \text{for some } C_0, c_0 > 0 \text{ and } \gamma^u \in \Gamma^u,$$

then $g$ has a *Markov tower extension* with exponential return times.

**Fact 2.** A number of statistical properties have been proved for maps $g$ that admit such Markov tower extensions (see e.g. [C, DSV, MN, RbY, SV, Y1, Y2]).

We now return to $T \in G$:

**Theorem 5 (Markov tower extensions)** Every $T \in G$ admits a positive measure horseshoe $T^R: \Lambda \to \Lambda$ of the type in [Y1]. This construction meets conditions (i) and (ii) in Fact 1 above; in particular, the return time function $R$ has an exponential tail.

We state three corollaries of Theorem 5 of the type indicated in Fact 2. In Corollaries 1, 2 and 3 below, $\nu$ is an SRB measure of $T$, $\varphi$ is a real-valued observable, and $S_n \varphi = \sum_{i=0}^{n-1} \varphi \circ T^i$.

**Corollary 1 (Central Limit Theorem)** Let $(T, \nu)$ be ergodic. Then a Central Limit Theorem holds for every H"older continuous observable $\varphi$, i.e.

$$\frac{1}{\sqrt{n}} \left( S_n \varphi - \int \varphi d\nu \right)$$

converges in distribution to a Gaussian except where $\varphi \circ T = \psi \circ T - \psi$ for some $\psi \in L^2$.

Corollary 1 follows immediately from Theorem 5 above and Theorem 3 in [Y1].

**Corollary 2 (Exponential decay of correlations)** Assume $(T, \nu)$ is mixing, equivalently, assume $(T^n, \nu)$ is ergodic for all $n \in \mathbb{Z}^+$. For $\eta > 0$, let $F_\eta$ be the set of all H"older functions with exponent $\eta$. Then $(T, \nu)$ has exponential decay of correlations for all test functions in $F_\eta$, that is to say, there exists $\tau = \tau(\eta) \in (0, 1)$ such that for all $\varphi, \psi \in F_\eta$, there exists $C = C(\varphi, \psi)$ such that for all $n \geq 1$,

$$\left| \int (\varphi \circ T^n) \psi d\nu - \int \varphi d\nu \int \psi d\nu \right| \leq C \tau^n.$$

Corollary 2 follows immediately from Theorem 5 above and Theorem 2 in [Y1].
Corollary 3 (Large deviations) Assume \((T, \nu)\) is ergodic, and \(\varphi\) is Hölder. Let \(\theta_{\text{max}}\) be any number \(< c_0/(\max \varphi - \min \varphi)\) where \(c_0\) is as in (1). We define the logarithmic moment generating function
\[
e(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \nu(e^{\theta S_n \varphi}) ,
\]
and let \(I(t)\) be its Legendre transform. Then for any interval \([a, b] \subset [\varphi(-\theta_{\text{max}}), \varphi(\theta_{\text{max}})]\), an interval which contains in its interior the point \(\varphi'(0) = \int \varphi d\nu\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \nu \left\{ z : \frac{1}{n} S_n \varphi(z) \in [a, b] \right\} = - \inf_{t \in [a, b]} I(t) .
\]
Corollary 3 follows immediately from Theorem 5 above and Theorem 2 in [RbY].

1.4 Geometric and combinatorial properties

Let \(\Omega = \cap_{n \geq 0} T^n(R)\) be the attractor and \(C \subset \Omega\) the critical set. The approximations in part (b) of Theorem 6 are in the direction of the type considered in [Kt].

Theorem 6 (Approximation by uniformly hyperbolic invariant sets)

(a) At every \(z \in C\), there is a vector \(v\) such that \(|DT_z^+(v)| \to 0\) exponentially fast as \(n\) tends to both \(+\infty\) and \(-\infty\).

(b) Let \(\Omega_\varepsilon := \{ z \in \Omega : d(T^n z, C) \geq \varepsilon \forall n \in \mathbb{Z} \}\). Then \(T\) is uniformly hyperbolic on \(\Omega_\varepsilon\) for every \(\varepsilon > 0\).

It will follow from the discussion that the strength of hyperbolicity on \(\Omega_\varepsilon\), including the minimum angle between stable and unstable subspaces and the constant in front of the exponential growth rate, deteriorates as \(\varepsilon \to 0\).

We turn next to a special coding of orbits on \(\Omega\). For notational simplicity, we state the next result only for the case \(I = S^1\); the interval case is treated similarly. The following notation will be used: \(x_1 < x_2 < \cdots < x_q < x_{q+1} = x_1\) are the critical points of \(f_0\), and \(C_i\) is the part of \(C\) near \((x_i, 0) \in R\). Let \(\Sigma_q = \Pi_{i=1}^q \{1, 2, \cdots, q\}\), and let \(\sigma : \Sigma_q \to \Sigma_q\) be the shift map.

Theorem 7 (Symbolic coding of orbits relative to \(C\))

(a) There is a natural partition of \(\Omega \setminus C\) into disjoint sets \(A_1, A_2, \cdots, A_q\) so that \(z \in A_i\) can be thought of as being “to the right” of \(C_i\) and “to the left” of \(C_{i+1}\).

(b) Under the additional assumption that \(f_0[x_j, x_{j+1}] \not\supset S^1\) for any \(j\), there is a closed subset \(\Sigma \subset \Sigma_q\) with \(\sigma(\Sigma) \subset \Sigma\) and a map \(\pi : \Sigma \to \Omega\) with the property that
   - for all \(s = (s_i) \in \Sigma\), \(\pi(s) = z\) implies that \(T^i z \in \tilde{A}_{s_i}\) for all \(i\);
   - \(\pi\) is a continuous surjection that is 1-1 except on \(\cup_{i=-\infty}^{\infty} T^i C\), where it is 2-1.

If the assumption in Part (b) does not hold, the statement can be amended by increasing the number of symbols as follows: For \(f_0\), use the partition given by \(C(f_0) \cup f_0^{-1} C(f_0)\) instead of that by \(C(f_0)\), and do likewise with \(T\).

Remark. We have seen three attempts to code the orbits of \(T \in G\): The construction in Sect. 1.3 has an infinite alphabet and variable return times, and misses a measure zero set
of points. (Measure in this remark refers to SRB measure.) The set \( \Omega_v \) in Theorem 6(b) contains a horseshoe in the sense of [S] (see [Kt]) but lives on a set of measure zero. The coding in Theorem 7 accounts for all orbits, and is the most natural of all in that it respects the geometry of the map, but the resulting shift \( \sigma : \Sigma \rightarrow \Sigma \) is not of finite type.

The following results are in part consequences of the coding in Theorem 7.

**Theorem 8 (Equilibrium states, entropy and periodic points)**

(a) For every continuous function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \), there exists an invariant measure \( \mu \) maximizing the quantity \( h_\mu(T) - \int \varphi d\nu \) where \( \nu \) ranges over all invariant probability measures. In particular, \( T \) has a measure of maximal entropy.

(b) Let \( P_n \) be the number of periodic points of \( T \) of (minimal) period \( n \). Then

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log P_n = h_{\text{top}}(T) > 0.
\]

The results in Theorem 8 are known to hold for Axiom A basic sets [B1, B2]. The relation between growth rate of periodic points and topological entropy in (2) is known to be false for generic diffeomorphisms [Kl]; see also [HK].

**Related works in 2D.** The first results were proved for Hénon attractors: Building on [BC2], Theorems 1 and 2 (in the context of Hénon maps) were proved in [BY1], Theorem 5 along with Corollaries 1 and 2 were proved in [BY2], and Theorem 3 was proved in [BV] for Hénon-like maps, which are slightly more general than Hénon maps. At about the same time, the next generation of results was presented in [WY1], in which the authors introduced the class of rank one attractors in two dimensions, replacing the formulas of the Hénon maps by generic geometric conditions. Part II of [WY1] contains the 2D-version of all of the results in this paper (with the exception of Corollary 3). See also [T] in relation to Theorem 3.

## 2 Properties of \( T \in \mathcal{G} \) from [WY2]

The aims of this section are twofold. The first is to provide a description of the maps \( T \in \mathcal{G} \) in a way that bypasses the very long and complicated construction in [WY2]. The second is to collect in one place the technical facts from [WY2] that are needed in the present paper. The exposition in Sect. 2.1 is informal, intended to convey an intuitive understanding of “good” rank-one maps. Sects. 2.2 and 2.3 contain mathematically precise statements to which we will refer in the proofs to follow.

### 2.1 From 1D to rank-one maps: an informal discussion

We first introduce a class of “good maps” in 1D, and then explain (i) how maps in \( \mathcal{G} \) are modeled after these 1D maps and (ii) how they differ.

**A. A class of good maps in 1D:** For \( f_0 \in \mathcal{M} \), let \( C(f_0) \) and \( C_\delta \) be as in Sect. 1.1. Two of the important properties of \( f_0 \) are the following: There exist \( \delta_0, \lambda_0, \epsilon_0 > 0 \) so that for all \( x_0 \in C(f_0) \) and \( n > 0 \),
(M1) (Distance to critical set) $f_0^n(x_0) \not\in C_{\delta_0}$, and

(M2) (Exponential growth of derivatives) $|(f_0^n)'(f_0(x_0))| > c_0 e^{\lambda_0 n}$.

The maps in $\mathcal{M}$ are among the simplest with non-uniform expansion. Unfortunately, (M1) severely limits the scope of $\mathcal{M}$ as a subset of $C^2(I, I)$. A larger set of non-uniformly expanding maps is obtained by relaxing (M1) and (M2) respectively to (G1) and (G2) below. Let $f \in C^2(I, I)$ be sufficiently near $f_0$. We say $f$ is a “good 1D map” if for all $x_0 \in C(f)$ and $n > 0$,

(G1) (Distance to critical set) $d(f^n(x_0), C(f)) \geq \min\{\frac{1}{2}\delta_0, e^{-\alpha n}\}$ and

(G2) (Exponential growth of derivatives) $|(f^n)'(f(x_0))| > ce^{\lambda n}$

where $\alpha < \frac{1}{100}\lambda$ and $\lambda < \frac{1}{5}\lambda_0$. In particular, critical orbits of $f$ are allowed to approach $C(f)$, but only slowly. The class of maps satisfying (G1) and (G2) is “large” in the sense that they occupy positive measure sets of parameters in certain generic 1-parameter families through $f_0$.

Conditions along the lines of (G1) and (G2) were first introduced in [CE] and [BC1], and used in [TTY]. In the generality discussed here, they were used in [WY3].

B. From good 1D to good rank-one maps: We fix $f_0 \in \mathcal{M}$ and consider, to begin with, an arbitrary $T \in \mathcal{G}_0$. Let $R_1$ be a tubular neighborhood of $I \times \{0\}$ in $X$ of size $\sim b$, and assume that $T(R_1) \subset R_1$. Geometrically, we may think of $T : R_1 \rightarrow R_1$ as a slightly “thickened up” version of $f_0 : I \rightarrow I$. Our plan is to put $T$ in $\mathcal{G}$ if it satisfies two conditions analogous to (G1) and (G2) above. A problem arises immediately: What plays the role of the critical set for $T \in \mathcal{G}_0$?

There is a number, $\theta = \text{constant} \cdot \log \frac{1}{\theta}$, that is of note: For $n \leq \theta^{-1}$, the geometry of $R_n = T^{n-1}(R_1)$ follows closely that of the piecewise monotone curve $T^{n-1}(I \times \{0\})$. This means in particular that $R_n$ is the union of well defined monotone branches separated by well defined “turns”. We take advantage of the relative simplicity of this geometry, and introduce in Paragraph B two important ideas: critical blobs and monotone branches. The discussion is limited to $R_n$ for $n \leq \theta^{-1}$.

First generation critical blobs. For each $x_i \in C(f_0)$, let $B_i^{(1)} = \{(x, y) \in R_1 : |x - x_i| < b^{\frac{1}{2}}\}$. Temporarily, let us think of $\cup_k B_k^{(1)}$ as the critical set, and try to impose (G1) and (G2) on its orbits: With the constant in the definition of $\theta$ suitably chosen, one can arrange the following for all $j < \theta^{-1}$: (i) the diameter of $T^j(B_i^{(1)})$ is $< e^{-\alpha j}$ and (ii) $DT^j_{z_1} \approx DT^j_{z_2}$ for all $z_1, z_2 \in B_i^{(1)}$. Thus we may view each $B_i^{(1)}$ as an (enlarged) critical point, and the orbits starting from it as representing a single trajectory. Moreover, it makes sense to impose (G1) and (G2) on this trajectory up to time $\theta^{-1}$, taking $|(f^n)'|$ in (G2) to mean $\|DT^n\|$. The sets $B_i^{(1)}, i = 1, 2, \cdots, q$, are called first generation critical blobs in [WY2].

As a hint of what is to come, notice that (G1) cannot possibly hold for all times: when away from $\cup_k B_k^{(1)}$, $T^j(B_i^{(1)})$ grows exponentially in size with $j$ due to the expanding properties of $f_0$; it must intersect $\cup_k B_k^{(1)}$ eventually. Therefore, as far as (G1) and (G2) are concerned, the critical blobs $\{B_i^{(1)}\}$ are objects that are meaningful for only finite time; their period of activity is defined to be $\theta^{-1}$.
Critical blobs and monotone branches of higher generations. The geometry of $R_2$ is simple: It is the union of a finite number of monotone branches, i.e. thin horizontal tubes ("horizontal" here means roughly aligned with the $I$-direction) with cross-sectional diameter $\sim b^2$, each one of which begins and ends at two different $T(B_i^{(1)})$s. The "turns" or "folds" are contained in $\cup iT_i(B_i^{(1)})$. Moreover, with $T(B_i^{(1)})$ kept away from $\cup k B_k^{(1)}$, each of the monotone branches either does not intersect a particular $B_k^{(1)}$, or it cuts across it completely in a clean way. This sets the stage for constructing critical blobs of the second generation: Fix $k$, and consider all of the monotone branches in $R_2$ that cross $B_k^{(1)}$. (If $f_0$ has $q_1$ pre-image points at $x_k$, then there will be $q_1$ such branches.) It is shown in [WY2] that in the intersection of each of these branches with $B_k^{(1)}$, one can identify a section of length $b^{2s}$ whose forward image will fold in the $x$-direction. These sets, which we call $B_j^{(2)}$ ($j$ is just a label here), are our critical blobs of generation 2.

Likewise, we have that $R_3$ is the union of a finite number of monotone branches, each one of which begins and ends at either a $T^2(B_i^{(1)})$ or a $T(B_j^{(2)})$. For the same reasons as before, each of these horizontal tubes either does not meet a $B_k^{(2)}$ or it crosses it completely, enabling us to define in each of the components of $R_3 \cap B_k^{(2)}$ a critical blob of generation 3, and so on.

The procedure above is used to define critical blobs and monotone branches of generation $n = 1, 2, \cdots, \theta^{-1}$. Each $B_i^{(n)}$ is contained in some $B_j^{(n-1)}$. It has horizontal length $\sim b^{2s}$ and cross-sectional diameter $\sim b^n$. We define $n\theta^{-1}$ to be its period of activity. Up to this time, all orbits starting from a $B_i^{(n)}$ can be viewed as representing a single orbit for which one can make sense of the statements in (G1) and (G2) (see Sect. 2.3). Each monotone branch of generation $n$ is a subset of $R_n$. It consists of a horizontal part called the "main body" of the branch; at each of the two ends of the main body is an image of a critical blob of the form $T^j(B_i^{n-j})$ for some $j < n$.

As a precondition for putting $T$ into $\mathcal{G}$, conditions (G1) and (G2) are imposed on each blob of generation $n$ up to the time $n\theta^{-1}$.

C. Control of global geometry after $\theta^{-1}$ generations: As explained earlier, the piece-wise monotone structure of $R_n$ cannot be maintained for all $n$: once the orbit of a critical blob returns to meet the critical structure – and that is inevitable – the geometry of $R_n$ will start to get complicated.

The following scheme is employed in [WY2] to retain control of the geometry: We continue to construct monotone branches and critical blobs beyond generation $\theta^{-1}$ as in Paragraph B, with the proviso that once the period of activity of a critical blob expires, the two branches which share it as an "end" are "discontinued", meaning we stop considering them from that time on. Thus the first time a branch is discontinued is $\theta^{-1}$, when the first generation critical blobs cease to be active. Following this scheme – assuming it can be carried out in a meaningful way – the geometry of all the monotone branches kept will remain intact.

\[3\text{In reality, each } B_i^{(n)} \text{ is a thin tube with structure inside. The word "blob" was used in [WY2] to suggest that conceptually this geometry or structure is entirely irrelevant.}\]
Figure 1. Replacing a branch with branches of higher generations

The deletion of monotone branches raises immediately some serious concerns: a part of the attractor $\Omega$ may be exposed; in general, one cannot afford to lose control completely of a region of the phase space. The remedy used in [WY2] is to replace each branch deleted by monotone branches of higher generations that lie inside of and run parallel to the discontinued branch, to ensure that no part of $\Omega$ is exposed. See Figure 1. It is not obvious a priori that such replacement branches are available, or that they will be adequate for our purposes. It is proved in [WY2] that this procedure works. We proved inductively that when a branch of generation $n$ is discontinued, one can systematically choose replacement branches to be of generation between $n$ and $(1 + 2\theta)n$, so that

$$R_{(1+2\theta)n} \subset \bigcup_{n \leq i \leq (1+2\theta)n} R_i$$

where $R_i$ is the collection of $i$th generation monotone branches constructed.

A side effect of this construction is that after generation $\theta^{-1}$ critical blobs do not necessarily come in consecutive generations: some generations may be missed due to the branch replacement process (e.g., it can in principle go from $n$ to $(1 + 2\theta)n$), but given the long period of activity of each blob, these relatively small gaps in generations do not lead to any loss of control in the dynamics. For $T$ to be in $\mathcal{G}$, conditions analogous to (G1) and (G2) are imposed on each critical blob of generation $n$ up to time $n\theta^{-1}$ as before.

D. A different generation of critical blobs for each epoch: For time $\leq \theta^{-1}$, critical points are represented by critical blobs of the first generation. At time $\theta^{-1}$, the image of a first generation critical blob may become too large to represent a single critical orbit; more precisely, we are no longer able to guarantee that it is small enough. At this time, we “retire” $T^{\theta^{-1}}(B_1^{(1)})$, and replace it by the cluster of $T^{\theta^{-1}}(B_j^{(2)})$ that lie within it. That is to say, between times $\theta^{-1}$ and $2\theta^{-1}$, second generation critical blobs are our surrogate critical points. Similarly, at time $2\theta^{-1}$, the critical blobs of generation 2 are retired, and those of generation 3 emerge, and so on. The answer to the question posed at the beginning of Paragraph B is therefore: between times $k\theta^{-1}$ and $(k+1)\theta^{-1}$, the role of critical points in 1D is played by critical blobs of generation $\sim k$, the small amount of fuzziness in generation being due to the replacement procedure above.

To summarize, unlike 1D maps, which have a fixed number of critical points, the effective number of critical blobs of $T \in \mathcal{G}$ grows slowly with time, at an exponential rate $\sim \theta$ (notice that $\theta \to 0$ as $b \to 0$). Conditions analogous to (G1) and (G2) hold for each of these critical blobs during the relevant periods.

Sect. 2.1 contains a very abbreviated version of the overall picture from A to Z. We have omitted a great deal, including (i) the supporting analysis needed to justify the steps
outlined above, and (ii) the geometric structures surrounding the critical blobs that make this analysis possible. (ii) is reviewed in detail in Sect. 2.2, and (i) in Sect. 2.3. All statements in Sects. 2.2 and 2.3 are mathematically accurate.

2.2 Geometry of $T \in \mathcal{G}$

We start with some frequently used notation: Let $R_1 = \{(x, y) : x \in I, y \in \mathbb{R}^{m-1} : |y| \leq (m-1)^{\frac{3}{2}} b\}$, and for $k \geq 2$, let $R_k = T^{k-1}R_1$. The attractor $\Omega$ is defined to be $\Omega = \cap_{k \geq 0} R_k$.

We will refer to the direction of $x$ the “horizontal” direction and that of $y$ the “vertical” directions. For $z_0 \in R_1$ and a tangent vector $w_0 \in \mathbb{R}^m$ at $z_0$, we write $z_n = T^n(z_0)$ and $w_n = DT^n_w(z_0)$. The items below marked with asterisks are to be made precise later.

- $C^{(1)} = \{(x, y) : x \in C_\delta(f_0)\}$ is our first critical region.
- $\lambda < \frac{1}{2} \lambda_0$ is a lower bound for the rate of derivative growth for critical orbits.*
- $\alpha << \min(\lambda, 1)$ is the rate of approach of critical orbits to the set of critical points.*
- $\theta$ is defined by $b^\theta = ||DT||^{-20}$.
- The letter $K$ is used as a generic constant throughout; its values depend on $f_0, K_0, \lambda$ and $m$ but not on $\delta, a, b, \alpha$.
- $v$ is a fixed vertical vector (see Sect. 1.1).
- $b$-horizontal vectors: Let $u = (u_x, u_y) \in \mathbb{R}^m$. We define $s(u) = |u_y| |u_x|$ to be the slope of $u$, and say $u$ is $b$-horizontal if $s(u) < \frac{3Kb}{2}$.
- $C^2(b)$-curves: Let $\gamma$ be a curve in $R_1$ and $\gamma'$ its tangent vectors. Let $\kappa = |\gamma' \wedge \gamma''|/|\gamma'|^3$ be the curvature. We call $\gamma$ a $C^2(b)$-curve if all its tangent vectors are $b$-horizontal and the curvatures are everywhere $< \frac{Kb}{3\theta}$ for a certain $K$.
- The foliations $\mathcal{F}_k$: $\mathcal{F}_1$ is the 1-D foliation in $R_1$ whose leaves are $\{y = \text{constant}\}$, and $\mathcal{F}_k = T^{k-1}_w(\mathcal{F}_1)$ is the foliation on $R_k$ obtained by pushing forward $\mathcal{F}_1$.
- Horizontal sections: A subset $H \subset R_k$ is called a horizontal section if it is a connected component of $R_k \cap \{x \in J\}$ for some interval $J \subset I$ and the leaves of $\mathcal{F}_k$ restricted to $H$ are $C^2(b)$-curves.

Two kinds of geometric structures are reviewed here. The first is the critical structure, a layered dynamical structure that lies in each of the component of $C^{(1)}$. This structure is used not only to define the critical blobs discussed in Sect. 2.1 but also to relate orbits entering $C^{(1)}$ to the critical blobs. The second are global geometric structures. We will systematically construct a tree of monotone branches from which the branches discussed in Sect. 2.1 are derived. Both of these structures were introduced in 2D in [WY1] and extended in [WY2] to all finite dimensions. We stress that they are well defined only for $T \in \mathcal{G}$, not for arbitrary rank one maps.

A. Critical regions, critical points and critical blobs: The first critical region $C^{(1)}$ is defined in the first bullet above. In each component $Q^{(1)}$ of $C^{(1)}$, let $x \in C(f_0)$ be such that $(x, 0) \in Q^{(1)}$, and call $z_0^*(Q^{(1)}) = (x, 0)$ “the critical point of $Q^{(1)}$”. For $T \in \mathcal{G}$, critical regions of generation $k$ for all $k > 1$ are defined and are denoted by $C^{(k)}$. In each component $Q^{(k)}$ of $C^{(k)}$, a representative critical point called $z_0^*(Q^{(k)})$ is selected. Properties of $C^{(k)}$ and $z_0^*(Q^{(k)})$ are summarized as follows:
(1) (Geometry of $C^{(k)}$) The $k$th critical region, $C^{(k)}$, is the union of a finite number of disjoint horizontal sections $Q^{(k)}_1$ of $R_k$. Each $Q^{(k)}$ has horizontal length $\min(2\delta, 2e^{-\lambda^*k})$ and cross-sectional diameter $< b\frac{r}{2}$.

(2) (Nested structure of $C^{(k)}$) For $k \leq \theta^{-1}$, $R_k$ meets each $Q^{(k-1)}$ in a finite number of horizontal sections, each one of which contains exactly one component $Q^{(k)}_1$ of $C^{(k)}$; see Figure 2. For $k \geq \theta^{-1}$, one does not always have $C^{(k)} \subset C^{(k-1)}$. Instead, for each $Q^{(k)}$, there exists $k' \in [(1 + 2\theta)^{-1}k, k)$ and a $Q^{(k')}$ such that $Q^{(k)} \subset Q^{(k')}$, $Q^{(k)}$ lying in a horizontal section of $R_k$ that stretches across $Q^{(k')}$ as before. This skipping of generations is due to the discontinuation of monotone branches alluded to in Sect. 2.1C and is discussed again in Part B below.

(3) (Alignment of critical regions and critical points) In the $x$-direction, each $Q^{(k)}$ is centered at a point $z^*_0 = z^*_0(Q^{(k)})$ with the following properties: $z^*_0$ lies on an arbitrarily chosen $F_k$-leaf, e.g., $T^{k-1}(\{y = 0\})$. If $\tau_k$ is the positively oriented unit vector at $z^*_0$ tangent to $F_k$, and $S = S(v, \tau_k)$ is the 2D subspace spanned by $v$ and $\tau_k$, then among all unit vectors in $S$, $\tau_k$ is contracted the most by $Df_{z_0}$. The point $z^*_0(Q^{(k)})$ is called a critical point of generation $k$.

It is proved in [WY2] that whenever $Q^{(k)} \subset Q^{(k)}$, we have

$$|z^*_0(Q^{(k)}) - z^*_0(Q^{(k)})| < b\frac{r}{2}k$$

and

$$\angle(\tau_k, \tau_k') < Kb\frac{r}{2}k.$$

(4) (Distance $d_C(\cdot)$ to critical set) For $z \in R_1 \setminus C^{(1)}$, let $d_C(z) = \delta + d(z, C^{(1)})$. For $z \in C^{(1)}$, we define $d_C(z) = |z - z^*_0(Q^{(k)})|$ where $k$ is the largest integer so that $z \in Q^{(k)}$.

Let $\Gamma_k$ be the set of all critical points of generation $k$. The critical set $C$ of $T$ is defined to be the set of accumulation points of $\Gamma = \cup_k \Gamma_k$. Equivalently, $C = \cap_{n>0} \cup_{k \geq n} C^{(k)}$. Inside each $Q^{(k)}$, we let

$$B^{(k)} = \{z \in Q^{(k)} : |z - z^*_0(Q^{(k)})| < b\frac{r}{2}k\}.$$  

$B^{(k)}$ is a critical blob of generation $k$. The analogs of (G1) and (G2) for critical blobs (as discussed in Sect. 2.1) are imposed on the critical points inside each blob.

(5) (Properties of critical orbits) Let $z_0 \in \Gamma_k$, and let $w_i = DT_z^i v$. Then for all $0 < i \leq \theta^{-1}k$,

(a) $d_C(z_i) \geq \min(\delta, e^{-\alpha i})$;

(b) $|w_i| > K^{-1}e^{\lambda^*i}$ where $\lambda^* = \lambda - 2\alpha\theta$ for a certain $K$. 

Figure 2. Geometry of critical regions
We say the period of activity for $B^{(k)}$ and $z_0 \in \Gamma_k$ ends at time $k\theta^{-1}$; they become irrelevant after this time. We remark also that it is the two properties in (5) that determine whether or not $T \in \mathcal{G}$, i.e. once these properties are met, the rest are shown in [WY2] to follow automatically.

B. Tree of monotone branches: We now define simultaneously subsets of $R_k$ called monotone branches and a “tree” $\{T_k\}_{k=1,2,\ldots}$ where each $T_k$ is a collection of monotone branches.

For definiteness, we consider $I = S^1$, and let $T_1 = \{R_1\}$. Suppose $B_1, \ldots, B_q$ are the critical blobs in $C^{(1)}$, indexed in the order in which they are located along $R_1$. Then the monotone branches of generation 2 are $M_1, \ldots, M_q$ where for each $i < q$, $M_i$ is the part of $R_2$ connecting $T(B_i)$ to $T(B_{i+1})$ including these two sets called the “ends” of the branch, and $M_q$ is the part of $R_2$ connecting $T(B_r)$ to $T(B_1)$. Let $T_2 = \{M_i\}$. For $T \in \mathcal{G}$, each $M_i \in T_2$ reproduces as follows: either $T(M_i) \cap C^{(2)} = \emptyset$, or it contains a $Q^{(2)}$ in its interior. In the first case, $T(M_i) \in T_3$. In the second, let $B'_1, \ldots, B'_r$ be the components of $B^{(2)}$ in $M_i$, in that order from $T(B_i)$ to $T(B_{i+1})$. The offspring of $M_i$ in $T_3$ are then $M'_0, \ldots, M'_r$ where for $0 < i < r$, $M'_i$ is the section of $R_3$ connecting $T(B'_i)$ to $T(B'_{i+1})$, $M'_0$ connects $T^2(B_i)$ to $T(B'_1)$, and $M'_r$ connects $T(B'_r)$ to $T^2(B_{i+1})$. This reproduction process continues, defining inductively $T_4, T_5, \ldots$, and so on, according to the following rules:

(0) (Structure of monotone branch) Each $M \in T_k$ is the disjoint union of a main body which we denote by $M^\circ$ and two end pieces each one of which is of the form $T^{k-i}(B^{(i)})$ for some critical blob $B^{(i)}$.

(1) (Viability of monotone branches) Reproduction continues for $M \in T_k$ if the following holds for each of its two end pieces: $k - i < \theta^{-1}i$ if the critical blob is of generation $i$. When $k - i = \theta^{-1}i$ is reached, the branch is discontinued, i.e., we stop considering it.

(2) (Geometry in relation to critical regions) For $M \in T_k$ and $Q^{(k')}$ with $(1+2\theta)^{-1}k < k' < k$, either $M^\circ \cap Q^{(k')} = \emptyset$ or $M^\circ$ contains a horizontal section of $R_k$ of length $> e^{-\alpha k}$ centered at $Q^{(k')}$. This section contains exactly one $Q^{(k)}$, and it lies in $M^\circ \cap Q^{(k')}$. This is how all components of $C^{(k)}$ are created.

(3) (Neighborhoods of $\Omega$) Let $\mathcal{R}_k = \bigcup_{M \in T_k} M$. It is proved in [WY2] that $R_{k(1+2\theta)} \subset \bigcup_{k \leq j < k(1+2\theta)} \mathcal{R}_j$.

References to [WY2]: The structure of critical regions and monotone branches are discussed in detail in Sects. 4.1 and 7.2 of [WY2]. Specifically, in Part A, (5)(a) is Condition (A2) in [WY2], and (5)(b) follows from (A4) and Lemma 5.2 in [WY2].

2.3 Dynamics of $T \in \mathcal{G}$

A. Control of individual orbits: Critical orbits of $T \in \mathcal{G}$ have certain properties by design (Sect. 2.2A(5)). Roughly speaking, we say an arbitrary orbit is controlled – a term with precise technical meaning – if every time it enters $C^{(1)}$, it follows a critical orbit and copies its behavior. The precise definition is as follows:

A number $\hat{\beta}$ satisfying $\alpha << \hat{\beta} << 1$ is fixed. Given $\xi_0 \in C^{(1)}$, $z_0 \in \bigcup_k \Gamma_k$ is a guiding critical point for $\xi_0$, written $\phi(\xi_0) = z_0$, if
Then the angle between $w$ and a fixed constant $K_n (\text{Bound period}) (1)$ returns. We say that $w$ is controlled up to time $n_1 \geq 0$ be the first time $\xi_n \in C^{(1)}$. We say the orbit segment $\xi_0, ..., \xi_n$ is free, and call $\xi_n$ the first free return. Assume that $\phi(\xi_n)$ exists, and let $p_1 = \hat{p}(\xi_n, \phi(\xi_n))$. The orbit segment from $\xi_{n+1}$ to $\xi_{n+p_1-1}$ as said to be bound to that of $\phi(\xi_n)$. Let $n_2 \geq n_1 + p_1$ be the first time $\xi_n \in C^{(1)}$ again, i.e., $n_2$ is the time of the next free return. Under the assumption that $\phi(\xi_n)$ exists, we have the second bound period $p_2$, and so on. In general, if guiding critical points exist at all free returns, we can divide the orbit of $\xi_0$ into an alternating sequence of free and bound segments. Such an orbit is said to be controlled.

**B. Control of tangent vectors:** Outside of $C^{(1)}$, $b$-horizontal vectors are transformed essentially by the derivative of the 1D map $f_0$. More precisely:

1. (Derivative growth out of $C^{(1)}$) Assume that $\xi_i \in R_1 \setminus C^{(1)}$ for $0 \leq i \leq n-1$, and let $w_0$ be a tangent vector at $\xi_0$ that is $b$-horizontal. Then $w_n$ is $b$-horizontal, and $|w_n| > c\delta^{\frac{1}{4}}|w_0|$. We can drop the factor $\delta$ if, in addition, $w_n \in C^{(1)}$.

2. Next we consider a pair $(\xi_0, w_0)$ where $w_0$ is a $b$-horizontal tangent vector at $\xi_0$ if $\xi_0 \not\in C^{(1)}$ and $w_0 = v$ if $\xi_0 \in C^{(1)}$. Assume that $\{\xi_i\}_{i=1}^{n}$ is controlled up to time $n$, and $\xi_n$ is a free return. We say that $w_n$ is aligned correctly if the following holds: Suppose $\phi(\xi_n) = z_0^*(Q^{(j)})$. Then the angle between $w_n$ and the tangent direction to $F_j$ at $\xi_n$ is $< \varepsilon_1 d_C(\xi_n)$ where $\varepsilon_1$ is a fixed constant $<< K^{-1}$ for a certain $K > 1$. We say the pair $(\xi_0, w_0)$ is controlled up to time $n$ if the orbit of $\xi_0$ is controlled up to the time $n$ and for $i \leq n$, the tangent vectors $w_i$ are correctly aligned at all free returns.

Suppose $(\xi_0, w_0)$ is controlled up to and including time $i$. Let $\xi_i$ be a free return, and let $p = \hat{p}(\xi_i, \phi(\xi_i))$. The following is proved in [WY2]:

(i) $\xi_0 \in Q^{(k)} \setminus B^{(k)}$ where $z_0 = z_0^*(Q^{(k)})$, and

(ii) if $\hat{p}(z_0, \xi_0)$ is the smallest $j > 0$ so that $|z_j - \xi_j| \geq e^{-\delta j}$, then $\hat{p}(z_0, \xi_0) \leq k\theta^{-1}$.

The number $\hat{p} = \hat{p}(z_0, \xi_0)$ is called the bound period between $\xi_0$ and $z_0$. (ii) above ensures that the period of activity of the guiding critical point does not expire before the end of the bound period, i.e. before the copying process is completed. From Sect. 2.2A(3) and the definition of $B^{(k)}$ in Sect. 2.2A, (i) implies that $d_C(\xi_0) \approx |\xi_0 - z_0|$. Note that for arbitrary $\xi_0$, $\phi(\xi_0)$ need not exist. It is also often the case that there are many candidates for $\phi(\xi_0)$.

We can drop the factor $\delta$ if, in addition, $w_n \in C^{(1)}$.

3. Assume that $\phi(\xi_n) = z_0^*(Q^{(j)})$. Then the angle between $w_n$ and the tangent direction to $F_j$ at $\xi_n$ is $< \varepsilon_1 d_C(\xi_n)$ where $\varepsilon_1$ is a fixed constant $<< K^{-1}$ for a certain $K > 1$. We say the pair $(\xi_0, w_0)$ is controlled up to time $n$ if the orbit of $\xi_0$ is controlled up to the time $n$ and for $i \leq n$, the tangent vectors $w_i$ are correctly aligned at all free returns.

Suppose $(\xi_0, w_0)$ is controlled up to and including time $i$. Let $\xi_i$ be a free return, and let $p = \hat{p}(\xi_i, \phi(\xi_i))$. The following is proved in [WY2]:

1. (Bound period) $\frac{1}{3\ln|DT^\alpha|} \ln d_C(\xi) < p < \frac{3}{4} \ln d_C(\xi)$.

2. (Direction at end of bound period) $w_{i+p}$ is $b$-horizontal.

3. (Partial derivative recovery) $|w_{i+p}| > K^{-1} e^{\frac{1}{4}p\lambda_0}|w_i|$.

4. We also have the following estimate at free returns:

   (4) (Alignment at free returns) Assume that $(\xi_0, w_0)$ is controlled up to time $n-1$, and $\xi_n$ is a free return with $\phi(\xi_n) = z_0^*(Q^{(k)})$ for some $k < n$. Then the angle between $w_n$ and the tangent direction of the foliation $F_k$ at $\xi_n$ is $< (Kb)^k$.

**C. Evolution of curves:** In connection with the control scheme outlined in Parts A and B, we review the following two sets of results from [WY2] on the evolution of curves. In the first, control is proved for leaves of $F_k$ that lie in the main bodies of monotone branches:
(1) (Control on main body of monotone branch) Let $\xi_1 \in R_1$ be such that $\xi_k \in M^0$ for some $M \in T_k$, and let $\tau_1$ be a vector tangent to the leaf of $F_1$ at $\xi_1$. Then $(\xi_1, \tau_1)$ is controlled up to time $k - 1$.

The rest of Part C discusses properties of curves that are assumed to be controlled. Let $\gamma_0$ be an arbitrary $C^2(b)$-curve in $R_1$, and let $\gamma_i = T^i(\gamma)$.

(2) (Outside of $C^{(1)}$) If $\gamma_j \cap C^{(1)} = \emptyset$ for all $0 \leq j < i$, then $\gamma_i$ is a $C^2(b)$-curve.

Our problem is what happens after $\gamma_i$ meets $C^{(1)}$. To control distortion near the “turns”, it is necessary to cut $\gamma_i$ into short segments and to treat each one separately. For bookkeeping purposes, we use the following partition $\mathcal{P} = \{I_{\mu,j}\}$ on $[-\delta, \delta]$ introduced in [BC1]: first partition $[0, \delta]$ into intervals of the form $I_\mu = (e^{-\mu}, e^{-\mu+1})$, then subdivide each $I_\mu$ into $\frac{1}{\mu^2}$ subintervals of equal length called $I_{\mu,j}$, and finally extend $\mathcal{P}$ to $[-\delta, 0]$ by reflection, i.e. $I_{\mu,j} = -I_{-\mu,j}$ for $\mu < 0$. We also use $I_{\mu,j}^+$ to denote the union of the 3 consecutive elements of $\mathcal{P}$ centered at $I_{\mu,j}$.

Let $\gamma_0$ be a $C^2(b)$-curve in $C^{(1)}$. We assume all points in $\gamma_0$ have a common guiding critical point $z_0$, and all tangent vectors are correctly aligned. Let $I(\gamma_0) = \{\|\xi - z_0\|, \xi \in \gamma_0\}$. We say $\gamma_0$ is an $I_{\mu,j}$-segment if $I(\gamma_0) \subset I_{\mu,j}^+$. If additionally $I(\gamma_0) \supset I_{\mu,j}$, then we call $\gamma_0$ a full $I_{\mu,j}$-segment.

(3) (End of bound period) If $\gamma_0$ is an $I_{\mu,j}$-segment, then $\gamma_p$ is a $C^2(b)$-curve where $p = p(\gamma_0)$ is the minimum of the bound period of all $\xi_0 \in \gamma_0$. If $\gamma_0$ is a full $I_{\mu,j}$-segment, then $|\gamma_p| > e^{-K_0|\mu|}$.

We say all the points in a $C^2(b)$-curve $\gamma_0$ have the same itinerary through step $n - 1$ if (i) all $(\xi, \tau(\xi))$ where $\xi \in \gamma_0$ and $\tau(\xi)$ a unit tangent vector to $\gamma_0$ at $\xi$ are controlled through step $n - 1$, (ii) for every $i < n$, either all $\xi \in \gamma_i$ are free, or they are all in bound state, and (iii) at each free return, $\gamma_i$ is an $I_{\mu,j}$-segment with a common guiding critical point.

(4) (Distortion estimate) There exists a constant $K$ such that if $\gamma_0$ is a $C^2(b)$-curve consisting of points with the same itinerary through step $n - 1$, and $\gamma_n$ is free, then for all $\xi, \xi' \in \gamma_0$,

$$\frac{|DT_\xi^\mu(\tau(\xi))|}{|DT_\xi^\mu(\tau(\xi'))|} < K.$$ 

D. Stable manifolds: The results in Part D hold for all $T \in \mathcal{G}_0$ (i.e. $T$ need not be in $\mathcal{G}$). The following construction produces, under suitable conditions for $z_0 \in R_1$, a codimension one temporary stable manifold $W_n(z_0)$ of order $n$ through $z_0$.

We need the following terminology: Let $S$ be a 2D plane in $I \times \mathbb{R}^{m-1}$. Abusing notation slightly, we also use $S$ to denote the 2D subspace of the tangent space at $z \in S \cap R_1$. Let $e_i(z, S)$ be a unit vector in the most contracted direction of $DT_z^i$ in $S$, i.e., $|DT_z^i e_i(z, S)| \leq |DT_z^i u|$ for all unit vectors $u \in S$. A curve $\gamma$ in $S \cap R_1$ is called a stable curve of order $i$ if $e_i(z, S)$ is well defined at every $z \in \gamma$ and is tangent to $\gamma$.

The following is proved in [WY2]. Consider either

(i) $z_0 \in R_1$ with $d_{C}(z_0) > \delta^2$, and $w_0$ a $b$-horizontal tangent vector at $z_0$, or
(ii) $z_0 \in C^{(1)}$ and $w_0 = v$.

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(1) (Order 1 stable curves) Let $S$ be a 2D plane containing both $z_0$ and $z_0 + w_0$. Then through $z_0$ there exists a stable curve of order 1 in $S$. This curve, maximally extended, is denoted by $\gamma_1(z_0, S)$. In Case (i), the slope of $\gamma_1(z_0, S)$ is everywhere $> K^{-1}\delta^2$. In Case (ii), $\gamma_1(z_0, S)$ has the shape of a parabola with curvature $> K^{-1}$.

(2) (Stable curves $\gamma_i$ of different orders) Let $\kappa$ be a constant such that $1 > \kappa > b^{\frac{1}{5}}$. In both cases (i) and (ii), if $|w_i| > \kappa^i$ for all $i \leq n$, then stable curves $\gamma_i = \gamma_i(z_0, S)$ of order $i$ exist for all $i \leq n$, and $\|\gamma_i - \gamma_{i-1}\|_{C^2} < (K\kappa^{-4})^{i-1}$.

(3) (Contraction along $\gamma_n$) Assuming $\gamma_n$ is defined, then for $z'_0 \in \gamma_n(z_0, S)$, we have $|DT_{z'_0}e_n(z'_0, S)| < (K\kappa^{-2})^i$ for all $i < n$. It follows that $|z'_i - z_i| < (K\kappa^{-2})^i$.

To obtain $W_n(z_0)$ under the conditions above, we let $S$ vary over all 2D planes containing $z_0$ and $z_0 + w_0$, and take the union of all the $\gamma_n(z_0, S)$.

References to [WY2]: Parts A and B of Sect. 2.3 are mainly from Sect. 5.2 of [WY2]. Specifically, B(0) is from Lemmas 3.4(a) and 3.5 of [WY2]. B(1) is Proposition 5.2(1)(i) of [WY2]. See also (A5)(i) of Sect. 4.1 in [WY2]. B(2) is Lemma 3.10 of [WY2] and B(3) is Proposition 5.2(2) of [WY2]. Observe that in this exposition we have avoided introducing splitting period and the splitting algorithm. To properly interpret the corresponding statements in [WY2], the reader should be aware that at a free return, the bound period is much larger than the splitting period and, outside of the splitting period, $w^* = w$. B(4) follows from the computation in the proof of Proposition 6.2(i) of [WY2].

For Part C: C(1) is (IA6)′(II) in Sect. 7.2 of [WY2]. C(2) is Lemma 3.4(b) of [WY2]. C(3) and C(4) are (P2)′(iii) and (P3)′ respectively in Sect. 9.2 of [WY2].

For Part D: D(1) is from Sect. 3.6 of [WY2]: (i) is Lemma 3.6 and (ii) is lemma 3.7. (It makes no difference in the proofs whether one uses $\delta^2$ or $\delta$.) The construction of temporary stable curves and temporary stable manifolds are in Sect. 3.3 of [WY2]. D(2) and D(3) are from Lemma 3.1(a) of [WY2].

With this we conclude our review of the properties of $T \in \mathcal{G}$. There will still be a few technical facts we must cite directly from [WY2], but the vast majority of the results we need in the proofs to follow are contained in the preceding pages.

3 Building Hyperbolic Behavior

This section contains a number of preliminary results that will be used in the proofs of several of the theorems.

3.1 A no-deep-returns condition

We prove here a sufficient condition for prolonging control from one free return to the next for a pair $(z_0, w_0)$. For the definition of “control”, see Sect. 2.3A,B. Roughly speaking, our result says that once control is established, we will continue to have it if the orbit does not venture into too deep a layer of the critical structure too quickly.

Let $K_4 = \frac{6}{X} \log \frac{1}{p^*}$. Within each $Q^{(k)}$, define

$$Z^{(k)} := \{z \in Q^{(k)} : |z - z_0^*(Q^{(k)})| < b^{\frac{k}{K_4^2}}\}.$$ 

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Notice that $Z^{(k)}$ is larger than the critical blob $B^{(k)}$ (Recall that $\theta$ is defined by $b^\theta = \|DT\|^{-20}$, and $B^{(k)}$ is defined in Sect. 2.2A). Our choice of $K_4$ ensures that for $\xi \in Q^{(k)} \setminus Z^{(k)}$ and $z_0 = z_0^*(Q^{(k)})$, the bound period $p(z_0, \xi)$ is $< \frac{1}{2} \theta^{-1} k$, which is shorter than the period of activity of $z_0$. This is not true for all the $\xi \in Q^{(k)} \setminus B^{(k)}$. (See Sects. 2.2A(4),(5) and 2.3A,B).

**Lemma 3.1** Let $z_1 \in R_1$ and $w_1 \in \mathbb{R}^n$ be such that $(z_1, w_1)$ is controlled by $\Gamma := \cup_j \Gamma_j$ through time $k - 1$, and let $z_k$ be a free return. If $z_k \in Q^{(i)} \setminus Z^{(i)}$ for some $Q^{(i)}, i \leq k$, then $w_k$ is correctly aligned. It follows immediately that control is extended to $(z_k, w_k)$.

**Proof:** Let $F_i$ be the foliation in $R_i$ as defined at the beginning of Sect. 2.2, and let $\tau_i$ be tangent to the leaf of $F_i$ at $z_k$. By Sect. 2.3B(4),

$$\angle(w_k, \tau_i) < (Kb)^i << K^{-1} b^\frac{x_i}{4} < K^{-1} d_c(z_k),$$

which is what is required for the correct alignment of $w_k$ (Sect. 2.3B).

### 3.2 Eventual control of almost every $z \in R_1$

The goal of this subsection is to prove

**Proposition 3.1** Assume (*) in Sect. 1.1. Then for Lebesgue-a.e. $z \in R_1$, there exist $v \in \mathbb{R}^n$ and $n = n(z) \in \mathbb{Z}^+$ such that if $x_1 = T^n z$ and $w_1 = DT^n v$, then $(x_1, w_1)$ is controlled by $\Gamma$ for all times.

We introduce the following notation: Let

$$\hat{Z}^{(k)} := \{ z \in Q^{(k)} : |z - z_0^*(Q^{(k)})| < b^{\frac{m}{4} k K_4} \}.$$

Let $Z^{(k)}$ denote the union of all the $Z^{(k)}$, $\hat{Z}^{(k)}$ the union of the $\hat{Z}^{(k)}$, and define

$$\hat{Z}_k = \cup_{k(1+2\theta)^{-1} < j \leq k} \hat{Z}^{(j)}.$$

The need for a set that includes multiple generations is that $Q^{(k)}$ may not be present in consecutive generations (Sect. 2.2A(2)). A key ingredient in the proof of Proposition 3.1 is

**Lemma 3.2** Assume (*). Then

$$\text{Leb}\{z_1 \in R_1 : z_k \in \hat{Z}_k \text{ infinitely often} \} = 0.$$

Lemma 3.2 is proved using a Borel-Cantelli argument. To make this argument, we need a better estimate on the volume of $Q^{(k)}$ than what is in Sect. 2.2A(1). Such an estimate is given in Lemma 3.3 below. The following language is used: We say a codimension one manifold $W$ spans the cross-section of $R_1$ (abbrev. “spans $R_1$”) and has diameter $< r$ if for every 2D plane $S$ containing the $x$-axis, $W \cap S$ is a curve of length $< r$ connecting the two components of $S \cap \partial R_1$. For $z \in R_1$, let $V(z)$ denote the $(m - 1)$-dimensional vertical subspace through $z$, and $D_r(z)$ the disk of radius $r$ centered at $z$ in $V(z)$. We call $A \subset Q^{(k)}$ a vertical slice of $Q^{(k)}$ if $A = Q^{(k)} \cap V(z)$ for some $z \in Q^{(k)}$. Finally, let $\varepsilon_2 < \frac{1}{3K_4(m-1)}$ be a positive number independent of $b$.
Lemma 3.3 Assume (*). Then there exists $K > 0$ such that for every $Q^{(k)}$ and every vertical slice $A$ of $Q^{(k)}$, there exists $z \in A$ such that

$$D_{(K-b)^{k+1}}(z) \subset A \subset D_{(Kb)^{(1-\epsilon_2)k}}(z).$$

Proof of Lemma 3.3: To prove the first inclusion, a crucial fact is that there exists $K > 0$ such that $|DTu| \geq K^{-1}b|u|$ for all $u$. This follows from (*) together with the fact that $|DT(\partial_x)| < K_0$ and $|DT(\partial_y)| < K_0b$, $1 \leq i < m$, for some $K_0$.

Let $z \in T^k\{y = 0\} \cap A$. We claim that $D_{(K-b)^{k+1}}(z)$ is contained in $A$. Suppose not. Then there exists $z' \in A \cap \partial Q^{(k)}$ with $|z - z'| < (K-b)^{k+1}$ such that $z'$ can be joined to $z$ by a straight line $\ell$ lying entirely in $Q^{(k)}$. Let $\ell_0 = T^{-k}\ell$, and let $\gamma$ be a parametrization of $\ell_0$ by arclength. Using the lower bound on $|DTu|$ above and the fact that $\ell_0$ connects $\{y = 0\}$ to $\partial R_1$ and therefore has length $> K^{-1}b$, we see that

$$|z - z'| = \int |DT^k\gamma'| > (K-b)^{k+1},$$

contradicting the supposition above.

The idea for the proof of the second inclusion is as follows: Let $z_k$ be an arbitrary point in $Q^{(k)}$. Suppose we know that through $z_1$ there is a temporary stable manifold $W$ of order $k-1$ (Sect. 2.3D) that spans $R_1$. We may then argue that $T^{k-1}(W)$, which contains $z_k$ and whose diameter we can estimate using Sect. 2.3D(3), spans a horizontal section containing $Q^{(k)}$. That would allow us to estimate the diameter of the vertical slice of $R_k$ through $z_k$. There are, however, technical issues in making this line of reasoning precise. We start with the existence of a codimension 1 temporary stable manifold.

Sublemma 3.1 There exists $j$ with $(1 - \frac{1}{2}\epsilon_2)k < j < k$ and $\epsilon(b) > 0$ with $\epsilon(b) \to 0$ as $b \to 0$ such that through $z_{k-j}$, there is a stable manifold $W_j$ of order $j$ with the following properties:

(i) $W_j$ spans $R_j$ and has diameter $< K\delta^{-1}b$;

(ii) for all $\xi \in W_j$, $|T^i\xi - T^i z_{k-j}| < (Kb)^{1-\epsilon(b)}$ for all $i \leq j$.

Since $k-1$ may not be a viable choice of $j$ in Sublemma 3.1, a second complication arises: we do not know that $Q^{(k)}$ is contained in a horizontal section of $R_j$, so cannot use $T^j(W_j(z_{k-j}))$ directly to estimate the diameter of the vertical section of $Q^{(k)}$ as proposed earlier. Hence we need

Sublemma 3.2 Let $j$ be as in Sublemma 3.1. Then there exist $j'$ with $(1+2\theta)^{-1}j < j' < j$ and a $Q^{(j')}$ such that $Q^{(k)} \subset Q^{(j')}$ and $T^{j'}(W_j)$ meets every leaf of $F_j'$ in $Q^{(j')}$.

We first complete the proof of Lemma 3.3 assuming these two sublemmas. Let $A'$ denote the vertical slice of $Q^{(j')}$ through $z_k$. It suffices to show $A' \subset D_{(Kb)^{(1-\epsilon_2)k}}(z)$. Fix $\xi \in (A' \cap \partial Q^{(j')}).$ Let $\ell$ be the leaf of $F_j'$ through $\xi$, $\hat{\ell} \subset (\ell \cap T^{j'}(W_j))$, and let $\hat{\ell}$ denote the segment of $\ell$ between $\xi$ and $\hat{\xi}.$ Since $\hat{\ell}$ is a $C^2$ curve, its length is approximately the horizontal distance between $\xi$ and $\hat{\xi}$, which is $\leq |z_k - \hat{\xi}|$. Thus

$$|z_k - \xi| < |z_k - \hat{\xi}| + |\hat{\xi} - \xi| \lesssim 2|z_k - \hat{\xi}| < 2(Kb)^{(1-\epsilon(b))j'},$$

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which for \( b \) small enough is \(< (Kb)^{(1-\varepsilon_2)k} \).

**Proof of Sublemma 3.1:** Let \( \tau_1 \) be the tangent vector to \( F_1 \) at \( z_1 \), and note that \((z_1, \tau_1)\) is controlled up to time \( k-1 \) (Sect. 2.3C(1)). We will show that for some \( j \), \((1-\frac{1}{2}\varepsilon_2)k < j < k\), for \( \xi_0 = z_{k-j} \) and \( u = \frac{\tau_{k-j}}{|\tau_{k-j}|} \), we have

\[
|DT_{\xi_0}^u u| > |DT|^{-\frac{4}{\varepsilon_2}} s
\]

for all \( s < j \). (3) is proved as follows: Consider the graph \( G \) of \( i \to \log |\tau_i| \) for \( 1 \leq i < k \). Let \( L \) be the (infinite) line through \( (k, \log |\tau_k|) \) of slope \( = \log \|DT\| \). Clearly all points in \( G \) (other than \((k, \log |\tau_k|)\)) lie above \( L \). Let \( P \) be the intersection of \( L \) with the line \( i = \frac{\varepsilon_2}{2} k \). We let \( L \) be pivoted at \( P \), and rotate it clockwise until it hits some point on \( G \). Let \( k-j \) be the first coordinate of the first point hit. Then \( 1 \leq k-j < \frac{1}{2}\varepsilon_2 k \), and (3) is proved if we can show that in its final position, the slope of \( L \) is \( > \frac{4}{\varepsilon_2} \log \|DT\| \). This is true because \( z_k \) being a free return, \( |\tau_1| > |\tau_k| \), so the slope of the straight line joining the two points \((1, \log |\tau_1|) \) and \( P \) is \( > \frac{4}{\varepsilon_2} \log \|DT\| \).

Let us assume without loss of generality that \( z_{k-j} \not\in \mathcal{C}(1) \) (consider \( z_{k-j+1} \) if \( z_{k-j} \in \mathcal{C}(1) \)). Let \( w = \partial_x \). We argue that we in fact have, for all \( s < j \),

\[
|DT^s w| \geq K^{-1}\delta |DT|^{-\frac{4}{\varepsilon_2}} s.
\]

To prove (4), we need only to consider the case \( w \neq \pm u \). Let \( S = S(u, w) \). From (3), \( e_s(z_{k-j}, S) \) is well defined at \( z_{k-j} \) and \(|e_s(z_{k-j}, S) - e_1(z_{k-j}, S)| = O(b) \) for all \( s < j \) (Sect. 2.3D(2)). Since \( z_{k-j} \not\in \mathcal{C}(1) \), it follows from Sect. 2.3D(1) that

\[
\angle(e_s(z_{k-j}, S), w) > K^{-1}\delta.
\]

Combining (3) and (5), we get (4). Using (4) we obtain \( |DT^s w| > \kappa_s \) for all \( s < j \) where \( \kappa = K^{-1}\delta |DT|^{-\frac{4}{\varepsilon_2}} > b^2 \). Sect. 2.3D then guarantees the existence of a stable manifold \( W_j \) through \( z_{k-j} \) of order \( j \). The other properties of \( W_j \) asserted are evident.

The proof of Sublemma 3.2 uses some ideas on monotone branches developed in Sect. 8.3 of [WY2]. These ideas, which are also important for the proof of Theorem 7, are a little too detailed to be included in the review in Section 2. We recall them in the Appendix, which contains also a proof of Sublemma 3.2.

**Proof of Lemma 3.2:** By the Borel-Cantelli Lemma, it suffices to show that

\[
\sum_k \text{Leb} \left( T^{-k} (\cup_{k(1+2\theta)^{-1} < j \leq k} \hat{Z}^{(j)} \cap R_k) \right) < \infty.
\]

We estimate \( \hat{S}_k \), the \( k \)-th term of this sum, by \( \hat{S}_k \leq \sum_{j=k(1+2\theta)^{-1}+1}^k \hat{S}_{kj} \) where

\[
\hat{S}_{kj} := \text{Leb} \left( T^{-(k-j)} T^{-j} (\hat{Z}^{(j)} \cap R_k) \right) \leq \frac{1}{(K^{-1}\theta \theta^{-1})^{2\theta_k}} \text{Leb} \left( T^{-j} \hat{Z}^{(j)} \right).
\]

Now

\[
\text{Leb} \left( T^{-j} \hat{Z}^{(j)} \right) \leq \max \frac{\text{Leb}(T^{-j}Q^{(j)} \cap \hat{Z}^{(j)}))}{\text{Leb}(T^{-j}Q^{(j)})} \cdot \sum \text{Leb}(T^{-j}Q^{(j)})
\]

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where the max and sum above are over all $Q^{(j)}$. Using first the regularity assumption (*) on $\det(DT)$ and then Lemma 3.3, we obtain, for $\hat{Z}^{(j)} \subset Q^{(j)}$,

$$\max \frac{\text{Leb}(T^{-j} \hat{Z}^{(j)})}{\text{Leb}(T^{-j} Q^{(j)})} \leq K^{2j} \frac{\text{Leb}(\hat{Z}^{(j)})}{\text{Leb}(Q^{(j)})} \leq K^{2j} \frac{(Kb^{n-1})^{(1-\varepsilon_2)}b \lambda^{-j}}{(K-1)b^{m-1})+1+\varepsilon_3}.$$ 

With $3\varepsilon_2(m-1) < \frac{1}{K^4}$, it follows that $\sum_{j,k} S_{kj} < \infty$. \hfill $\Box$

**Proof of Proposition 3.1:** Let $Y_0 = \{z_1 \in R_1 : z_k \notin \hat{Z}_k \text{ for all } k \geq 1 \}$ and for $i > 0$, let

$$Y_i = \{z_1 \in R_1 : z_i \in \hat{Z}_i \text{ and } z_k \notin \hat{Z}_k \text{ for all } k > i \}.$$ 

By Lemma 3.2, $\cup_{i \geq 0} Y_i$ has full Lebesgue measure. We will prove the eventual control of all points in $\cup Y_i$ (in the sense of Proposition 3.1) one $Y_i$ at a time.

Fix $i \geq 1$. For $z_1 \in Y_i$, let $\xi_1 = z_1$ so $\xi_1 \in \hat{Z}_i$ by definition. Let $w_1 = v$. We will show that $(\xi_1, w_1)$ is controlled for all time. By Lemma 3.1, it suffices to check the no-deep-return condition for the orbit of $\xi_1$. Suppose $\xi_n \in C^{(1)}$. Let $j$ be the largest integer $\leq n$ such that $\xi_n \in Q^{(j)}$. If $j \leq (1+2\theta)^{-1}n$, then there exists $j', j < j' \leq n$, and a horizontal section $H$ of $R_{j'}$, such that $\xi_n \in Q^{(j')} \cap H$ but $\xi_n$ is not in the $Q^{(j')} \cap H$. This is clearly acceptable. It now suffices for us to prove

$$\xi_n \notin \cup_{n(1+2\theta)^{-1} < k \leq n} Z^{(k)}$$ \hspace{1cm} for all $n > 0$. \hspace{1cm} (6)

To prove (6), let $i_0$ be the largest $j$, $i(1+2\theta)^{-1} < j \leq i$, such that $\xi_1 \in \hat{Z}^{(j)}$. We regard $\xi_1$ as bound to $\hat{z} = z^*_0(\hat{Q}^{(i_0)})$ and let $n_0 = \min\{p(\hat{z}, \xi_1), \theta^{-1} i_0\}$. Then $n_0 > K^{-1}\theta^{-1}i_0$ by Sect. 2.3B(1). The relation in (6) holds for $n < n_0$ because $d_C(\hat{z}_n) > e^{-\alpha n}$. Suppose, to derive a contradiction, that it fails for some $n > n_0$. This implies there exists $k'$, $n(1+2\theta)^{-1} < k' \leq n$, such that $\xi_n \in Q^{(k')}$ and $|\xi_n - z^*_0(Q^{(k')})| < b^{\frac{1}{K^4}}$. If $k' > (n+i)(1+2\theta)^{-1}$, then $z_{n+i} = \xi_n \in \hat{Z}_{n+i}$, contradicting $z_{n} \in Y_i$. If not, there is $k'' \in ((n+i)(1+2\theta)^{-1}, (n+i)]$, which is only slightly larger than $k'$ since $i < K\theta n$, such that $\xi_n \in Q^{(k'')} \subset Q^{(k')}$. By Sect. 2.2A(3),

$$|z^*_0(Q^{(k'')}) - z^*_0(Q^{(k')})| < K b^{\frac{1}{K^4}(1+2\theta)^{-1}}.$$ 

Thus $|\xi_n - z^*_0(Q^{(k'')})| > b^{\frac{1}{K^4}K^4}$, again contradicting $z_{n} \in Y_i$.

For $i = 0$, we take $w_1 = \partial_x$ instead of $\mathbf{v}$, since $z_1 \in Y_0$ is away from $\mathcal{C}$. The argument is similar. \hfill $\Box$

### 3.3 Control of $C^2(b)$-curves

In this subsection, we focus on the iteration of $C^2(b)$-curves. Given a $C^2(b)$-curve $\gamma$, we will construct inductively (i) a decreasing sequence of subsets $\gamma \supset \Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset \cdots$ so that for all $z_1 \in \Lambda_i$, $(z_1, \tau_1)$ is controlled up to time $i$, and (ii) a sequence of canonical subdivisions $\{Q_i\}$ on $\Lambda_i$ so that points in each $\omega \in Q_i$ have the same itinerary up to time $i$. The constructions in (i) and (ii) will proceed simultaneously.

Our construction goes as follows:
If $\gamma \cap (\cup Z^{(1)}) = \emptyset$, we let $\Lambda_0 = \gamma$. To construct $Q_0$ on $\Lambda_0$, first we divide $\gamma$ into connected components that lie entirely in or out of $C^{(1)}$. If a component of $\gamma \setminus C^{(1)}$ has length $> 3\delta$, we divide it into subsegments of length between $\delta$ and $3\delta$; otherwise we leave it alone. For each component of $\gamma \cap Q^{(1)}$, we use $z^*_0(Q^{(1)})$ as the guiding critical point (see Sect. 2.3C). If it is an $I_{\mu,j}$-segment, we leave it alone; if not, we divide it into full $I_{\mu,j}$-segments. If any segment that results from the subdivisions above is “short”, meaning $< \delta$ for a segment outside of $C^{(1)}$ or less than a full $I_{\mu,j}$-segment for a segment inside $C^{(1)}$, we attach it to a neighbor (if there is one), and call the resulting partition $Q_0$.

If $\gamma \cap (\cup Z^{(1)}) \neq \emptyset$, we let $\Lambda_0 = \gamma \setminus (\cup Z^{(1)})$, and construct $Q_0$ on this set as is done in the last paragraph – except that if there is a “short” component of $\gamma \setminus (\cup Z^{(1)})$ just outside of $Z^{(1)}$, we will remove this short piece from $\Lambda_0$.

We assume inductively that $Q_k, k \leq i - 1$, have been defined, and $\Lambda_k = \cup_{\omega \in Q_k} \omega$. Moreover, we assume (a) for $z_1 \in \Lambda_{i-1}$, $(z_1, \tau_1)$ is controlled up to and including step $i - 1$, (b) all the points in each $\omega \in Q_{i-1}$ have the same itinerary through step $i - 1$, and (c) if $T^{i-1}(\omega)$ is a free return, then a common guiding critical point has been assigned. We now define $\Lambda_i$ and $Q_i$ on each $\omega \in Q_{i-1}$ as follows:

(i) If $T^i(\omega)$ is in a bound state, we put $\omega$ in $Q_i$.

(ii) Suppose $T^i(\omega)$ is free, so in particular it is a $C^2(b)$ curve. In each component of $T^i(\omega) \cap Q^{(1)}$, we let $k$ be the largest number $\leq i + 1$ such that $T^i(\omega) \cap Q^{(k)} \neq \emptyset$, delete $z \in \omega$ such that $T^i(z) \in Z^{(k)}$, and assign $z^*_0(Q^{(k)})$ as guiding critical point for $(T^i(\omega) \cap Q^{(1)}) \setminus Z^{(k)}$. The set that remains after these deletions is $\omega \cup \Lambda_i$ – except where a deletion leaves a short segment of $(T^i(\omega) \cap Q^{(1)}) \setminus Z^{(k)}$ just outside of some $Z^{(k)}$; that we delete as in the definition of $\Lambda_0$. To construct $Q_i$ on $\omega \cup \Lambda_i$, we divide the $T^i$-image of each connected component of $\omega \cup \Lambda_i$ into full $I_{\mu,j}$-segments for the part inside $C^{(1)}$ and into segments of size $\delta$ for the part outside of $C^{(1)}$, and attach short end pieces to their neighbors as before. The pullback of this partition by $T^{-i}$ is our partition $Q_i$ on $\omega \cup \Lambda_i$.

Observe that for points in $\Lambda_i$, control is extended one more step by Lemma 3.1 since $k \leq i + 1$. So the inductive assumption (a) holds for all $z \in \Lambda_i$. Inductive assumptions (b) and (c) for $z \in \Lambda_i$ follow directly from (i) and (ii).

Remarks: (1) For the construction above, it is possible for $\Lambda_i$ to be empty for all $i \geq i_0$. Further assumptions are needed to guarantee that $\Lambda_i \neq \emptyset$ for all $i$.

(2) We observe that the deletions in (ii) can take place only when $k(1 + 2\theta) > i$. To prove this claim, we assume the contrary, and let $z \in \omega$ be such that $T^i(z)$ is just inside $Z^{(k)}$. Then $T^i(z) \subset R_{i+1} \subset R_{k(1+2\theta)}$, so it lies in a monotone branch $M$ of generation $k'$ for some $k' \in (k, k(1 + 2\theta)]$ (Sect. 2.2B(3)). It then follows from Sect. 2.2B(2) that $M^c \cap Q^{(k)}$ contains a $Q^{(k')}$. But by assumption, $z \in Z^{(k')} \setminus Q^{(k')}$, which is impossible.

(3) It follow from (2) that for $z \in \Lambda_n$,

$$d_C(z_i) > b_i^\theta b_i^{-(1+2\theta)^{-1}} \text{ for all } i \leq n.$$ 

This rule of distance exclusion is the least stringent we use to guarantee that control can be extended to the next step. It is useful when we wish to control as large a set as possible.
integer such that $\omega$ and (ii) it is a full $IT$ so the fraction of $\gamma (\Lambda_\infty )$ deleted initially , we are guaranteed that $m_\gamma (\Lambda_0 ) > \frac{99}{100} |\gamma |$. We will estimate $m_\gamma (\Lambda_i )/m_\gamma (\Lambda_{i-1} )$, treating separately the two cases $i \leq i_1$ and $i > i_1$ where $e^{-\alpha i_1} \approx \delta^2$.

Fix $i \leq i_1$, and let $\omega \in Q_{i-1}$ be such that part of $T^i (\omega )$ is deleted. Let $\ell$ be the smallest integer such that $\omega \in Q_{i+\ell}$. Then there are two possibilities for $T^i (\omega )$: (i) it is outside of $C^{(1)}$, and (ii) it is a full $I_{\mu ,\gamma}$-segment. In case (ii), $T^i (\omega)$ has length $e^{-K\alpha |\mu |}$ by Sect. 2.3C(3), so the fraction of $T^i (\omega)$ deleted is

$$\frac{|\{ z \in T^i \omega : d_\gamma (z) < \delta^2 \}|}{|T^i \omega |} < \frac{2 \delta^2}{e^{-K\alpha |\mu |}} .$$

(8)

Now $\delta^2 \approx e^{-\alpha i_1}$ and $|\mu | \leq \alpha i_1$ by definition. Pulled back to $\omega$, the estimate above gives, by Sect. 2.3C(4),

$$\frac{|\{ z \in \omega : d_\gamma (T^i z) < \delta^2 \}|}{|\omega |} < \frac{K e^{-\alpha i_1}}{e^{-K\alpha^2 i_1}} < K e^{-\frac{1}{2} \alpha i_1} = K \delta .$$

(9)

Case (i) is treated similarly, except that we have $|T^i (\omega )| > \frac{1}{10} \delta$ and $|T^i (\omega )| > \frac{7}{10} \delta$ by Sect. 2.3B(0). The fraction in (9) is also $< K \delta$. Pooling these estimates for all $\omega \in Q_{i-1}$, we have $m_\gamma (\Lambda_i )/m_\gamma (\Lambda_{i-1} ) > 1 - K \delta$.

Consider now $i > i_1$, and proceed as before. In case (ii), the numerator in (8) is $\leq 2 e^{-\alpha i}$, and $|\mu | \leq \alpha \ell$. Using the weaker bound $|\mu | < \alpha i$, we obtain

$$\frac{\text{fraction of } \omega \text{ deleted}}{K e^{-\frac{1}{2} \alpha i} < K \delta e^{-\frac{1}{2} \alpha (i - i_1)} .}$$

The same bound holds for case (i). Hence $m_\gamma (\Lambda_i )/m_\gamma (\Lambda_{i-1} ) > 1 - K \delta e^{-\frac{1}{2} \alpha (i - i_1)}$.

To finish,

$$\log \frac{m_\gamma (\Lambda_\infty )}{m_\gamma (\Lambda_0 )} = \log \left( \prod_{i=1}^\infty \frac{m_\gamma (\Lambda_i )}{m_\gamma (\Lambda_{i-1} )} \right) > - \operatorname{const} \cdot K \delta \left( i_1 + \sum_{i > i_1} \frac{1}{2} \alpha (i - i_1) \right) ,$$

which is convergent, proving $m_\gamma (\Lambda_\infty ) > \sigma m_\gamma (\Lambda_0 )$ for some $\sigma > 0$. □
3.4 A hyperbolic set and its stable manifolds

In this subsection we describe a positive Lebesgue measure set $\Lambda$ with the following properties: (i) orbits starting from it are uniformly hyperbolic in forward time, and (ii) it is the union of stable manifolds, each one of which spans the cross-section of $R_1$. Let

$$\Lambda := \{ z_0 \in R_1 : d_C(z_i) > \min(\delta^2, e^{-\alpha i}) \text{ for all } i \geq 0 \}.$$

**Lemma 3.5** The following hold for all $z \in \Lambda$:

(a) $|DT^i_z(\partial_x)| > K^{-1}\delta^2 e^{2\lambda i}$ for all $i > 0$.

(b) Through $z$ passes a $C^1$ embedded disk $W^s(z)$ with the property that

(i) $W^s(z)$ has codimension 1 and spans the cross-section of $R_1$;

(ii) all tangent vectors to $W^s(z)$ make angles $> K^{-1}\delta^2$ with the $x$-axis;

(iii) for all $\xi \in W^s(z)$, $|T^i \xi - T^i z| < \frac{\alpha}{\delta^2}(K\delta)^i$ for all $i \geq 0$.

(c) For $z, z' \in \Lambda$, either $W^s(z) = W^s(z')$ or $W^s(z) \cap W^s(z') = \emptyset$.

Lemma 3.5(b)(ii) implies that the $W^s$-disks above are uniformly transversal to all $C^2(b)$-curves, each one meeting a $C^2(b)$-curve in at most one point. We remark also that assertions in Lemma 3.5 do not follow directly from standard theories such as those in [HPS] or [R].

**Proof:** (a): Observe that $(z, \partial_x)$ is controlled for all time. If $z_i$ is free, then $|DT^i_z(\partial_x)| > K^{-1}\delta^2 e^{2\lambda i}$ (Sects. 2.3B(0) and 2.3B(3)). If $z_i$ is in a bound state, let $j$ be the next time it is free. By Sect. 2.3B(1) we have $j - i < K\alpha j$. It then follows that

$$|DT^i_z(\partial_x)| > K^{-(j-i)}K^{-1}\delta^2 e^{2\lambda j} > K^{-1}\delta^2 e^{2\lambda j}.$$

(b) and (c): Because of (a), all conclusions of Sect. 2.3D apply. First, we fix a 2D-plane $S$ through $z$ containing a vector parallel to the $x$-axis. We construct for $n = 1, 2, \ldots$, temporary stable curves of order $n$ in $S$ through $z$. These curves make angles $> K^{-1}\delta^2$ with the $x$-axis (Sect. 2.3D(1)) and therefore connect the two components of $S \cap \partial R_1$. Call them $\gamma_n^s(z, S)$. Letting $n \to \infty$, $\gamma_n^s(z, S)$ converges in $C^1$ to a curve $\gamma^s_\infty(z, S)$ (Sect. 2.3D(2)).

Next we rotate $S$ through all 2D-planes containing a vector parallel to the $x$-axis, and define $W_n^s(z) := \cup \gamma_n^s(z, S)$ and $W^s(z) := \cup \gamma_n^s(z, S)$. View each $W_n^s(z)$ as the graph of a continuous function $\varphi_n^z : D \to \mathbb{R}$ where $D$ is the vertical slice of $R_1$ through $z$ and the range is the $x$-axis. Since $\varphi_n^z$ converges uniformly to some $\varphi^z$, it follows that $W_n^s(z)$, which is the graph of $\varphi^z$, spans the cross-section of $R_1$.

We prove next the compatibility property in (c). Consider $z' \in W_n^s(z)$. Since it is contracted to $\Lambda$ rapidly, the arguments above apply also to $z'$. Suppose, to derive a contradiction, that $\varphi^z(y_0) \neq \varphi^{z'}(y_0)$ for some $y_0 \in D$. Let $\gamma$ be a segment parallel to the $x$-axis connecting $\xi := (\varphi^z(y_0), y_0)$ and $\xi' := (\varphi^{z'}(y_0), y_0)$. On the one hand, we have

$$|T^i \xi - T^i \xi'| \leq |T^i \xi - T^i z| + |T^i z - T^i z'| + |T^i z' - T^i \xi'| < \frac{3Kb}{\delta^2}(Kb)^i$$

for all $i$ by Sect. 2.3D(3). This, however, is not possible, since iterates of the curve $\gamma$ grow— not and shrink exponentially— according to Sect. 3.3.

It remains to show for arbitrary $z \in \Lambda$ that $W^s(z)$ is a $C^1$ embedded disk, and to verify (b)(ii). At every $y \in D$, from the construction of $W^s(z)$ at $(\varphi(y), y)$, we see that $\varphi^s(y)$ is
given by \( e_\infty((\varphi(y), y), S(\partial_x, \partial_y)) \). (b)(ii) follows. The continuity of \( y \mapsto \partial_y \varphi(y) \) follows from the continuity of \( y \mapsto e_n((\varphi(y), y), S(\partial_x, \partial_y)) \) and the uniform convergence of these mappings as \( n \to \infty \).

A set that is geometrically nicer to work with than \( \Lambda \) is
\[
\Lambda^s := \bigcup_{z \in \Lambda} W^s_s(z).
\]

### 3.5 Absolute continuity of stable foliations

This property is crucial for the passage of information on unstable curves to positive Lebesgue measure sets in the phase space. The result is well known, but given its importance and (slightly) different formulations by different authors, we have elected to include a proof for the convenience of the reader.

Let \( \gamma_1 \) and \( \gamma_2 \) be two \( C^2(b) \)-curves, and let \( \Lambda^s \) be as above. We say the holonomy map \( \Theta : \gamma_1 \cap \Lambda^s \to \gamma_2 \) is well defined if for every \( z \in \gamma_1 \cap \Lambda^s \), \( W^s_s(z) \) meets \( \gamma_2 \). The intersection, which we denote by \( \Theta(z) \), is a single point by transversality.

**Proposition 3.2** There exists \( K > 0 \) for which the following hold: Let \( \gamma_1 \) and \( \gamma_2 \) be as above, and assume that the holonomy map \( \Theta : \gamma_1 \cap \Lambda^s \to \gamma_2 \) is well defined. Let \( A \) be a Borel subset of \( \gamma_1 \cap \Lambda^s \). Then
\[
K^{-1}m_{\gamma_1}(A) \leq m_{\gamma_2}(\Theta(A)) \leq Km_{\gamma_1}(A). \tag{10}
\]

**Proof:** Since the roles of \( \gamma_1 \) and \( \gamma_2 \) are interchangeable, it suffices to prove the second inequality in (10). Assume without loss of generality that \( A \) is closed. Given \( \varepsilon > 0 \), it suffices to show that \( m_{\gamma_2}(\Theta(A)) < K(m_{\gamma_1}(A) + \varepsilon) \) for some \( K \) independent of \( A \) or \( \varepsilon \). Fix a neighborhood \( U \) of \( A \) in \( \gamma_1 \) such that \( m_{\gamma_1}(U) < m_{\gamma_1}(A) + \varepsilon \). We will show that \( \Theta(A) \) is covered by segments whose lengths sum to \( < Km_{\gamma_1}(U) \).

We iterate \( \gamma_1 \), defining on it canonical subdivisions as in Sect. 3.3 and deleting those \( \omega \in Q_i \) that do not meet \( A \); this ensures that all segments that remain are controlled. A stopping time \( S \) is introduced as follows. Fix \( S_0 \) large enough that for all \( i \geq S_0 \), the following hold for all \( \omega \in Q_i \) that are free at time \( i \): \( \omega \subset U \) and \( |T^i(\omega)| \gg (K\delta^{-2}b)^i \) where \( K \) is as in Lemma 3.5. Define \( S(x) \) to be the smallest integer \( i \geq S_0 \) when \( x \) is free.

For \( \omega \in Q_i \) with \( S|\omega| = i \), we seek to compare \( m_{\gamma_1}(A \cap \omega) \) and \( m_{\gamma_2}(\Theta(A \cap \omega)) \). Let \( \tilde{\omega} \) be the shortest subsegment of \( \omega \) containing \( \omega \cap A \), and let \( \tilde{\Theta}(\tilde{\omega}) \subset \gamma_2 \) be the subsegment whose endpoints are the \( \Theta \)-images of \( \partial \tilde{\omega} \). Then \( \tilde{\omega} \) and \( \tilde{\Theta}(\tilde{\omega}) \) have the same itinerary up to time \( i \), and the Hausdorff distance between \( T^i(\tilde{\omega}) \) and \( T^i(\tilde{\Theta}(\tilde{\omega})) \) is \( \lesssim (K\delta^{-2}b)^i \). If \( |T^i(\tilde{\omega})| \) is long enough that the distance between the two \( C^2(b) \) curves \( T^i(\tilde{\omega}) \) and \( T^i(\tilde{\Theta}(\tilde{\omega})) \) is much smaller than their lengths, and we may conclude \( |T^i(\tilde{\omega})| \approx |T^i(\tilde{\Theta}(\tilde{\omega}))| \). In general, \( T^i(\tilde{\omega}) \) can be considerably shorter than \( T^i(\omega) \), so we settle for the weaker inequality \( |T^i(\tilde{\Theta}(\tilde{\omega}))| \lesssim |T^i(\omega)| \).

To finish, it remains to show \( |\tilde{\Theta}(\tilde{\omega})| \lesssim K|\omega| \). To that end, we pick \( z \in \omega \cap A \) and let \( \tau \) denote tangent vectors to \( \gamma_1 \) or \( \gamma_2 \). Using the shorthand \( a_1 \sim a_2 \) to mean \( K^{-1}a_1 < a_2 < Ka_1 \), we have
\[
\frac{|T^i(\omega)|}{|\omega|} \sim |DT^i_\omega \tau|, \quad \frac{|T^i(\tilde{\Theta}(\tilde{\omega}))|}{|\tilde{\Theta}(\tilde{\omega})|} \sim |DT^i_{\tilde{\Theta}(\tilde{\omega})} \tau|, \quad \text{and} \quad \frac{|DT^i_{\tilde{\Theta}(\tilde{\omega})} \tau|}{|DT^i_\omega \tau|} \sim 1.
\]
The first two \( \sim \) comes from Sect. 2.3C(4). The last is from Lemma 3.2(a) of [WY2], the proof of which is elementary and self-contained (See A.6 of [WY2]); this lemma is applicable here because Lemma 3.5(a) holds for all \( z \in \omega \cap A \). Finally, we have the distortion estimate in Sect. 2.3C(4). The relations above together with \(|T^i(\hat{\Theta}(\tilde{\omega}))| \lesssim |T^i(\omega)|\) imply \(|\Theta(\tilde{\omega})| < K|\omega|\). \(\Box\)

4 Ergodic Components and Basin Properties

This section contains the proofs of Theorems 2, 3 and 4. Our strategy is to prove Theorems 2 and 3 simultaneously: We construct explicitly \( r \) ergodic SRB measures, \( \nu_1, \cdots, \nu_r, r \) no greater than the number of critical points of \( f_0 \), and show that Lebesgue-a.e. point in \( R_1 \) lies in \( \bigcup_{i=1}^r W^s(\nu_i) \). It will then follow that there can be no other ergodic SRB measure since a full Lebesgue measure set in the basin has been accounted for.

4.1 Collection of SRB measures and gateways to their basins

As before we let \( C = \{x_1, \cdots, x_q\} \) be the critical set of \( f_0 \), the 1D map of which \( T \) is a perturbation, and let \( \pi_x : R_1 \to I \) be projection onto the \( x \)-axis. The following notation is used in this subsection: \( \rho^+_1 = [x_1 + \delta, x_1 + \delta]; \rho^- = [x_i - \delta, x_i - \delta] \) is the interval with \( f_0(\rho^-_1) = f_0(\rho^+_1) \), \( \frac{1}{2}\rho^-_1 \) is defined similarly, and \( U^-_i = \pi_x^{-1}(\rho^-_1) \). The aim of this subsection is to prove

**Proposition 4.1** There exist ergodic SRB measures \( \nu_1, \cdots, \nu_r, r \leq q \), and a mapping \( \sigma : \{1, \cdots, q\} \to \{1, \cdots, r\} \) which “assigns” the \( i \)-th component of \( C^{(1)} \) to \( \nu_{\sigma(i)} \) in the sense below: For each \( i \), there are two closed sets \( V^+_i \subset (\Lambda^s \cap U^+_i) \) such that

(i) each \( V^+_i \) is a union of \( W^s \)-disks,

(ii) there exists \( \varepsilon > 0 \) such that \( m_{\gamma}(\gamma \cap V^+_i) > \varepsilon m_{\gamma}(\gamma) \) for every \( C^1 \)-curve \( \gamma \)

that runs the full length of \( U^+_i \) or \( U^-_i \); and

(iii) for \( i \) as above, \( m_{\gamma} \)-a.e. \( z \in (\gamma \cap V^+_i) \) lies in the stable manifold of a \( \nu_{\sigma(i)} \)-typical point.

We think of the sets \( V^+_i \) as “gateways” to \( W^s(\nu_{\sigma(i)}) \) because of property (iii) and the fact that most curves when iterated will cross some \( U^+_i \).

**Proof:** We divide this proof into three steps.

**Step 1. Combinatorial relationship between the different components of \( C^{(1)} \).**

**Lemma 4.1** There exists \( \hat{K} = O(\log \delta^{-1}) \) for which the following holds: For \( i = 1, 2, \cdots, q \), there is a subinterval \( \rho \subset \frac{1}{2}\rho^+_i \) (equivalently \( \frac{1}{2}\rho^-_i \)) and \( n \leq \hat{K} \) such that

(i) \( f^n_i(\rho) \cap C^i_0 = \emptyset \) for all \( k < n \);

(ii) \( f^n_i(\rho) = \rho^+_j \) or \( \rho^-_j \) for some \( j \).

This is a 1D statement, the proof of which we leave as an exercise. Because of the \( a \) priori bound on the number of iterates involved, we may assume \( b \) is small enough that this result passes to \( T \), that is to say, there is a section \( A \subset \pi_x^{-1}(\frac{1}{2}\rho^+_i) \) of \( R_1 \) such that \( T^k(A) \cap C^{(1)} = \emptyset \) for \( k < n \) and \( T^n(A) \) crosses \( U^+_j \).
Consider the directed graph on the set of vertices \{1, 2, \ldots, q\} defined by \(i \rightarrow j\) if \(i\) and \(j\) are related as in Lemma 4.1. We write \(i \rightarrow \cdots \rightarrow j\) if there is a chain of arrows from \(i\) to \(j\), and let \(I\) be a minimal set of vertices with the property that for every \(i \in \{1, 2, \ldots, q\}\), there exists \(j \in I\) such that \(i \rightarrow \cdots \rightarrow j\). Notice that each \(i \in I\) lies in a chain \(i \rightarrow \cdots \rightarrow i\). This is because by definition, starting from \(i\) there is a chain that leads back to \(I\), and if \(i \rightarrow \cdots \rightarrow j\) for some \(j \in I\), \(j \neq i\), then \(i\) is not needed, contradicting the minimality of \(I\).

**Step 2. Construction of SRB measures.**

Let \(i \in I\). From the fact that \(i\) lies on a cycle, it follows that \(T\) has a periodic point \(z \in \pi_{\pm}^{-1}(1/2)^+\) or \(z \in \pi_{\pm}^{-1}(1/2)^-\) with the property that \(d_c(z_j) > \frac{1}{3}\delta\) for all \(j\). We let \(\ell_0\) be the local unstable curve centered at \(z\) of length \(\frac{1}{3}\delta\), and push forward \(m_0 = \text{Lebesgue measure on } \ell_0\) as in the construction of SRB measures discussed in detail in Section 9 of [WY2]. This SRB measure, which we call \(\nu\), is constructed by “catching” a positive fraction of \(\frac{1}{n} \sum_{i=0}^{n-1} T^i(m_0)\) on a closed set \(\mathcal{N} = \bigcup D_t\) where \(\{D_t\}\) is a continuous family of unstable curves with the property that \(\pi_x(D_t) = L\), \(L\) being an interval of length \(\frac{1}{3}\delta\). In fact, we construct a canonical subdivision on \(\ell_0\), and when estimating the measure “caught” on \(\mathcal{N}\), we count only those parts of \(T^i(m_0)\) carried by \(T^i(\omega)\) on \(\omega \in \mathcal{Q}_t\) and \(T^i(\omega)\) is free and contains some \(D_t\). For reasons to become clear momentarily, we modify this construction slightly, deleting points with \(d_c(z) \leq \min(\delta^2, e^{-\alpha})\). Since a positive \(m_0\)-measure of \(\ell_0\) remains after this deletion (Lemma 3.4), the procedure above continues to be valid, producing an SRB measure \(\nu\) with \(\nu(\mathcal{N}) > 0\).

We claim that \(\nu|_{\mathcal{N}}\) is part of an ergodic component of \(\nu\). By Hopf’s argument, it suffices to show there is a collection of stable manifolds that meet each \(D_t\)-curve in \(\mathcal{N}\) in a set of positive Lebesgue measure. Let \(\Lambda_{\mathcal{X}}\) be the union of \(W^s\)-leaves \(\gamma^s\) in \(\Lambda^s\) such that \(\gamma^s\) meets the middle half of some \(D_t\). Given the slope of \(\gamma^s\) (Lemma 3.5(b)(ii)) and the lengths of the \(D_t\)-curves, \(\gamma^s\) meets all \(D_t\)-curves in \(\mathcal{N}\). It then follows from Lemma 3.4 that \(\Lambda_{\mathcal{X}}\) meets every \(D_t\) in a set of positive 1D Lebesgue measure.

Let \(\tilde{\nu}_i\) be this ergodic component in the last paragraph, normalized. Distinct elements of the set \(\{\tilde{\nu}_i, i \in I\}\) are renamed as \(\nu_1, \ldots, \nu_r\).

**Step 3. Construction of \(V_i^\pm\) for all \(i\).**

Consider first \(i \in I\). Let \(\ell_0, \mathcal{N}\) and \(\tilde{\nu}_i\) be as in Step 2, assuming without loss of generality that \(\hat{z} \in U_i^+\). We pick an (arbitrary) \(\omega \in \mathcal{Q}_n\), the canonical subdivision on \(\ell_0\), such that \(\omega\) is free at time \(n\) and \(T^n(\omega) \supseteq\) some \(D_t\). Let \(\tilde{\omega} = \{z \in \omega : T^n z \in \Lambda_{\mathcal{X}}\}\). First we check that \(\tilde{\omega} \subset \Lambda\), the hyperbolic set in Sect. 3.4: Let \(z \in \tilde{\omega}\). For \(j < n\), we have \(d_c(T^j z) > \min(\delta^2, e^{-\alpha j})\) by our deletion rule; for \(j = n + k\), since \(T^n z \in \Lambda_{\mathcal{X}}\), we have

\[
d_c(T^{n+k} z) \geq d_c(T^k(T^n z)) > e^{-\alpha k} > e^{-\alpha(n+k)} .
\]

Now define \(V_i^+ := \cup \{W^s_i(z) : z \in \tilde{\omega}\}\). By Lemma 3.4, \(T^n(\omega)\) meets \(\Lambda_{\mathcal{X}}\) in a set of positive measure, so \(m_{\omega}(\tilde{\omega}) > 0\) (where \(m_{\omega}\) is the Riemannian measure on \(\omega\)). From the positivity of the conditional densities of \(\tilde{\nu}_i\) on \(D_t\)-curves, we have that \(m_{\omega}\)-a.e. \(z \in \tilde{\omega}\) lies in the stable manifold of a \(\tilde{\nu}_i\)-typical point. The existence of \(\varepsilon_3\) follows from the definition of \(V_i^+\), Lemma 3.4 and Proposition 3.2. This completes the proof of our assertions for \(V_i^+\).

One checks easily that for each \(W^s_i(z)\) in \(V_i^+\), there exists \(z' \in U_i^- \cap \Lambda^s\) such that \(T(W^s_i(z')) \subset W^s(Tz)\). The union of \(W^s_i(z')\) for all such \(z'\) is \(V_i^-\).
For \( i \not\in \mathcal{I} \), let \( j \in \mathcal{I} \) be such that \( i \to \cdots \to j \). We define \( V_i^+ \) as follows: Let \( \rho \subset \rho_i^+ \) be as in Lemma 4.1, i.e. there exists \( n \) be such that \( f_{0}(\rho) \cap C_{0} = \emptyset \) for \( k < n \) and \( f_{0}(\rho) = \rho_j^\pm \) (say). By the remark following Lemma 4.1, these properties are inherited by the horizontal section \( H = \pi_x^{-1}(\rho) \). We take \( V_i^+ := H \cap T^{-n}(V_j^+) \), and leave it to the reader to verify that this set has the desired properties. \( \square \)

### 4.2 Almost-everywhere behavior in basin

To prove Theorems 2 and 3, we first show that Lebesgue-a.e. \( z \in X \) lies in the stable manifold of a point that is typical with respect to one of the ergodic SRB measure constructed in Sect. 4.1.

**Proposition 4.2** Let \( \nu_1, \ldots, \nu_r \) be the ergodic SRB measures of \( T \in \mathcal{G} \) constructed in Sect. 4.1. Then \( \bigcup_{i=1}^r W^s(\nu_i) \) occupies a set of full Lebesgue measure in \( X \).

**Proof:** As usual, it suffices to work in \( R_1 \). Let \( B = R_1 \setminus \bigcup_{i=1}^r W^s(\nu_i) \). Our objective here is to prove that \( B \) has Lebesgue measure 0. Let \( Y_i \) be as defined in the proof of Proposition 3.1. Since \( \cup Y_i \) has full Lebesgue measure (Lemma 3.2), it suffices to show that \( \text{Leb}(B \cap Y_i) = 0 \) for all \( i \). Moreover, since \( T^{i}(B) \subset B \), it suffices to derive a contradiction assuming \( \text{Leb}(B \cap T^{-i}Y_i) > 0 \) for some \( i \). This is what we will do.

Assume for definiteness \( i \geq 1 \). We foliate \( C(1) \) with line segments parallel to \( v \). Since \( T^{i}Y_i \subset C(1) \), and \( \text{Leb}(B \cap T^{-i}Y_i) > 0 \), there is a segment \( \gamma \) with \( m_{\gamma}(B \cap T^{-i}Y_i) > 0 \). Let \( \delta \ll \varepsilon_3 \) where \( \varepsilon_3 \) is as in Proposition 4.1(ii). By the Lebesgue density theorem, there exists a short segment \( \gamma_0 \subset \gamma \) such that

\[
m_{\gamma}(B \cap T^{-i}Y_i \cap \gamma_0) > (1 - \delta)m_{\gamma}(\gamma_0).
\]

The segment \( \gamma_0 \) will be iterated forward. Recall from the proof of Proposition 3.1 that all \( \xi_1 \in \gamma_0 \cap B \cap T^{-i}Y_i \) satisfy the no-deep-returns rule of Lemma 3.1. With \( \gamma_0(\xi_1) = v \), \((\xi_1, \gamma_0(\xi_1))\) is controlled for all future times. (The argument is similar in the case \( i = 0 \), except that \( \gamma_0 \) is parallel to the \( x \)-axis.)

We now iterate \( \gamma_0 \) forward, constructing on it a canonical subdivision \( Q_n \) as described in Sect. 3.3 – subject to the following rule of deletion and stopping time:

- **Deletion rule:** \( \omega \in Q_n \) is deleted if it does not intersect \( B \cap T^{-i}Y_i \).

- **Stopping time \( S \):** If \( \omega \in Q_{n-1} \) is such that \( T^n(\omega) \) crosses some \( U_j^\pm \), then we set \( S = n \) on \( T^{-n}(T^n(\omega) \cap U_j^\pm) \) and do not iterate this segment further.

This deletion rule ensures that the process as described in Sect. 3.3 can be continued, i.e. all segments that remain are controlled, since \( \xi_1 \in T^{-i}Y_i \) satisfies the no-deep-returns rule. The purpose of the stopping rule is to ensure, as we will see, that a definite fraction of every stopped segment meets \( V_j^\pm \) for some \( j \).

Let \( \gamma_\infty \) denote the subset of \( \gamma_0 \) that is not deleted in the procedure above. Observe that \( \gamma_\infty \supset \gamma_0 \cap B \cap T^{-i}Y_i \).

**Lemma 4.2** \( S(z) < \infty \) for \( m_\gamma \)-a.e. \( z \in \gamma_\infty \).

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Proof: For 1D maps satisfying (G1) and (G2) in Sect. 2.1(A), this result follows from Lemma 2.5 and Corollary 2.1 of [WY2] (see Appendix A.2 of [WY2]), which assert that if we keep iterating a segment, almost every point will eventually be in a free segment of length $> \delta$, and such a segment will cross $p_j^\pm$ for some $j$ within a fixed number of iterates. The same argument holds verbatim for controlled curves in the present setting (see Sect. 2.3C). □

We have proved that modulo $\gamma_\infty$ is the union of a countable number of intervals $\{\omega\}$ on each one of which $S$ is constant. At time $S = S(\omega)$, $T^S(\omega)$ crosses some $U_j^\pm$ and has a fixed upper bound in length. By Sect. 2.3C, $T^S$ has uniformly bounded distortion on $\omega$. It then follows by Proposition 4.1(ii) that $m_\gamma(\omega \cap T^{-S}(V_j^\pm)) > K^{-1}\varepsilon_3 m_\gamma(\omega)$. However, modulo a set of $m_\gamma$-measure zero, $\omega \cap T^{-S}(V_j^\pm) \subset W^s(\nu_{\sigma(j)})$. Thus we obtain, summing over $\omega$, that

$$m_\gamma(\gamma_\infty \cap B) < (1 - K^{-1}\varepsilon_3)m_\gamma(\gamma_\infty) \leq (1 - K^{-1}\varepsilon_3)m_\gamma(\gamma_0).$$

Since $(\gamma_\infty \cap B) \supset (B \cap T^{i-1}Y \cap \gamma_0)$, this contradicts (11) provided $\hat{\varepsilon} < K^{-1}\varepsilon_3$. □

Proof of Theorems 2 and 3: Let $\{\nu_1, \cdots, \nu_r\}$ be the ergodic SRB measures constructed in Sect. 4.1. Then Theorem 3(a) and (b) follow from Proposition 4.2 because properties (1) and (2) in the definition of $\Gamma_\nu$ in Sect. 1.2 pass from $z \in \Gamma_\nu$ to all points in $W^s(z)$; see [R, Y3]. To prove Theorem 2, suppose there is an ergodic SRB measure $\nu \notin \{\nu_1, \cdots, \nu_r\}$. As explained in Sect. 1.2, $\text{Leb}(W^s(\nu)) > 0$. This is impossible, for Lebesgue-a.e. $z \in R_1$ belongs in $\bigcup_{i=1}^r W^s(\nu_i)$. □

4.3 Conditions for ergodicity and mixing

Let $C_{\delta^2} = \{(x, y) \in R_1 : d(x, C) < \delta^2\}$.

Lemma 4.3 Assume $2 < e^\lambda$. Then there exists $N$ depending only on $\delta$ and $\lambda$ such that if $H$ is a horizontal section of $R_1$ of length $\frac{1}{10}\delta$, then there exist a subsection $H \subset H$ and $n = n(H) \leq N$ such that

(i) $T^k(H) \cap C_{\delta^2} = \emptyset$ for all $k < n$ and

(ii) $T^n(H)$ crosses two adjacent components of $C_{\delta^2}$.

Proof: This is strictly a 1D result, since the orbits of interest do not pass through $C_{\delta^2}$. Let $\omega \subset R_1$ be a $C^2$-segment such that $\pi_x(\omega) = \pi_x(H)$, and consider $T^i(\omega)$, $i = 1, 2, \cdots$, deleting all parts that fall into $C_{\delta^2}$ -- until one of the deleted segments crosses two components of $C_{\delta^2}$. Observe that after $k - 1$ iterates, the number of segments remaining is $\leq 2^{k-1}$, and the number of segments deleted at step $k$ is $\leq 2^k$. We estimate the average length of the subsegments of $T^n(\omega)$ that remain: First, the pullback to $\omega$ of all the deleted parts has total measure $\leq \sum_{k=1}^n 2^k e^{-\lambda k}(2\delta^2)$. Since $2 < e^\lambda$, this is $< \frac{1}{2} |\omega|$ assuming $\delta$ is sufficiently small. Thus the average length of the connected segments that are present at step $n$ is $> 2^{-(n-1)} e^{\lambda n} \frac{1}{2} |\omega|$. The process therefore must stop, since this length cannot grow indefinitely. □
Proof of Theorem 4: Let \( \{ J_i \} \) be the intervals of monotonicity for \( f_0 \), and \( \tilde{J}_i \) the union of \( \pi_{-1}^{-1}(J_i) \) with the two components of \( \mathcal{C}_2 \) between which it lies. We assume in the definition of \( \pi_{ij} \) in Sect. 1.2 that \( f_0(J_i) \supset J_j \) may be replaced by \( f_0(J_i) \supset J_j \).

(a) To derive a contradiction, we suppose there are two distinct ergodic SRB measures, \( \nu_1 \) and \( \nu_2 \). Let \( \mathcal{N}_i \) be the set of unstable curves used to “catch” \( \nu_i \) (Step 2 in the proof of Proposition 4.1), and let \( H_i = \pi_{-1}^{-1}(\mathcal{N}_i) \), \( i = 1, 2 \). Lemma 4.3 tells us that \( T^n(H_1) \) crosses \( \tilde{J}_t \) for some \( n \) and \( \ell \). Once a \( \tilde{J}_t \) is crossed, the hypothesis in part (a) tells that for every \( k \), there exists \( j \) such that \( T^j(H_1) \) crosses \( \tilde{J}_k \). In particular, if \( H_2 \subset \tilde{J}_k \), then \( T^j(D_t) \) crosses \( H_2 \) for every \( D_t \in \mathcal{N}_1 \). Thus we have a positive \( \nu_1 \)-measure set of points that are future-generic with respect to \( \nu_2 \). Hence \( \nu_1 = \nu_2 \).

(b) Let \( \nu \) be the unique SRB measure for \( T \). The mixing property of \( (T, \nu) \) is equivalent to \( (T^n, \nu) \) being ergodic for all \( n \geq 1 \). Let \( \mathcal{N} = \cup D_t \) and \( H = \pi_{-1}^{-1} \mathcal{N} \) be as above. Reasoning as above, we deduce from the hypothesis of part (b) that for all large enough \( N \), the \( T^N \)-image of every \( D_t \) in \( \mathcal{N} \) crosses \( H \). This proves the ergodicity of \( (T^n, \nu) \) for all \( n \geq 1 \). \( \square \)

5 Markov Tower Extensions

This section contains a proof of Theorem 5. We will review the basic setup, and refer the reader to Sects. 1.1 and 1.2 of [Y1] for precise definitions and conditions. The construction here is considerably more elaborate than those in Section 4 and can be used to deduce some of the results discussed earlier, such as the existence of the SRB measures with the properties in Proposition 4.1.

5.1 Reference sets and return maps

We recall the basic setup \( T^R : \Lambda \rightarrow \Lambda \) of a positive measure horseshoe with infinitely many branches and variable return times:

(i) (Hyperbolic product set) Here \( \Lambda \) is a compact hyperbolic product set, i.e. there are two families \( \Gamma^s \) and \( \Gamma^u \) of local stable and unstable disks with the property that each \( \gamma^u \in \Gamma^u \) meets each \( \gamma^s \in \Gamma^s \) transversally in exactly one point, and \( \Lambda = \Gamma^u \cap \Gamma^s \), our shorthand for \( (\cup \{ \gamma^u \in \Gamma^u \}) \cap (\cup \{ \gamma^s \in \Gamma^s \}) \).

(ii) (Positive measure on unstable leaves) \( m_{\gamma^u}(\gamma^u \cap \Lambda) > 0 \) for every \( \gamma^u \in \Gamma^u \) where \( m_{\gamma^u} \) is the induced Riemannian measure on \( \gamma^u \).

(iii) (Infinite horseshoe structure and return time function) A subset \( \Lambda_0 \subset \Lambda \) is called an s-subset if \( \Lambda_0 = \Gamma^u \cap \Gamma^s_0 \) for some \( \Gamma^s_0 \subset \Gamma^s \); u-subsets are defined similarly. Then modulo a set whose conditional measures on \( \gamma^u \)-leaves have \( m_{\gamma^u} \)-measure 0, \( \Lambda \) is the disjoint union of (countably many) s-subsets \( \{ \Lambda_i \} \) where for each \( i \), \( T^{R_i} \Lambda_i \) is a u-subset of \( \Lambda \) for some \( R_i \in \mathbb{Z}^+ \). The function \( R : \Lambda \rightarrow \mathbb{Z}^+ \) with \( R|_{\Lambda_i} = R_i \) is called the return time function.

Theorem 5 asserts that \( T \) admits a structure as above with some additional properties. Here is an outline of the main steps in its construction:
(1) **Multi-component horseshoe** $T^R : \cup_i \Lambda(i)^{\pm} \to \cup_i \Lambda(i)^{\pm}$. This structure is constructed first. It is similar to $T^R : \Lambda \to \Lambda$ except that in the place of $\Lambda$ there are $2q$ hyperbolic product sets $\Lambda(i)^{\pm}$ where $\Lambda(i)^{+,\prec}$ and $\Lambda(i)^{+,\succ}$ are located in the $i$th component of $C^{(i)}$, one on each side of the critical set. Here the return time function is called $\tilde{R}$. Each of the $2q$ hyperbolic product sets is subdivided into $s$-subsets, each one of which is carried, under $T^R$, to a $u$-subset of possibly a different component of $\{\Lambda(i)^{\pm}\}$. The horseshoe in Theorem 5 is obtained by inducing on one of these $2q$ components.

(2) **Expanding horseshoe map** $\tilde{T}^R : \cup_i \bar{\Lambda}(i)^{\pm} \to \cup_i \bar{\Lambda}(i)^{\pm}$. Given $T^R : \cup_i \Lambda(i)^{\pm} \to \cup_i \Lambda(i)^{\pm}$, let $\Gamma(i)^{\pm,\ast}$ be the defining families of stable manifolds for $\Lambda(i)^{\pm}$. By collapsing each leaf in $\Gamma(i)^{\pm}$ to a point, one obtains quotient sets which we denote by $\bar{\Lambda}(i)^{\pm}$ and a quotient map $\tilde{T}^R : \cup_i \bar{\Lambda}(i)^{\pm} \to \cup_i \bar{\Lambda}(i)^{\pm}$. In the construction to follow, we will proceed in the reverse order, i.e. first construct a quotient system and then extend it to a multi-component horseshoe.

To simplify notation slightly, we will omit the superscript $\pm$ and write $\Lambda(i)$ to mean $\Lambda(i)^{+}$ and/or $\Lambda(i)^{-}$ when either (i) it does not matter whether one reads it to be $+$ or $-$, or (ii) it matters but there is no ambiguity from context.

A. **Construction of expanding map**

The first step is to specify the stable family $\Gamma(i)^{\ast}$ used to define $\Lambda(i)^{\ast}$ for a fixed $i$ (and choice of $+$ or $-$). Let $\gamma = \frac{1}{2}\rho_i \times \{0\}$ where $\frac{1}{2}\rho_i$ is as in Sect. 4.1. We define a decreasing sequence of sets $\gamma = \gamma_0 \supset \gamma_1 \supset \gamma_2 \supset \cdots$ as follows: Let $k$ be the first time $T^k(\gamma)$ meets the forbidden region $D_k := \{d < \min(d^2, e^{-\alpha k})\}$. We set $\gamma_j = \gamma_0$ for $j < k$. For $\gamma_k$, we consider the connected components of $\gamma \setminus T^{-k}D_k$, discard those whose $T^k$-images do not contain an $I_U$, (e.g. when the image crosses a forbidden region with a tiny piece on one side), and let $\gamma_k$ be the union of the rest. For $\gamma_{k+1}$, we look at $T(\gamma_k)$, make the same type of deletions with respect to $D_{k+1}$, and pull back the remaining set by $T^{-(k+1)}$. The process is continued, leading to $\gamma_\infty = \cap_k \gamma_k$. The set $\gamma_\infty$ is nonempty, compact, and has positive Lebesgue measure (see Lemma 3.4). We define $\Gamma(i)^{\ast} = \{W^s_\ast(z), z \in \gamma_\infty\}$ where $W^s_\ast$ is as in Sect. 3.4, let $\Lambda(i)^{\ast}$ be the union of the leaves in $\Gamma(i)^{\ast}$, and let $\bar{\Lambda}(i)^{\ast}$ be the quotient of $\Lambda(i)^{\ast}$ obtained by collapsing each $W^s_\ast$-leaf to a point.

Next we define the return time $\tilde{R}$ and return map $\tilde{T}^R : \cup_i \bar{\Lambda}(i)^{\pm} \to \cup_i \bar{\Lambda}(i)^{\pm}$. Let $i$ be fixed as before. Viewing $\gamma \cap \Lambda(i)^{\ast}$ as representing $\bar{\Lambda}(i)^{\ast}$, we will (informally) define $\tilde{R}$ and $\tilde{T}^R$ on $\gamma \cap \Lambda(i)^{\ast} = \gamma_\infty$. To define $\tilde{R}$ on $\gamma_\infty$, we consider the entire curve $\gamma$, and construct for each $j$, a partition $\hat{Q}_j$ on $\gamma_j \setminus \{\tilde{R} \leq j\}$. The $\hat{Q}_j$’s are similar to the canonical subdivisions in Sect. 3.3, but differ in two ways: The first is that we do not divide $\omega \in \hat{Q}_j$ when $T^j(\omega)$ is outside of $C^{(i)}$, i.e. we allow the segment to grow long. The second has to do with the parts that fall into the “gaps” of $\Lambda(k)$; we will get to that momentarily. Let $U_k = \pi^{-1}(\rho_k)$ be as in Sect. 4.1, and let $U_k$ be the largest region in $U_k$ bounded by two $W^s_\ast$-leaves.

The first stopping time $S_0$ is defined as follows: Let $S_{-1}$ be a large number the purpose of which will become clear. For $z \in \gamma$, we define $S_0(z) = n$ if $n$ is the first time $> S_{-1}$ when $\omega$, the element of $\hat{Q}_n$ containing $z$, has the property that $T^n(\omega)$ is free and crosses some $U^+_k$ or $U^-_k$ (or both). When that happens, we let $\Theta_\omega = \omega \cap T^{-n}\Lambda(i)^{\ast}$, and define $\tilde{R}|_{\Theta_\omega \cap \gamma_\infty} = n$. For $\eta \in \Theta_\omega \cap \gamma_\infty$, we define $\tilde{T}^R(\eta)$ to be the point in $\bar{\Lambda}(k)^{\ast}$ obtained by collapsing the $W^s_\ast$-leaf containing $T^n(\eta)$. We also define $\hat{Q}_n|_{\omega \cap \Theta_\omega}$ to be the partition into
connected components, i.e. subsegments of \( \omega \) which under \( T^n \) fall into distinct gaps of \( \tilde{\Lambda}^{(k)} \) will be treated as unrelated from here on.

We continue to iterate \( \omega \setminus \Theta_\omega \) until the next stopping time \( S_1 \): For \( z' \in \tilde{\omega} \), \( S_1(z') = n' \) if \( n' \) is the smallest integer \( > n \) when \( \omega' \), the element of \( \hat{Q}_{n'-1} \) containing \( z' \), has the property that \( T^{n'}(\omega') \) is free and crosses some \( U_k \). As with the first stopping time, we let \( \Theta_\omega' = \omega' \cap T^{-n'}\Lambda^{(k),s} \), set \( \tilde{R}|_{\Theta_\omega' \cap \gamma_\infty} = n' \), and define \( \tilde{T}^{\tilde{R}}(\eta) \) for \( \eta \in \Theta_\omega' \cap \gamma_\infty \) as before. The process is continued \textit{ad infinitum}. On the set of \( \gamma_\infty \) on which \( \tilde{R} \) is not defined in this process, we set \( \tilde{R} = \infty \). The return map \( \tilde{T}^{\tilde{R}} \) is not defined on this set.

For arbitrary \( j \in \mathbb{Z}^+ \), suppose \( \omega \in \hat{Q}_{n_j-1} \) is such that \( S_j|_\omega = n_j \), \( T^{n_j}(\omega) \) crosses \( U_k \), and \( \Theta_\omega \) is as above. The following is in anticipation of the proof of the horseshoe property in Part B:

**Lemma 5.1** \( \Theta_\omega \subset \gamma_\infty \), equivalently \( \tilde{T}^{n_j}(\gamma_\infty \cap \Theta_\omega) = \tilde{\Lambda}^{(k)} \)

**Proof:** Let \( z' \in \Lambda^{(k),s} \cap T^{n_j}(\omega) \), and let \( z \in \omega \) be such that \( T^{n_j}z = z' \). We need to show \( z \in \gamma_\infty \). First, \( z \in \gamma_{n_j-1} \) by definition, and second, \( z_{n_j} \in W^u_s(\xi_0) \) for some \( \xi_0 \) satisfying \( d_C(\xi_\ell) = \min(\delta_\ell, e^{-\alpha_\ell}) \) for all \( \ell \geq 0 \). Thus for \( n \geq n_j \), \( d_C(z_n) = \min(\delta_\ell, e^{-\alpha(n-n_j)}) \), which is \( > 2e^{-\alpha n} \) assuming \( S_{-1} \) is large enough. This ensures that \( z \in \gamma_n \) for all \( n \geq n_j \), and therefore \( z \in \gamma_\infty \). \( \square \)

We record also the following observation for later use: In the setting of Lemma 5.1, if \( \hat{\omega} \subset \omega \) is the shortest subsegment with \( T^\infty(\hat{\omega}) \) stretched across \( U_k \), then the end points of \( \hat{\omega} \) lie in \( \Lambda^{(i),s} \). This is because \( U_k \) is bounded by \( W^s \)-curves.

**B. Construction of multi-component horseshoe map**

Our first order of business is to specify \( \Gamma^{(i),u} \), the families of unstable curves used to define the hyperbolic product sets \( \Lambda^{(i)} \). Let \( \hat{\omega} \) be a periodic point whose orbit satisfies \( d_C(\hat{\omega}_j) \geq \frac{1}{2} \delta \) for all \( j \), and let \( \ell_0 \) be the local unstable curve of length \( \frac{1}{4} \) centered at \( \hat{\omega} \). (There is an abundance of such points: see Sect. 6.1 or Step 2 in the proof of Proposition 4.1.) For \( n = 1, 2, \ldots \), we let \( \ell_n = \{ z_n : z_0 \in \ell_0, d_C(z_j) > \min(\delta^2, e^{-\alpha j}) \} \) for \( j = 1, \ldots, n \).

From Sect. 3.3, we know that points in \( T^{-n}\ell_n \) and their tangent vectors are controlled up to time \( n \). As a first try, we let \( \Gamma^{(i),u} \) be the collection of curves \( \gamma^u \) with the property that \( \text{(i)} \gamma^u \) is a free subsegment of \( \ell_n \), any \( n \), and \( \text{(ii)} \gamma^u \) connects the two \( W^s \)-manifolds in \( \partial U_k \). Observe that \( \{ \gamma^u \} \) are unstable curves with uniform estimates, meaning there exist \( \kappa < 1 \) and \( K > 1 \) such that for all \( z \in \gamma^u \) and tangent vectors \( \tau \) to \( \gamma^u \) at \( z \), we have \( |DT_z\gamma^u\tau| \leq K\kappa^n|\tau| \) for all \( n \geq 0 \). This is because going forward in time from \( \ell_0 \) to \( \gamma^u \), tangent vectors are controlled; their derivatives therefore grow with uniform estimates (Sect. 2.3B). Going backward in time from \( \ell_0 \), the images stay outside of \( \mathcal{C}_{g2} \) by design.

Temporarily, let \( \Lambda^{(i)} = \Gamma^{(i),u} \cap \Gamma^{(i),s} \) where \( \Gamma^{(i),u} \) is as above and \( \Gamma^{(i),s} \) is as in Paragraph A. We seek to verify the horseshoe property of \( T^{\tilde{R}} \). Consider the situation of Lemma 5.1, and let \( \Gamma^{(i),u} \cap \Gamma^{(i),s} \) be the s-subset of \( \Lambda^{(i)} \) defined by \( \gamma_\infty \cap \Theta_\omega \). Extending the return time function \( \tilde{R} = n_j \) to this s-subset by declaring it to be constant on \( W^s \)-manifolds, we ask if \( T^{\tilde{R}}(\Gamma^{(i),u} \cap \Gamma^{(i),s}) \) is a u-subset of \( \Lambda^{(k)} \). That is equivalent to asking that the following two conditions hold: \( \text{(a)} \) \( T^{\tilde{R}} \) defines a bijection from \( \Gamma^{(i),s} \) to \( \Gamma^{(k),s} \), and \( \text{(b)} \) for every \( \gamma^u \in \Gamma^{(i),u} \), \( T^{\tilde{R}}(\gamma^u) \) contains an element of \( \Gamma^{(k),u} \). \( \text{(a)} \) is exactly Lemma 5.1. Let us call two \( C^2(b) \)-curves
\( \sigma \) and \( \sigma' \) *s-related* if they begin and end in the same pair of \( W^s_i \)-manifolds, and let \( \hat{\omega} \) be as in the observation following the proof of Lemma 5.1. For (b) to hold, it suffices to have \( T^{n_j}(\sigma) \in \Gamma^i,u \) for all subsegments \( \sigma \) of \( \gamma^u \in \Gamma^i,u \) that are \( s \)-related to \( \hat{\omega} \). Clearly, \( T^{n_j}(\sigma) \) has to be very close to being free, since the first \( n_j \) images of \( \sigma \) are close to corresponding images of \( \hat{\omega} \), but it may also just miss the cutoff by accident. To be precise, we amend the definition of \( \Gamma^i,u \) above by including all segments of this type.

Finally, the definition in [Y1] requires that the set \( \cup \{ \gamma^u : \gamma^u \in \Gamma^i,u \} \) be closed. As defined above, that may not be the case. We let \( \Lambda^{(i),u} \) be the closure of this set, and claim that \( \Lambda^{(i),u} \) so defined is itself the union of a continuous family of unstable curves. The proof of this claim follows *verbatim* the argument in [WY2], Sect. 9.4. Note that the uniform estimates for \( \gamma^u \) above are crucial for this argument.

Summarizing, we have completed the construction of hyperbolic product sets \( \Lambda^{(i),\pm}, i = 1, 2, \cdots, q \), and have introduced a mapping \( T^R : (\cup_i \Lambda^{(i),\pm}) \cap \{ R < \infty \} \to \cup_i \Lambda^{(i),\pm} \) with the dynamical structure of a multi-component horseshoe.

**C. Inducing to obtain \( T^R : \Lambda \to \Lambda \)**

Let \( \mathcal{A} = \{ \Lambda^{(i),\pm} \} \) be the hyperbolic product sets constructed that are nonempty \( (\Gamma^{(i),u,\pm} \) may be empty for some \( i \)). Renaming the elements of \( \mathcal{A} \) as \( \{ \hat{\Lambda}^{(i)} \} \), we write \( \hat{\Lambda}^{(i)} \to \hat{\Lambda}^{(j)} \) if \( T^R(\hat{\Lambda}^{(i)}) \cap \hat{\Lambda}^{(j)} \neq \emptyset \). Observe that by the growth properties of controlled curves, for every \( \hat{\Lambda}^{(i)} \in \mathcal{A} \), there exists \( \hat{\Lambda}^{(j)} \in \mathcal{A} \) such that \( \hat{\Lambda}^{(i)} \to \hat{\Lambda}^{(j)} \). Using finite-state Markov chain ideas, one sees that there is a closed, transitive subset \( A_0 \subset \mathcal{A} \), i.e. (a) for all \( \hat{\Lambda}^{(i)} \in A_0 \), \( \hat{\Lambda}^{(i)} \to \hat{\Lambda}^{(j)} \) implies \( \hat{\Lambda}^{(j)} \in A_0 \), and (b) for every \( \hat{\Lambda}^{(i)} \in A_0 \) (including the case \( i = j \)), there is a sequence of arrows that goes from \( \hat{\Lambda}^{(i)} \) to \( \hat{\Lambda}^{(j)} \). Now pick an arbitrary \( \hat{\Lambda}^{(i)} \in A_0 \) and call it \( \Lambda \).

The return time \( R \) from \( \Lambda \) to itself is defined as follows: Suppose that for \( z \in \Lambda \), its trajectory under \( T^R \) is \( z = z_0, z_1, \cdots, z_k \) where \( k \) is the first time when \( z_k \in \Lambda \). Then we define \( R(z) = \tilde{R}(z_0) + \tilde{R}(z_1) + \cdots + \tilde{R}(z_{k-1}) \). If the sequence \( z_i \) above is not defined, or if it never returns to \( \Lambda \), then we set \( R(z) = \infty \). It is easy to see that the mapping \( T^R : \Lambda \cap \{ R < \infty \} \to \Lambda \) also has the horseshoe property, i.e. it sends \( s \)-subsets of \( \Lambda \) to \( u \)-subsets of \( \Lambda \).

### 5.2 Verification of (P1)–(P5) in [Y1]

For the precise formulation of (P1)–(P5), see [Y1], Sects. 1.1 and 1.2.

With the exception of (a) and (b) below, (P1) and (P2) are immediate consequences of the construction in Sect. 5.1. In particular, we have shown that \( \Lambda \cap \{ R < \infty \} = \cup_i \Lambda_i \) where each \( \Lambda_i \) is an \( s \)-subset of \( \Lambda \) which under \( T^R \) is mapped to a \( u \)-subset stretching across \( \Lambda \) exactly once.

(a) for any given \( n \), there are at most finitely many \( s \)-subsets \( \Lambda_i \) with \( R|_{\Lambda_i} = n \);

(b) on each \( \gamma^u \in \Gamma^u \), \( m_{\gamma^u}(\Lambda \setminus \cup \Lambda_i) \cap \gamma^u = 0 \).

(a) is true because at any finite time \( n \), at most finitely many \( \omega \in \hat{\mathcal{Q}}_{n-1} \) (see Paragraph A of Sect. 5.1) can have the property that \( T^n(\omega) \) is long enough to cross completely any \( U_k \), let alone to make a return to \( \Lambda \). (b) follows from Proposition 5.1, the proof of which occupies all of Sect. 5.3.
(P3) follows from Lemma 3.5(b)(iii). For \(z, z'\) such that \(z' \in W^s_*(z)\), we have, in fact, 
\(|T^nz - T^n z'| < b^n\) for all \(n > 0\).

Separation times between pairs of points depend only on the \(W^s_*\)-manifolds in which they reside, so it suffices to define \(s(z, z')\) for \(z, z' \in \gamma_\infty\) where \(\gamma_\infty\) is as in Paragraph A of Sect. 5.1. We choose canonical subdivisions \(Q_j\) on \(\gamma_j\) in a way that is compatible with \(Q_j\), meaning: (i) when introducing subdivisions on segments outside of \(C^{(1)}\) in the definition of \(Q_j\), we make sure they leave intact segments that are eventually mapped across some \(\mathcal{U}_k\); and (ii) in the setting of Lemma 5.1, we require that elements of \(Q_{n_j}\) contain entire components of \(\omega \setminus \Theta_\omega\). For \(z, z' \in \gamma_\infty\), we define \(s(z, z')\) to be the largest \(n\) so that for all \(k < n\), \(T^R(z), T^R(z) > k\) and \(z\) and \(z'\) lie in the same element of \(Q_k\). This definition is easily checked to satisfy the conditions required of the separation time \(s(\cdot, \cdot)\).

(P4) follows from the following lemma:

**Lemma 5.2** Let \(z, z' \in \gamma^u\) for some \(\gamma^u \in \Gamma^u\). Then for \(0 \leq j \leq n \leq s(z, z')\):

- (a) \(d(T^nz, T^n z') \leq Ke^{-\frac{1}{4}\lambda(s(z, z') - n)}\);
- (b) \(\log \prod_{k=j}^n \frac{|DT_{\gamma_k} \tau_k|}{|DT_{\gamma_k'} \tau_k'|} \leq Ke^{-\frac{1}{4}\lambda(s(z, z') - n)}\) where \(\tau_k\) and \(\tau_k'\) are unit tangent vectors to \(T^k \gamma^u\) at \(T^k z\) and \(T^k z'\) respectively.

**Proof:** (a) Let \(\hat{\gamma}\) be the subsegment of \(\gamma\) between \(z\) and \(z'\). Then \(\hat{\gamma}\) is controlled up to time \(s := s(z, z')\). By Lemma 5.5 of [WY2], we have

\[d(z_s, z'_s) \geq Ke^{-\frac{1}{4}\lambda(s - n)}d(z_n, z'_n),\]

from which (a) follows.

(b) By the proof of (P3') in Sect. 9.2 of [WY2], we have

\[\log \prod_{k=j}^n \frac{|DT_{\gamma_k} \tau_k|}{|DT_{\gamma_k'} \tau_k'|} \leq Kd(z_n, z'_n).\]

(a) is then used to prove (b). \(\square\)

(P5)(a) is a version of Lemma 3.2 of [WY2] and has the same proof. (b) is Proposition 3.2 of this paper.

### 5.3 Tail estimate for return time

To complete the proof of Theorem 5, it remains to prove

**Proposition 5.1** There exist \(C_0\) and \(c_0 > 0\) such that for all \(\gamma^u \in \Gamma^u\), we have

\[m_{\gamma^u}\{R > n\} < C_0 e^{-c_0 n}\] for all \(n \geq 0\).

The constants in this proposition may depend on \(\delta\). We begin with a simple and quite general result that will be used twice in the proof:

Let \((\Omega_0, P)\) be a probability space, and let \(\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \cdots\) be a decreasing sequence of measurable sets. For each \(k\), there are two disjoint measurable sets \(\Delta_k, \Theta_k \subset \Omega_k\), and a
Z^+\text{-valued measurable function } S_k \text{ defined on } \Omega_k \setminus \Delta_k. \text{ We assume } 0 < S_0 < S_1 < S_2 < \cdots < S_k \text{ on } \Omega_k, \text{ set } S^* = S_k \text{ on } \Theta_k, \text{ and let } \Omega_{k+1} = \Omega_k \setminus (\Delta_k \cup \Theta_k). \text{ This setup has the following interpretation: } \Delta_k \text{ is the set deleted at step } k, S_k \text{ is the time it takes to complete the first } k \text{ steps, } \Theta_k \text{ is the set on which the process terminates (in a meaningful way) at the conclusion of step } k, \text{ and } S^* \text{ is this termination time. Finally, we define } S^* = \infty \text{ on } \cap_k \Omega_k. \text{ Mindful of the fact that the domains of functions vary, we define } \{S^* > n\} \text{ to mean } \{z \in \Omega_0 : S^*(z) \text{ is defined and is } > n\}, \text{ and so on.}

Lemma 5.3 Assume there exist $C, c, \sigma > 0$ and $N_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$,

(a) $P\{S_{k+1} - S_k > n \mid \Omega_k\} < C e^{-cn}$ for all $n > 0$;

(b) $P\{j \in \mathbb{N}^0 \Theta_j \mid \Omega_k\} > \sigma$.

Then there exist $C', c' > 0$ such that

$$P\{S^* > n\} < C' e^{-c'n} \quad \text{for all } n > 0.$$  \hspace{1cm} (12)

Proof: From assumption (b), we have

$$P(\Omega_k) < (1 - \sigma) \frac{k}{N_0}. \hspace{1cm} (13)$$

We will show below that for small enough $\varepsilon > 0$, there exist $C'', c'' > 0$ such that

$$P(\cup_{k<\varepsilon n}\{S_k > n\}) < C'' e^{c''n} \quad \text{for all } n > 0.$$  \hspace{1cm} (13)

Since $\{S^* > n\} \subset \Omega_{[\varepsilon n]} \cup (\cup_{k<\varepsilon n}\{S_k > n\})$, the desired result follows from a combined use of (12) and (13).

To prove (13), we let $A_j = \{z : S_j(z) \leq n \text{ and } S_{j+1}(z) > n\}$ for $j < \varepsilon n$. We partition $A_j$ into $A_j(k_1, \cdots, k_j) = \{S_1 = k_1, \cdots, S_j = k_j\}$ for all possible combinations of admissible (k_1, \cdots, k_j), and use assumption (a) to estimate the size of each of these sets. Recombining and summing over $j$, we obtain

$$P(\cup_{k<\varepsilon n}\{S_k > n\}) \leq \sum_{j<\varepsilon n} P(A_j) \leq \sum_{j<\varepsilon n} \binom{n}{j} C e^{-cn} < C'' e^{c''n}$$

provided that $\varepsilon$ is small enough. \hfill \Box

Proof of Proposition 5.1: We divide the proof into the following two steps. All notation is as in Sect. 5.1.

Step 1. Tail estimate for the return time function $\tilde{R}$. Invoking Proposition 3.2, we note that it suffices to consider the return map $\tilde{T}^R$ on the segment $\gamma = \frac{1}{2} \rho_i \times \{0\}$ in Paragraph A of Sect. 5.1. More precisely, if we let $S_j$ be as in Sect. 5.1, and define $S^* = S_j$ on $\Theta_\omega$ where $\omega$ and $\Theta_\omega$ are as in Lemma 5.1, then $\tilde{R} = S^*$, and it suffices to prove an exponential bound for $m_\gamma\{S^* > n\}$. This setup fits perfectly that of Lemma 5.3 with $\Omega_0 = \gamma$, $P = (m_\gamma(\gamma))^{-1} m_\gamma$, $\Delta_k$ the deleted parts, and $\Theta_k$ the sets on which $S^* = S_k$. To obtain the desired result, then, it suffices to check assumptions (a) and (b) in Lemma 5.3, which is what the rest of the proof is about.
First, we check that \( S_0 < \infty \) for \( m_\gamma \) a.e. \( z \in \gamma_\infty \). We will show, in fact, that there exist \( C_1', c_1' > 0 \) such that for all \( n > 0 \),

\[
m_\gamma \{ S_0 > n \} < C_1' e^{-c_1'n} m_\gamma(\gamma).\]

The proof of this estimate is a slight variant of the proof of Lemma 2.5 of \([WY2]\). The setting here is that of an iterated \( C^2(b) \) curve, but the 1D arguments in Lemma 2.5 of \([WY2]\) apply since all points in \( \gamma_n \) are controlled up to time \( n \) (see Sect. 3.3). Note also that the argument remains valid even though \( S_0 \) here is not the stopping time \( S \) in \([WY2]\) (we do not stop when the size of an image is \( > \delta \), but continue to iterate until the image crosses the entire length of a \( \Lambda^{(k)} \)). The difference in the definitions of stopping time, however, is not without consequence: the distortion constant, and hence the constants \( C_1' \) and \( c_1' \) in the above, may now depend on \( \delta \).

Next we investigate the distribution of \( S_{j+1} - S_j \) for \( j \geq 0 \). Let \( \omega \) be as in Lemma 5.1. We consider a component \( \omega' \) of \( \omega \setminus \Theta_\omega \), and let \( T = T(\omega') \geq 0 \) be the smallest integer such that \( T^{S_j(\omega)+T}(\omega') \) is free and contains either an \( I_{\mu} \) or a segment of length \( \frac{1}{6}\delta \). There are two possibilities: (i) \( T^{S_j(\omega)}(\omega') \) lies entirely to the left or the right of \( \Lambda^{(k)},s \); in this case, \( T(\omega') = 0 \), since the distance between \( \partial \Sigma_k \) and \( \Lambda^{(k),s} \) is \( > \frac{1}{6}\delta \). (ii) \( T^{S_j}(\omega)(\omega') \) lies in a gap of \( \Lambda^{(k),s} \). Suppose this gap is of generation \( p \), meaning it was first created at step \( p \) in the construction of \( \gamma_\infty \). We claim that in this case, \( T(\omega') = p \), and the reasons are as follows: In order for a gap of generation \( p \) to be created, \( T^p(\gamma_{p-1}) \) must cross completely a component of \( dC < \min(\delta^2, e^{-\alpha p}) \). A deletion may also occur if \( T^p(\gamma_{p-1}) \) meets such a forbidden region but does not cross it completely, but such a deletion would increase the size of a gap of an earlier generation, not create a new one. Thus the \( T^p \)-image of a gap of generation \( p \) must contain an \( I_{\mu} \).

To estimate \( m_\gamma(\omega \cap \{ S_{j+1} - S_j > n \}) \), we break up the set as follows:

\[
(S_{j+1} - S_j > n) \subset \left( \{ S_{j+1} - S_j > n \} \cap \{ T < \frac{n}{2} \} \right) \cup \{ T \geq \frac{n}{2} \}.
\]

First, we claim that

\[
m_\gamma(\omega \cap \{ T \geq \frac{n}{2} \}) < Ce^{-\alpha \frac{1}{2} n} m_\gamma(\omega). \tag{14}
\]

This is because \( \omega \cap \{ T \geq \frac{n}{2} \} \) is contained in the union of all components of \( \hat{\omega} \setminus \Theta_\omega \) whose \( T^{n_j} \)-images lie in gaps of \( \Lambda^{(k),s} \) of generation \( \geq \frac{n}{2} \). The assertion in (14) is obtained by an estimate similar to those in the proof of Lemma 3.4 (see also the measure estimate for the set of parameters that survive the deletion process in Sect. 13 of \([WY2]\)

For \( \omega' \subset \omega \) with \( T(\omega') < \frac{n}{2} \), \( S_{j+1} - S_j > n \) means \( S_{j+1} - (S_j + T)(\omega') > \frac{n}{2} \). We claim there exist \( C_2', c_2' > 0 \) (independent of \( j,n \)) such that for these \( \omega' \) and these \( n \),

\[
m_\gamma(\omega' \cap \{ S_{j+1} - (S_j + T) > \frac{n}{2} \}) < C_2' e^{-c_2'n} m_\gamma(\omega').
\]

Let \( \omega = T(S_{j+1} + T)(\omega')(\omega') \), and observe that we either have \( |\omega| > \frac{1}{6}\delta \) (case (i) above) or \( \omega \) contains an \( I_{\mu} \)-interval for some \( \mu \leq \frac{\alpha}{\gamma_\infty} \) (case (ii)). To obtain the desired estimate, we prove a variant of Corollary 2.1 of \([WY2]\). We claim that the conclusion of this corollary holds with a slightly smaller exponent \( K^{-1} \) if one replaces the hypothesis of \( |\omega| > K^{-1} \delta \) in
by \(|\omega| > |I_{\mu,\ell}| > e^{-\alpha n}\). The only needed modification in the proof appears in the denominator of the term in parenthesis in the displayed formula.

This completes the verification of assumption (a) in Lemma 5.3.

The assumption in (b) clearly holds with \(N_0 = 0\). Thus Lemma 5.3 is applies, i.e., there exist \(C_3', \epsilon_3' > 0\) such that

\[
m_{\gamma} \{\hat{R} > n\} < C_{3}' e^{-\epsilon_3'n} \quad \text{for all } n > 0.
\]

Step 2. Tail estimate for the return time function \(R\). We again seek to apply Lemma 5.3, this time with \(\Omega_0 = \Lambda\) and \(P = m_{\gamma}|\Lambda \cap \gamma^n\) normalized where \(\gamma^n\) is an arbitrarily chosen unstable curve in \(\Gamma^u\). We define \(S_0(z) = \hat{R}(z)\), and for \(k \geq 1\), define

\[
S_k(z) = S_{k-1}(z) + \hat{R}(T^{S_{k-1}(z)}).
\]

No deletions are made here, i.e. \(\Delta_k = \emptyset\) for all \(k\), and \(S^*(z) = S_k\) where \(k\) is the smallest integer so that \((T^k\hat{R})(z) \in \Lambda\). (Notice that this is not necessarily the first return time of \(z\) to \(\Lambda\) under iterates of \(T\).) Clearly, \(R = S^*\).

Assumption (a) in Lemma 5.3 is exactly the result of Step 1. Recall that every \(\hat{\Lambda}^{(k)} \in \mathcal{A}_0\) (following the notation of Sect. 5.1, Paragraph C) is mapped back to \(\Lambda\) by some power of \(T^\hat{R}\). Assumption (b) is easily seen to be valid with \(N_0\) chosen to be the minimum of these powers as \(\hat{\Lambda}^{(k)}\) range over \(\mathcal{A}_0\).

This completes the proof of Step 2 and the proof of Proposition 5.1. □

6 Geometric and Combinatorial Properties

This section contains the proofs of Theorems 6, 7 and 8.

6.1 Approximation by uniformly hyperbolic invariant sets

In this subsection we prove Theorem 6. The reader is referred to Sect. 2.2 for notation.

Lemma 6.1 There exists \(K > 1\) such that at every \(z \in \mathcal{C}\), there exists a vector \(\tau\) with the property that for all \(n \geq 0\),

\[
|DT_z^n(\tau)| < (Kb)^n \quad \text{and} \quad |DT_z^{-n}(\tau)| \leq Ke^{-\frac{1}{4}\lambda n}.
\]

Proof: For \(z \in \mathcal{C}\), let \(Q^{(k_i)} \supset Q^{(k_{i+1})} \supset \cdots\) be such that \(z = \cap_i Q^{(k_i)}\). Let \(z^{(k_i)} = z^*_0(Q^{(k_i)})\), and let \(\tau_{k_i}\) be the positively oriented unit tangent vector to \(\mathcal{F}_{k_i}\) at \(z^{(k_i)}\). Then \(z^{(k_i)} \to z\) as \(i \to \infty\), and by Sect. 2.2A(3), \(\tau_{k_i}\) converges to a vector \(\tau\) at \(z\). Let \(n > 0\) be fixed. With \(\tau_{k_i}\) being the most contracted direction in \(S(\nu, \tau_{k_i})\), we have, for \(k_i > n\), \(|DT_z^n(\tau_{k_i})| < (Kb)^n\) (Sect. 2.3(D)(3)). Letting \(i \to \infty\), we obtain \(|DT_z^n(\tau)| < (Kb)^n\).

To prove the estimate for \(DT_z^{-n}\), recall that every \(Q^{(k_i)}\) is in the main body of a monotone branch (Sect. 2.2B(2)). Letting \(\xi_1 = T^{-k_1+1}z^{(k_i)}\) and \(\tau_1 = DT_{z^{(k_i)}}^{-k_1+1}(\tau_{k_i})\), we have that \((\xi_1, \tau_1)\) is controlled for \(k_i - 1\) iterates (Sect. 2.3C(1)), and \(z^{(k_i)} = T^{k_i-1}(\xi_1)\) is a free return
Given in (i), this number is in the range claimed (Sect. 2.2A(1)). We now use Lemma 5.5 of [WY2] to obtain

\[ |DT_{z}^{-n}(\tau_{i_{k}})| \leq K^{b_{\delta}n}e^{\lambda' n}|\tau_{i_{k}}| \]

assuming \( k_{i} \geq n \) where \( \lambda' \) is slightly smaller than \( \frac{1}{3}\lambda \). Since the contraction is uniform, this estimate is passed to \( \tau \) as before.

\[ \square \]

Theorem 6(a) follows immediately from Lemma 6.1. Theorem 6(b) asserts that \( \Omega_{\varepsilon} \), the set of points whose orbits stay at distances \( \geq \varepsilon \) from \( C \), is uniformly hyperbolic. For \( \varepsilon \geq \delta \), this follows from Sect. 2.3B(0). A similar result can, in fact, be used for \( \varepsilon \) as small as \( \sim \sqrt{b} \). For \( \varepsilon \ll \sqrt{b} \), the situation is considerably more delicate: the direction of \( E^{u} \) can be far from horizontal on \( \Omega_{\varepsilon} \). Our proof of uniform hyperbolicity is not based on \textit{a priori} knowledge of invariant cones.

While it is natural to define \( \Omega_{\varepsilon} \) using \( d(\cdot, C) \), ordinary distance between a point and the critical set \( C \), from the technical point of view it is more natural to work with \( d_{C}(\cdot) \). The following geometric picture is useful both here and in the next subsection.

**Lemma 6.2** For \( z \in (\Omega \cap C^{(1)}) \setminus C \), let \( k' \in (k, (1 + 2\theta)k) \) such that

(i) \( z \in (H \cap C^{(k)}) \setminus Q^{(k)} \) where \( Q^{(k)} \) is the component of \( C^{(k)} \) containing \( z \), \( H \) is a horizontal section of \( R_{\theta} \), which crosses \( Q^{(k)} \), and \( Q^{(k)} \subseteq H \cap C^{(k)} \);

(ii) \( e^{-\lambda k(1+2\theta)} < d_{C}(z) \leq e^{-\lambda k} \).

**Proof:** Since \( z \in \Omega \subseteq R_{(1+2\theta)k} \), it lies in a monotone branch of generation \( k' \) for some \( k' \in (k, (1 + 2\theta)k) \) (Sect. 2.2B(3)). Assertion (i) is a direct consequence of the geometry in Sect. 2.2B(2). As for (ii), by definition, \( d_{C}(z) = |z - z_{0}^{*}(Q^{(k)})| \) (Sect. 2.2A(4)), and for \( z \) as in (i), this number is in the range claimed (Sect. 2.2A(1)).

\[ \square \]

**Lemma 6.3** Given \( \varepsilon > 0 \), \( \exists \gamma(\varepsilon) > 0 \) such that for all \( z \in \Omega \), if \( d_{C}(z) < \gamma(\varepsilon) \), then \( d(z, C) < \varepsilon \).

**Proof:** For \( z \in (\Omega \cap C^{(1)}) \setminus C \), let \( Q^{(k)} \) be the deepest critical region containing \( z \). The lemma is easily verified if \( Q^{(k)} \cap C \neq \emptyset \), for \( d(z, C) \leq d_{C}(z) + d(z_{0}^{*}(Q^{(k)}), C) \), and in this case, \( d_{C}(z) = |z - z_{0}^{*}(Q^{(k)})| \geq e^{-\lambda k(1+2\theta)} \) (Lemma 6.2) while \( d(z_{0}^{*}(Q^{(k)}), C) < b_{\varepsilon}^{\frac{n}{2}} \) (Sect. 2.2A(3)).

It is possible, however, for \( Q^{(k)} \) not to contain any part of \( C \). (Recall that \( d_{C}(\cdot) \) is defined using \( \cup_{k} \Gamma_{k} \), while \( C \) is the set of accumulation points of \( \cup_{k} \Gamma_{k} \); see Sect. 2.2A.) An argument that includes this case goes as follows: Given \( \varepsilon \), let \( n \) be such that \( e^{-\lambda n} + 2b_{\varepsilon}^{\frac{n}{4}} < \varepsilon \). Let \( \mathcal{E} \) be the collection of \( Q^{(k)} \), \( n \leq k < n(1 + 2\theta) \), and let \( \mathcal{E}' = \{ Q \in \mathcal{E} : Q \cap C = \emptyset \} \). For each \( Q \in \mathcal{E}' \), let \( i(Q) \geq \text{generation}(Q) \) be such that \( Q \) contains no \( Q^{(i)} \) with \( i > i(Q) \); \( i(Q) \) must exist if \( Q \cap C = \emptyset \). Since \( \mathcal{E}' \) is finite, \( i_{0} := \max_{Q \in \mathcal{E}'} i(Q) \) is finite. Finally, let \( \gamma(\varepsilon) < \varepsilon \) be such that for \( z \in \Omega \), if \( d_{C}(z) < \gamma(\varepsilon) \), then \( z \in C^{(k)} \) for some \( k > i_{0} \); such a \( \gamma(\varepsilon) \) exists by Lemma 6.2. This implies \( z \in Q \) for some \( \hat{Q} \in \mathcal{E} \) (Sect. 2.2A(2)) with \( Q \cap C \neq \emptyset \). Assuming \( Q^{(k)} \) is the deepest critical region containing \( z \), we have

\[
\begin{align*}
d(z, C) & \leq |z - z_{0}^{*}(Q^{(k)})| + |z_{0}^{*}(Q^{(k)}) - z_{0}^{*}(\hat{Q})| + d(z_{0}^{*}(\hat{Q}), C) \\
&< e^{-\lambda k} + b_{\varepsilon}^{\frac{n}{2}} + b_{\varepsilon}^{\frac{n}{2}} < \varepsilon .
\end{align*}
\]

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Proof of Theorem 6(b): Let 

$$\tilde{\Omega}_\varepsilon := \{ z \in \Omega : d_C(T^n z) \geq \varepsilon \text{ for all } n \in \mathbb{Z} \}.$$ 

By Lemma 6.3, $$\Omega_\varepsilon \subset \tilde{\Omega}_\gamma(\varepsilon)$$, so it suffices to prove the uniform hyperbolicity of $$T$$ on the latter. We will prove it for $$\tilde{\Omega}_\varepsilon$$ for an arbitrary $$\varepsilon > 0$$.

Step 1. Existence of vectors with uniform expansion

We prove that there exists an $$N = N(T, \varepsilon) \in \mathbb{Z}^+$$ such that at every $$z \in \tilde{\Omega}_\varepsilon$$, there is a vector $$v$$ with

$$|DT_z^k(v)| > e^{\frac{1}{4} \lambda_k} |v| \quad \text{for all } k \geq N. \quad (15)$$

Let $$\tilde{\mathcal{Z}}_i$$ be as in Sect. 3.2. For $$z \in \tilde{\Omega}_\varepsilon$$, let $$k(z) \geq 0$$ be such that $$T^k(z) \in \tilde{\mathcal{Z}}_k(z)$$ and $$T^i z \notin \tilde{\mathcal{Z}}_i$$ for all $$i > k(z)$$. Since $$d_c(T^i z) > \varepsilon$$ for all $$i$$, $$k(z)$$ clearly exists. Let $$\xi_1 = T^{k(z)} z$$, and let $$w_1 = \partial_z$$ if $$k(z) = 0$$, $$w_1 = v$$ if $$k(z) > 0$$ ($$\xi_1$$ being in $$C^{(1)}$$ in the latter case). It is shown in the proof of Proposition 3.1 that the pair $$(\xi_1, w_1)$$ is controlled for all future times. We will argue that $$DT^k_{\xi_1}(w_1)$$ has the uniform growth in (15). Once that is established, uniform growth for $$z$$ and $$v = v(z) = DT^{-k(z)}_{\xi_1}(w_1)$$ will follow immediately from the observation that $$k(z)$$ is uniformly bounded on $$\tilde{\Omega}_\varepsilon$$.

Recall that the derivative growth along controlled orbits – with iterates during bound periods omitted – is given by Sect. 2.3B(0) and (3). To see that $$DT^k_{\xi_1}(w_1)$$ has the uniform growth in (15), we further observe that (i) for orbits in $$\tilde{\Omega}_\varepsilon$$, there is a finite upper bound to the length of bound periods, and (ii) $$T$$ being locally a diffeomorphism, there exists $$\hat{\varepsilon} > 0$$ such that at all points and for all vectors $$w$$, $$|DT(w)| \geq \hat{\varepsilon} |w|$$. Shrinking the exponent slightly, the dip before recovery in $$|DT^k_{\xi_1}(w_1)|$$ during bound periods can be offset by previous growth provided $$N$$ is sufficiently large.

Step 2. Identification of $$E^u$$ and $$E^s$$

For $$z \in \tilde{\Omega}_\varepsilon$$, we define $$E^u(z)$$ as follows: For $$k = 1, 2, \ldots$$, let

$$u_k = \frac{DT_{T^{-kN} z}^{kN} v(T^{-kN} z)}{|DT_{T^{-kN} z}^{kN} v(T^{-kN} z)|}$$

where $$v(\cdot)$$ and $$N$$ are as in Step 1. Let $$u$$ be a limit point of $$u_k$$ as $$k \to \infty$$. Then clearly $$|DT_{u^{-iN}(u)}| < e^{-\frac{1}{4} \lambda_i N}$$ for all $$i \geq 0$$. We define $$E^u(z)$$ to be the 1D subspace generated by $$u$$. To prove that the subspaces $$E^u$$ so defined are $$DT$$-invariant, suppose $$DT_z(E^u(z)) \neq E^u(Tz)$$ for some $$z$$, and deduce a contradiction as follows: For each $$i \in \mathbb{N}^+, DT_{T^{-iN} z}$$ increases the area of the parallelogram spanned by $$DT_{T^{-iN} z}(u(z))$$ and $$DT_{T^{-iN} z}^{-1}(u(Tz))$$ by a factor $$\geq e^{2 \frac{1}{4} \lambda_i N} \angle(DT_z(E^u(z)), E^u(Tz))$$. For $$i$$ large enough, this is $$>> (Kb)^{iN+1}$$, the maximum area growth in $$iN + 1$$ steps for maps in $$\mathcal{G}_0$$ (see Sect. 1.1).

Next we observe that at every $$z \in \tilde{\Omega}_\varepsilon$$, due to the presence of the expanding vector $$u$$ in the last paragraph, the set of vectors that are contracted under $$DT_z^n$$ as $$n \to \infty$$ constitutes at most a codimension one subspace. We now show that at each $$z$$, there is at least one codimension one subspace, called $$E^s(z)$$, such that for all $$v \in E^s(z), |DT^n_z v| < (Kb)^n$$.
for all $n \geq N + k(\varepsilon)$: We work with $\xi_1 = T^{k(z)}z$, and let $w_1$ be as in Step 1. Let $S$ be any 2D subspace containing $w_1$, and $e_n(\xi_1, S)$ the most contracted direction in $S$ at $\xi_1$. By Sects. 2.3D(2) and 2.3D(3), we have for all $n \geq N$, $|DT^n_\xi \xi_n(\xi_1, S)| < (Kb)^n$ and $\angle(e_n+1(\xi_1, S), e_n(\xi_1, S)) < (Kb)^n$. Thus $e_n(\xi_1, S) \to e_\infty(\xi_1, S)$ as $n \to \infty$. Since $e_\infty(\xi_1, S) \neq w_1$, we may vary $S$ (arbitrarily) to obtain $m - 1$ linearly independent vectors of the form $e_\infty(\xi_1, S)$. The pullback to $z$ of the codimension one subspace spanned by these vectors is $E^s(z)$. To invariance of $E^s$ follows immediately from its characterization at the beginning of the paragraph.

That $E^u$ and $E^s$ vary continuously on $\Omega_\varepsilon$ and are transversal at each point follows from the uniform expansion and contraction estimates in the preceding paragraphs.

\[\square\]

### 6.2 Symbolic coding of orbits relative to $C$

For $T \in \mathcal{G}$, there is a single partition that best reflects the geometry of $T$, namely the one defined by $C$. The coding in this subsection is exclusively with respect to this partition. For definiteness, we consider the case $I = S^1$, and let $x_1 < x_2 < \cdots < x_q < x_{q+1} = x_1$ be the critical points of $f_0$ in the order in which they appear on $S^1$. Let $C_i$ denote the part of $C$ in the component of $C^{(1)}$ containing $(x_i, 0)$. We would like to introduce a partition $A := \{A_1, \cdots, A_q\}$ where $A_i$ consists of points in $\Omega$ that lie “to the right” of $C_i$ and “to the left” of $C_{i+1}$. Such a partition is a priori not well defined, since the $C_i$ are Cantor sets. We address the issues involved.

### A. Coding of orbits in $\Omega$

For $z \in \Omega \setminus C$, we define the address of $z$, denoted $a(z)$ as follows: If $z \not\in C^{(1)}$, then $z$ is clearly in one of the $A_i$ above, and we write $a(z) = i$. For $z \in C^{(1)}$, let $Q^{(k)}$ be the deepest critical region containing $z$. The picture inside $Q^{(k)}$ is as described in Lemma 6.2(i). Depending on whether $z$ lies in the left chamber or right chamber of $(Q^{(k)} \cap H) \setminus Q^{(k')}$, we define the location of $z$ accordingly. Points on $C_i$ have two addresses: $i$ and $i + 1$.

We remark that while $a(z)$ is defined for all $z \in \Omega$, not all points in $R_1 \setminus \Omega$ have well defined addresses. For example, let $H_j$ be the set of all horizontal sections of higher generations crossing at least the middle part of $Q^{(k)}$. There is no meaningful way to assign addresses to points in $Q^{(k)} \setminus (\cup H_j)$.

We define next the itinerary of $z_0 \in \Omega$, denoted $\iota(z_0)$. For $z_0$ such that $z_i \not\in C$ for all $i$, let $\iota(z_0) = a = (a_i)_{i=-\infty}^\infty$ where $a_i = a(z_i)$. Orbits that pass through $C$ have exactly two itineraries; the reason there are not more than two is that for all $z \in C$, $T^i z \not\in C$ for all $i \neq 0$. Let $\Sigma = \{\iota(z_0) : z_0 \in \Omega\}$. By continuity arguments, $\Sigma$ is a closed subset of the full shift on $q$ symbols. It is clearly invariant under the shift operator $\sigma$, which commutes with $T$. For each $a \in \Sigma$, we let $\pi(a) = \{z_0 \in \Omega : \iota(z_0) = a\}$.

At issue is whether or not the partition $A$ separates points, equivalently whether or not $\pi(a)$ consists of a single orbit for each $a \in \Sigma$. To eliminate a trivial way for this to fail, we make the simplifying assumption that if $J$ is an interval of monotonicity, then $f_0(J) \not\subseteq S^1$. (If this does not hold, simply enlarge the partition by adding some pre-images of $C_i$.)

Under the assumption in the last paragraph, we prove

**Proposition 6.1** For every $a \in \Sigma$, $\pi(a)$ consists of exactly one point.
The proof of this proposition involves the use of monotone branches. We first discuss the symbolic coding for these branches. Monotone branches are treated in Sections 7 and 8 of [WY2] and reviewed in Sect. 2.2B.

B. Coding of monotone branches

For a monotone branch $M \in \mathcal{T}$, we define $\iota_-(M)$ as follows: For $M = R_1$, let $\iota_-(M) = [\emptyset]$. Let $k \geq 1$ and $M \in \mathcal{T}_k$ be such that $\iota_-(M) = [a_{-k+1}, \ldots, a_{-1}]$ has been assigned, and suppose $M$ is not discontinued. If $M$ does not meet any of the $B^{(k)}$, then it lies strictly between $C_j$ and $C_{j+1}$ for some $j$, and we define $\iota_-(T(M)) = [a_{-k+1}, \ldots, a_{-1}, j]$. If $M$ meets some $B^{(k)}$, and $M'$ is the child of $M$ with $T^{-1}(M')$ located between $C_j$ and $C_{j+1}$, then $\iota_-(M') = [a_{-k+1}, \ldots, a_{-1}, j]$. In other words, $\iota_-(M)$ gives the backward itinerary of $M$. Notice that the $f_0(J) \not\supset S^1$ condition we have assumed ensures that no two distinct monotone branches have the same backward itinerary.

We discuss next the relation between coding of individual orbits and that of monotone branches. First we introduce the following language: For $z_0 \in \Omega$ and $M \in \mathcal{T}_k$, we say $\iota(z_0)$ is compatible with $\iota_-(M)$ if $\iota(z_0) = (a_i)$ and $\iota_-(M) = [a_{-k+1}, \ldots, a_{-1}]$. Notice that (i) it is not the case that $\iota(z_0)$ is compatible with $\iota_-(M)$ for all $z_0 \in \Omega \cap M$; this is due to the fact that the ends of a monotone branch “wrap around the turns”; (ii) not every $n$-block is the itinerary of a monotone branch, not even when the block is a segment of $\iota(z_0)$ for some $z_0 \in \Omega$; this is due to the discontinuation of branches.

For $m < n$, let $V_m^n(z_0) := \{ \xi_0 \in \Omega : a(\xi_i) = a_i \text{ for } m \leq i \leq n \}.$

**Lemma 6.4** Let $z_0 \in \Omega$ with $\iota(z_0) = (a_i)$ be given.

(i) If $\iota_-(M) = [a_{-n+1}, \ldots, a_{-1}]$, then $V_{-n+1}^1(z_0) \subset M$.

(ii) For every $n \in \mathbb{Z}^+$, there exist $k$, $n \leq k < n(1 + 2\theta)$, and $M \in \mathcal{T}_k$ such that $\iota(z_0)$ is compatible with $\iota_-(M)$.

**Proof:** (i) is obvious. To prove (ii), it is necessary to get inside the “branch replacement” procedure introduced in [WY2] and used in the proof of the result in Sect. 2.2B(3). We review this procedure in the Appendix; the proof of Lemma 6.3(ii) is given at the end of this review. \qed

C. Proof of Proposition 6.1

Let $a = (a_i)_{i=0}^\infty \in \Sigma$. We fix $z_0 \in \pi(a)$ and $\varepsilon > 0$, and let $B(z_0, \varepsilon)$ denote the ball of radius $\varepsilon$ centered at $z_0$. We must show $V_m^n(z_0) \subset B(z_0, \varepsilon)$ for all $m, n$ sufficiently large. Our strategy is as follows. First we pick $m$ such that $(Kb)^m < \frac{\varepsilon}{2}$, and let $n = n(m, \varepsilon)$ be a large number to be determined. Applying (ii) in Lemma 6.4 with $z_n$ in the place of $z_0$ and $n + m$ in the place of $n$, we find $M \in \mathcal{T}_k$, $m + n \leq k \leq (m + n)(1 + 2\theta)$, with $\iota_-(M) = [a_{-(k-n)}, \ldots, a_0, a_1, \ldots, a_{n-1}]$. It suffices to show $T^{-n}(M) \subset B(z_0, \varepsilon)$, for by (i) in Lemma 6.4, $V_{-(k-n)}^{n-1}(z_0) \subset T^{-n}(M)$.

The diameter of $T^{-n}(M)$ is estimated as follows: Let $\xi_1 = T^{-(k-n)}z_0$. We show (1) there is a codimension one stable manifold $W$ through $\xi_1$ such that $T^{k-n}(W)$ has diameter $< \frac{\varepsilon}{2}$ and meets every $\mathcal{F}_{k-n}$-leaf in $T^{-n}M$, and (2) the lengths of all $\mathcal{F}_{k-n}$-leaves in $T^{-n}(M)$ are $< \frac{\varepsilon}{2}$. 40
To prove (1) we use Lemma 8.1 of [WY2]. Take a point $\xi_k$ in the main body of $M$. It follows from Lemma 8.1 of [WY2] that at $\xi_1$, there is a codimension one stable manifold spanning $R_1$ with the property that for $\xi_1, \xi'_1 \in W$, $|\xi_1 - \xi'_1| < (Kb^2)^i$ for $i < k$. For $m$ large enough as above, it follows that $T^{k-n}(W)$ has diameter $< \frac{\varepsilon}{2}$.

For (2), we prove the following lemma:

**Lemma 6.5** There exists $K > 0$ such that if $M \in T_k$ and $\gamma$ is a piece of $F_k$-leaf in $M$ such that $T^{-k}\gamma$ has length $\geq e^{-h}$, then $k < Kh$.

Lemma 6.5 is essentially a 1D statement, the proof of which we leave as an exercise.

Assuming Lemma 6.5, we finish the proof of Proposition 6.1 by observing that the lengths of the $F_k$-leaves in $T^{-n}M$ are 

$$< \|DT\|^{(m+n)(1+2h)}e^{-\frac{m+n}{K}} = \|DT\|^{m(1+2h)}e^{-\frac{m}{K}} \cdot \|DT\|^{n2h}e^{-\frac{n}{K}}.$$ 

Clearly, $n$ can be chosen so that the bound above is $< \frac{\varepsilon}{2}$. □

The results of this subsection together prove Theorem 7.

### 6.3 Periodic points, entropy and equilibrium states

We let $P(T, \varphi)$ denote the topological pressure of $T$ for the potential $\varphi : R_1 \to \mathbb{R}$, i.e. $P(T, \varphi) = \sup_\nu P_\nu(T, \varphi)$ where $P_\nu(T, \varphi) = h_\nu(T) + \int \varphi d\nu$ and the supremum is taken over all $T$-invariant Borel probability measures $\mu$. See [Wa] for more information.

**Proof of Theorem 8:** (1) Given a continuous function $\varphi$ on $R_1$, let $\tilde{\varphi} = \varphi \circ \pi$ be its lift to $\Sigma$. Then $P(T, \varphi) = P(T|_\Omega, \varphi|_\Omega) \leq P(\sigma, \tilde{\varphi})$. Using the time-zero-partition, we see that $\sigma : \Sigma \to \Sigma$ has an equilibrium state, which we call $\tilde{\nu}$. Let $\nu = \pi_*(\tilde{\nu})$. Now $\pi$ is one-to-one except on $\bigcup_{i=-\infty}^{\infty} T^i(C)$, a set which cannot carry any part of any invariant probability measure. We therefore have $P_\nu(T, \varphi) = P_\nu(\sigma, \tilde{\varphi})$, i.e. $\nu$ is an equilibrium state for $\varphi$.

(2) Let $N_n$ be the number of distinct blocks of length $n$ in $\Sigma$, and $Q_n$ the number of periodic $n$-blocks. Using again that no point in $C$ is periodic, we see, via the coding in Theorem 7, that $P_n \leq Q_n$. (Recall that $P_n$ is the number of fixed points of $T^m$.) The proof of (1) above tells us that $h_{top}(T) = h_{top}(\sigma)$. We have thus proved

$$\limsup_{n \to \infty} \frac{1}{n} \log P_n \leq \lim_{n \to \infty} \frac{1}{n} \log N_n = h_{top}(\sigma) = h_{top}(T).$$

The reverse inequality follows from [Kt], which shows that for all $C^2$ diffeomorphisms of compact manifolds and invariant Borel probability measures $\mu$ with no zero Lyapunov exponents,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_n \geq h_\mu.$$

□

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Appendix

Replacement of monotone branches: synopsis of \[WY2\], Sect. 8.3

As explained in Sects. 2.1C and 2.2B, a monotone branch is discontinued before either one of its ends becomes too large. The problem of “branch replacement”, roughly speaking, is one of finding a collection of branches of slightly higher generations that together cover the part of the attractor “exposed” by the removal of the discontinued branch.

More precisely, let \(M \in T_k\) and \(M_1 \in T_{k_1}\) with \(k_1 > k\). We say \(M_1\) is subordinate to \(M\) if (i) \(M_1 \subset M\), (ii) the ends of \(M_1\) and \(M\) are related as follows: Let \(E\) and \(E'\) be the ends of \(M\), and \(E_1\) and \(E'_1\) the ends of \(M_1\). Suppose \(T^{-i}E = B^{(k-i)}\), then \(T^{-i}E_1 = B^{(k_1-i)}_1\) with \(B^{(k_1-i)}_1 \subset B^{(k-i)}\); and \(E'\) and \(E'_1\) are related the same way. A collection of monotone branches \(\{M_j\}\) subordinate to \(M\) is called a viable replacement for \(M\) if \((M^\circ \cap \Omega) \subset \cup_j M_j\).

A procedure for systematically replacing discontinued branches is put forth in \[WY2\]. This procedure goes hand-in-hand with the inductive construction of \(\{S_m\}_{m=1,2,\ldots}\), where each \(S_m\) is a collection of monotone branches and these collections are chosen to have the properties

(a) \(S_m \subset \cup_{m \leq k < m(1+2\theta)} T_k\), and

(b) \(\cup\{M, M \in S_m\} \supset R_{m(1+2\theta)}\).

In this construction, one stipulates that for every \(M \in S_m\), if \(M \in T_k\) and \(E\) is an end of \(M\) with \(T^{-i}E = B^{(k-i)}\), then \(i \leq \frac{2}{\theta}(k-i)\theta^{-1}\). That is to say, once \(i = \frac{2}{\theta}(k-i)\theta^{-1}\) is reached for either of its two ends, it will be time to replace \(M\).

Let \(S_1 = \{R_1\}\). We assume \(S_k\) has been defined for all \(k \leq m\), and construct \(S_{m+1}\) as follows: Consider one \(M \in S_m\) at a time. If \(M\) has not reached its replacement time, then we put all the children of \(M\) into \(S_{m+1}\). If it has, then we put into \(S_{m+1}\) the children of \(\{P'\}\) where \(\{\{P'\}\}\) is the viable replacement for \(M\) described below.

Suppose that \(M \in T_k\) is to be replaced on account of its end \(E = T^iB^{(k-i)}\). To find a suitable replacement for \(M\), we go back to the moment when \(E\) is created. It is shown that \(T^{-i}M\) is contained in a horizontal section \(H\) of \(R_{k-1}\) of length \(< e^{-2m(k-i)}\) centered at \(B^{(k-i)}\). Suppose for definiteness that \(T^{-1}M\) lies on the right side of \(B^{(k-i)}\). For each \(P \in S_{k-i+1}\) that meets \(H\), we let \(\hat{H}\) be a component of \(P \cap H\), and let \(P_1\) be the child of \(P\) with the property that \(T^{-1}P_1\) contains the right half of \(\hat{H}\). This means that one of the ends of \(P_1\) is \(T(B^{(p)})\) for some \(B^{(p)} \subset B^{(k-i)}\), \(p\) being the generation of \(P\). Inductively, we let \(P_2\) be the child of \(P_1\) carrying the end \(T^2(B^{(p)})\), \(P_3\) the child of \(P_2\) carrying the end \(T^3(B^{(p)})\), and so on until \(P_i\) is reached. It is not hard to check that if well defined (i.e. the process is not interrupted by discontinuations), \(P_i\) is subordinate to \(M\). It will be a member of the collection \(\{P'\}\).

To generate the entire collection \(\{P'\}\), we run through all the \(P \in S_{k-i+1}\), and for each component of \(P \cap H\), we produce a \(P_i\) as above. One then verifies that (i) in each case \(P_i\) is well defined, (ii) the collection \(\{P'\}\) so obtained is a viable replacement for \(M\), and (iii) defining \(S_{m+1}\) using this procedure produces a sequence \(\{S_m\}_{m=1,2,\ldots}\) with properties (a) and (b) above.

**Proof of Sublemma 3.2:** We choose for \(i = 1, 2, \ldots, (1+2\theta)^{-1}j\) a branch \(M(i) \in S_i\) with the property that \(z_{k-(1+2\theta)^{-1}j+i} \in M(i)\) and \(M(i+1)\) is either a child of \(M(i)\) or it is a
child of $P$ where $P$ is a member of a family that is a viable replacement for $M(i)$. This ensures that if $T^W$ meets every leaf in $M(i)$, then $T^{i+1}W$ will have the same property with respect to $M(i+1)$. Finally, let $Q^{(j)}$ be the critical region in $M(j)$ containing $z_k$. □

**Proof of Lemma 6.4(ii):** We fix an arbitrary $\xi_0 \in \Omega$, and suppose we have identified a monotone branch $M = M(\xi_0, n) \in \mathcal{S}_n$ such that $\iota(M)$ is compatible with $\iota(\xi_0)$. If $M$ is not discontinued, then the choice of $M(\xi_0, n+1) \in \mathcal{S}_{n+1}$ is clear. Suppose one of the ends of $M$ expires. We back up $i$ steps to $Q^{(j)}$ where this end is created. By induction hypothesis, there exists $P \in \mathcal{S}_{k-i+1}$ such that $\iota(P)$ is compatible with $\iota(\xi_{n-i})$. Since $P$ is one of the branches that run across $Q^{(k-i)}$, it follows that among the collection $\{P'\}$ used to replace $M$, there is one, say $P'_*$, such that $\iota(M')$ is compatible with $\iota(\xi_n)$. One of the children of $P'_*$ is then a candidate for $M(\xi_0, n+1)$. Observe that if $\iota(M) = [a_{-k+1}, \cdots , a_{-1}]$ and $\{M'\}$ is a viable replacement for $M$, then each of the $M'$ has an itinerary of the form $\iota(M') = [\ast, \ast, \cdots , \ast , a_{-k+1}, \cdots , a_{-1}]$.

The assertion for $z_0$ is then proved by letting $\xi_0 = z_{-n}$ for each $n$. □

**References**


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