Dissipative Homoclinic Loops of Two-Dimensional Maps and Strange Attractors with One Direction of Instability

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Abstract

We prove that when subjected to periodic forcing of the form

$$p_{\mu,\rho,\omega}(t) = \mu(\rho h(x, y) + \sin(\omega t)),$$

certain two-dimensional vector fields with dissipative homoclinic loops generate strange attractors with Sinai-Ruelle-Bowen measures for a set of forcing parameters (μ, ρ, ω) of positive Lebesgue measure. The proof extends ideas of Afraimovich and Shilnikov and applies the recent theory of rank 1 maps developed by Wang and Young. We prove a general theorem and then apply this theorem to an explicit model: a forced Duffing equation of the form

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^3 = \mu \sin(\omega t).$$

Contents

1 Introduction

This paper is about proving the existence of sustained, observable chaos in explicit models of dynamical processes. We study the effects of periodic forcing on

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certain two-dimensional flows that admit homoclinic orbits. The aim is to formulate checkable hypotheses that imply the existence of sustained, observable chaos for a set of forcing parameters of positive Lebesgue measure. We formulate such hypotheses for a general class of systems and then apply the general result to an explicit model: a Duffing equation. By *sustained, observable chaos* we refer to a number of precisely defined dynamical, geometric, and statistical properties that will be made precise in what follows.

1.1 Background: Dynamical Systems

The general theory of hyperbolic dynamics is one of the most important components of the modern theory of dynamical systems. Individual orbits are typically unstable in systems with some degree of hyperbolicity. It is therefore natural to study such systems from a probabilistic point of view and ask the following questions:

(Q1) What mechanisms are creating the hyperbolicity?

(Q2) Does the system admit an invariant measure that describes the asymptotic distribution of a large set (positive Riemannian volume) of orbits? If so, how many such measures does the system admit?

(Q3) What are the ergodic and statistical properties of the invariant measures identified in (Q2)? For example, do correlations decay at an exponential rate?

For a conservative system (a system preserving a measure ν that is equivalent to Riemannian volume), the Birkhoff pointwise ergodic theorem answers (Q2). If ν is ergodic, then almost every point with respect to Riemannian volume produces an orbit that is asymptotically distributed according to ν. The situation is completely different for dissipative (volume-contracting) systems. For such systems, question (Q2) is a major challenge. We focus on dissipative systems in this paper.

Let M be a compact Riemannian manifold and let $F : M \to M$ be a C^2 embedding. A compact set Σ satisfying $F(\Sigma) = \Sigma$ is called an *attractor* if there exists an open set U, called its *basin*, such that $F^n(x) \to \Sigma$ as $n \to \infty$ for every $x \in U$. The attractor Σ is said to be

(1) *irreducible* if it cannot be written as a union of two disjoint attractors;
(2) *uniformly hyperbolic* if the tangent bundle over Σ admits a splitting into two DF-invariant subbundles E^s and E^u such that $DF|E^s$ is uniformly contracting and $DF|E^u$ is uniformly expanding. We assume that E^u is nontrivial.

An irreducible, uniformly hyperbolic attractor Σ supports a unique F-invariant Borel probability measure ν with the following property: there exists a set $S \subset U$ having full Riemannian volume in U such that for every continuous observable

$\varphi : U \to \mathbb{R}$ and for every $x \in S$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(F^i(x)) = \int_M \varphi \, d\nu. \tag{1.1}$$

This measure is known as the Sinai-Ruelle-Bowen, or SRB, measure of F. We adopt the point of view that sets of positive Riemannian volume correspond to observable events. In this sense, the SRB measure on a uniformly hyperbolic attractor is observable because time and space averages of observables coincide for a set of initial data of full Riemannian volume in the basin. Note that for dissipative systems, this does not follow from the Birkhoff ergodic theorem because if F is volume-contracting in the basin, then the SRB measure will be supported on a set of zero Riemannian volume.

Many models of physical and biological processes exhibit some degree of hyperbolicity but are not uniformly hyperbolic in character. The theory of nonuniform hyperbolicity can potentially be useful for the analysis of such systems. This theory relaxes the assumptions of uniform hyperbolicity by assuming that contraction and expansion occur only asymptotically in time and only almost everywhere with respect to an invariant measure. New notions of SRB measure have developed as the theory of nonuniform hyperbolicity has matured. The following definition is the state of the art:

DEFINITION 1.1. Let M be a compact Riemannian manifold and let $F : M \to M$ be a C^2 embedding. An F-invariant Borel probability measure ν is called an *SRB measure* if (F, ν) has a positive Lyapunov exponent ν almost everywhere (a.e.) and if ν has absolutely continuous conditional measures on unstable manifolds.

These more general SRB measures are important for many reasons, one of which is that members of a large class of them are observable: if ν is an ergodic SRB measure with no 0 Lyapunov exponents, then there exists a set S of positive Riemannian volume such that (1.1) holds for every continuous observable $\varphi : M \to \mathbb{R}$ and for every $x \in S$. For dissipative systems, proving the existence of genuinely nonuniformly hyperbolic dynamics and SRB measures is a major challenge. SRB measures were first constructed outside of the uniformly hyperbolic setting by Benedicks and Young relatively recently [6, 7]. Here SRB measures are constructed for certain Hénon maps. A major advance has been made in the last decade: the theory of rank 1 maps.

1.2 Theory of Rank 1 Maps

Developed by Wang and Young [48, 52, 53], the theory of rank 1 maps provides checkable conditions that imply the existence of strange attractors and SRB measures in parametrized families $\{F_a\}$ of dissipative embeddings in dimension N for any $N \geq 2$. The term "rank 1" refers to the local character of the embeddings: some instability in one direction and strong contraction in all other directions. Roughly speaking, the theory asserts that under certain checkable conditions, there

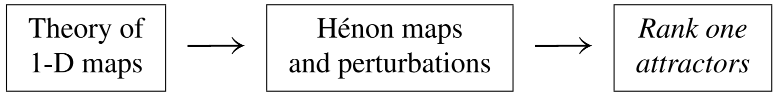


FIGURE 1.1. Progression of ideas leading to the theory of rank 1 maps.

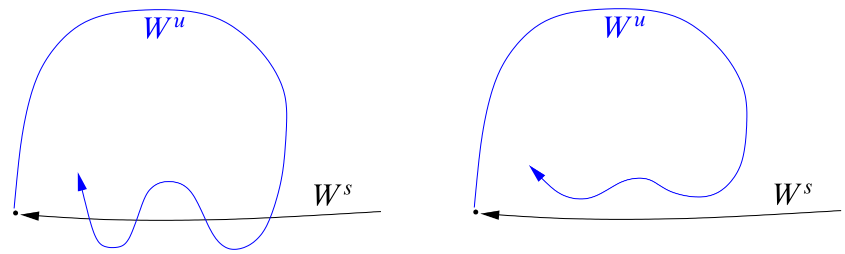
exists a set Δ of values of a of positive Lebesgue measure such that for $a \in \Delta$, F_a is a genuinely nonuniformly hyperbolic map that has a strange attractor and admits an SRB measure. A comprehensive dynamical profile is given for such F_a. In its strongest form, the profile is as follows.

The map F_a admits a unique SRB measure ν, and ν is mixing. Lebesgue almost every orbit in the basin of the strange attractor is asymptotically distributed according to ν in the sense of (1.1) and has a positive Lyapunov exponent. Thus the chaos associated with F_a is both observable and sustained in time. The system (F_a, ν) satisfies a central limit theorem, and correlations decay at an exponential rate for Hölder observables. The source of the nonuniform hyperbolicity is identified, and the geometric structure of the attractor is analyzed in detail.

Figure 1.1 illustrates the progression of ideas that has led to the development of the theory of rank 1 maps. The theory is ultimately based on theoretical developments concerning one-dimensional maps with critical points; see, e.g., [4, 9, 17, 24, 42, 43, 44]. We note in particular the parameter exclusion technique of Jakobson [17] and the analysis of the quadratic family by Benedicks and Carleson [4]. The analysis of the Hénon family by Benedicks and Carleson [5] provided a breakthrough from one-dimensional maps with critical points (the quadratic family) to two-dimensional diffeomorphisms (small perturbations of the quadratic family). Mora and Viana [25] generalized the work of Benedicks and Carleson to small perturbations of the Hénon family and proved the existence of Hénon-like attractors in parametrized families of diffeomorphisms that generically unfold a quadratic homoclinic tangency.

Wang and Young made several advances, two of which we discuss here. First, the theory of rank 1 maps replaces the formulas of the Hénon diffeomorphisms with a set of analytic and geometric hypotheses so that the theory can be applied to concrete systems of differential equations. Second, as discussed earlier, a comprehensive dynamical profile is given for the nonuniformly hyperbolic maps. Exponential decay of correlations for these maps is proved by applying the techniques of [54, 55].

The theory of rank 1 maps has been applied to several concrete models thus far. Examples include simple mechanical systems [49] and electronic circuits [29, 30, 46, 47]. Guckenheimer, Wechselberger, and Young [14] connect the theory of rank 1 maps and geometric singular perturbation theory by formulating a general technique for proving the existence of chaotic attractors for three-dimensional vector fields with two time scales. Lin [21] demonstrates how the theory of rank 1 maps can be combined with sophisticated computational techniques to analyze the



(a) Transverse homoclinic intersections. (b) Invariant manifolds do not intersect.

FIGURE 1.2. Some time-T maps that can occur when a system with a homoclinic loop is subjected to periodic forcing of period T.

response of concrete nonlinear oscillators of interest in biological applications to periodic pulsatile drives.

The dynamical scenario studied most extensively thus far is that of weakly stable structures subjected to periodic pulsatile forcing. Weakly stable equilibria [31], limit cycles [32, 49, 50], and supercritical Hopf bifurcations [50] in finite-dimensional systems have been treated. Here intrinsic shear in the unforced system is amplified by the cumulative effects of a pulsatile force followed by a long period of relaxation. This amplification produces rank 1 dynamics. Lu, Wang, and Young [22] use the theory of rank 1 maps and invariant manifold techniques to prove that certain parabolic partial differential equations undergoing supercritical Hopf bifurcations admit SRB measures when subjected to periodic pulsatile forcing. In this paper we analyze systems with dissipative homoclinic loops.

1.3 Periodically Forced Homoclinic Solutions

Homoclinic phenomena including homoclinic tangles were first observed by Poincaré [38, 39, 40] and have been studied extensively. Important systems in this context include the nonlinear pendulum, the Duffing equation, and the van der Pol oscillator [3, 11, 13, 19, 20, 45].

Consider a differential equation of the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \tag{1.2}$$

on $\mathbb{R}^N$. A *homoclinic orbit* is a solution $\boldsymbol{\Phi}$ of (1.2) that converges to a single stationary point of saddle type as $t \to \pm\infty$. The homoclinic orbit is therefore part of both the stable and unstable manifolds of the saddle. When a system with such an orbit is forced periodically, the stable and unstable manifolds that coincide in the unforced system will typically become distinct.

Figure 1.3 illustrates some of the possibilities. If the stable and unstable manifolds intersect transversely as in Figure 1.3(a), homoclinic tangles and horseshoes are produced (horseshoes are invariant sets on which the dynamics are conjugate to certain symbolic systems). This scenario has been studied extensively; see e.g., [23, 38, 39, 40, 41]. This paper focuses on the scenario in which the stable and unstable manifolds do not intersect (see Figure 1.3(b)). Afraimovich and Shilnikov [2] initiated the study of this scenario by proving that it is possible to define a flow-induced return map. They showed that this map admits horseshoes in certain parameter regimes and conjectured that it has a strange attractor. Our paper solves this conjecture in the affirmative.

Section 2 contains a complete description of our results. We summarize them briefly here. Consider an autonomous vector field on $\mathbb{R}^2$ that admits a homoclinic solution associated with a stationary point z_0 of saddle type. We assume that the saddle is dissipative, meaning that the eigenvalues $-\alpha$ and β of the linearization of the vector field at z_0 satisfy $0 < \beta < \alpha$, and nonresonant (see (H1) in Section 2). Suppose the autonomous system is subjected to periodic forcing of the form $p_{\mu,\rho,\omega}(t) = \mu(\rho h(x, y) + \sin(\omega t))$. Here μ, ρ, and the frequency ω are parameters. We convert the forced system into an autonomous system on the extended phase space $\mathbb{R}^2 \times \mathbb{S}^1$ by adding an angular variable θ. The main theorem (Theorem 1 in Section 2) of this paper provides checkable conditions that imply the existence of a strange attractor in the forced system for a set of forcing parameters (μ, ρ, ω) of positive (three-dimensional) Lebesgue measure. More precisely, for such parameters we show that the flow of the forced system admits a flow-induced return map to an annulus in the extended phase space and this discrete-time annulus map admits a strange attractor. By "strange attractor" we refer to the complete dynamical profile implied by the theory of rank 1 maps (see [48, 53]). In particular, the strange attractor supports a unique mixing SRB measure with exponential decay of correlations, and Lebesgue almost every point in the basin has a positive Lyapunov exponent. The chaos is therefore sustained in time and observable. We prove the main theorem by explicitly computing the flow-induced annulus maps and then using the theory of rank 1 maps.

Theorem 1 is designed so that it can be directly applied to given differential equations. In fact, to apply Theorem 1, all one must do is compute the Melnikov function and verify that it is a Morse function. In Section 8, we demonstrate the power of Theorem 1 by applying it to a concrete model: a forced Duffing equation of the form

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^3 = \mu \sin(\omega t). \tag{1.3}$$

Many natural systems possess homoclinic loops. The following is a partial list of systems to which the analysis behind Theorem 1 applies (see chapter 4 of [13] for details about these classical systems).

(1) *Pendulum.* Start with the pendulum described by the Hamiltonian $H = p^2/2 + (1 - \cos(q))$. Then perturb with time-periodic forcing and damping (damped sine-Gordon).

(2) *Pendulum oscillator.* Start with the Hamiltonian

$$H = \frac{p^2}{2} + (1 - \cos(q)) + \frac{x^2}{2} + \frac{\omega^2 y^2}{2}.$$

Restrict to an energy surface and consider a Poincaré map, e.g., $y = 0$ to $y = 0$. This Poincaré map has a separatrix loop. Now perturb in a natural fashion.

(3) *Perturbations of Hénon-Heiles Hamiltonians.*

We note that Theorem 1 is formulated for the forcing function p to enhance the readability of the proof and to allow for a direct application to the forced Duffing equation. The analysis on which Theorem 1 is based can be adapted to apply to more general forcing functions. Theorem 1 therefore applies (with the appropriate adaptations) to a rich variety of concrete physical models.

1.4 Outline

We state the main result precisely in Section 2. In Section 3 we discuss a simplified model due to Afraimovich and Shilnikov. Here we present some of the geometric and analytic ideas behind the main result in a setting with relatively low technical complexity. Sections 4–7 are devoted to the proof of the main theorem. We apply the main theorem to the forced Duffing equation in Section 8.

2 Statement of Results

2.1 Setting and Statement of the Main Theorem (Theorem 1)

Let $(x, y) \in \mathbb{R}^2$ be the phase variables and t be the time. We start with an autonomous system

$$\begin{cases} \frac{dx}{dt} = -\alpha x + f(x, y), \\ \frac{dy}{dt} = \beta y + g(x, y), \end{cases} \tag{2.1}$$

where f and g are real analytic at $(x, y) = (0, 0)$ and $f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0$. We assume that α and β satisfy a certain Diophantine nonresonance condition and that $(x, y) = (0, 0)$ is a dissipative saddle point. Namely, we assume the following:

(H1) Nonresonant dissipative saddle

(a) There exist $d_1, d_2 > 0$ such that for all $m, n \in \mathbb{Z}^+$ with $m + n > 0$, we have

$$|m\alpha - n\beta| > d_1(m + n)^{-d_2}.$$

(b) $0 < \beta < \alpha$.

We also assume that the positive x-side of the local stable manifold of $(0, 0)$ and the positive y-side of the local unstable manifold of $(0, 0)$ are included as part of a homoclinic solution, which we denote as $x = a(t)$, $y = b(t)$. Let

$$\ell = \{\ell(t) = (a(t),\ b(t)) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

We further assume that $f(x, y)$ and $g(x, y)$ are C^4 in a sufficiently small neighborhood of ℓ.

To the right side of equation (2.1) we add a time-periodic term to form a nonautonomous system

$$\begin{cases} \frac{dx}{dt} = -\alpha x + f(x, y) - \mu(\rho h(x, y) + \sin(\omega t)), \\ \frac{dy}{dt} = \beta y + g(x, y) + \mu(\rho h(x, y) + \sin(\omega t)), \end{cases} \tag{2.2}$$

where μ, ρ, and ω are parameters. We assume that $h(x, y)$ is analytic at $(x, y) = (0, 0)$ and C^4 in a small neighborhood of the homoclinic loop ℓ. The parameter μ satisfies $0 \leq \mu \ll 1$ and controls the magnitude of the forcing term. The prefactor ρ and the forcing frequency ω are much larger parameters, the ranges of which we will make explicit momentarily. Observe that the same forcing function is added to the equation for y but subtracted from the equation for x. We do this to facilitate the application of our theorem to the Duffing equation. The analysis in this work is by no means limited to these particular forcing functions.

To study (2.2), we introduce an angular variable $\theta \in \mathbb{S}^1$ and write it as

$$\begin{cases} \frac{dx}{dt} = -\alpha x + f(x, y) - \mu(\rho h(x, y) + \sin(\theta)), \\ \frac{dy}{dt} = \beta y + g(x, y) + \mu(\rho h(x, y) + \sin(\theta)), \\ \frac{d\theta}{dt} = \omega. \end{cases} \tag{2.3}$$

We define

$$(u(t), v(t)) = \left\| \frac{d}{dt} \ell(t) \right\|^{-1} \frac{d}{dt} \ell(t)$$

where $\ell(t) = (a(t), b(t))$ is the homoclinic loop of equation (2.1). The vector $(u(t), v(t))$ is a unit vector tangent to ℓ at $\ell(t)$. Define

$$\begin{aligned} E(t) = v^2(t)\big(-\alpha + \partial_x f(a(t), b(t))\big) + u^2(t)\big(\beta + \partial_y g(a(t), b(t))\big) \\ - u(t)v(t)\big(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))\big). \end{aligned} \tag{2.4}$$

The quantity $E(t)$ measures the rate of expansion of the solutions of equation (2.1) in the direction normal to ℓ at $\ell(t)$ (see Section 4.3). In matrix form, we have

$$E(t) = \begin{pmatrix} v(t) & -u(t) \end{pmatrix} \begin{pmatrix} -\alpha + \partial_x f(\ell(t)) & \partial_y f(\ell(t)) \\ \partial_x g(\ell(t)) & \beta + \partial_y g(\ell(t)) \end{pmatrix} \begin{pmatrix} v(t) \\ -u(t) \end{pmatrix}.$$

Define

$$\begin{aligned} A &= \int_{-\infty}^{\infty} (u(s) + v(s)) h(a(s), b(s)) e^{-\int_0^s E(\tau) d\tau} \, ds, \\ C &= \int_{-\infty}^{\infty} (u(s) + v(s)) \cos(\omega s) e^{-\int_0^s E(\tau) d\tau} \, ds, \\ S &= \int_{-\infty}^{\infty} (u(s) + v(s)) \sin(\omega s) e^{-\int_0^s E(\tau) d\tau} \, ds. \end{aligned} \tag{2.5}$$

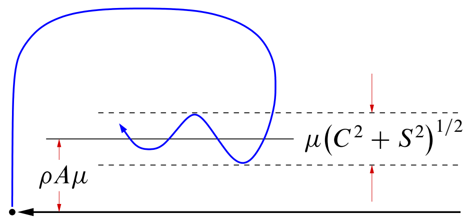


FIGURE 2.1. The geometric meaning of the integrals A, C, and S.

The integrals A, C, and S are all absolutely convergent (see Lemma 4.8). They describe the relative positions of the stable and unstable manifolds of the perturbed saddle. See Figure 2.1. The quantity $\rho A\mu$ measures the average distance between the stable and unstable manifolds, and $\mu(C^2+S^2)^{1/2}$ measures the magnitude of the oscillation of the unstable manifold relative to the stable manifold.

We assume that A, C, and S satisfy the following nondegeneracy conditions.

(H2) Nondegeneracy conditions on A, C, and S

(a) $A \neq 0$.
(b) $C^2+S^2 \neq 0$.

Given equation (2.2) satisfying (H1) and (H2), we let

$$\rho_1 = -\frac{202}{99}\frac{(C^2+S^2)^{1/2}}{A}, \quad \rho_2 = -\frac{396}{101}\frac{(C^2+S^2)^{1/2}}{A}.$$

We also let

$$I = \{z \in \mathbb{R} : |z| < K\mu\}$$

for some $K > 1$ sufficiently large and independent of μ, and

$$\Sigma = \{\ell(0) + (v(0), \; -u(0))z \in \mathbb{R}^2 : z \in I\} \times \mathbb{S}^1.$$

The following is the main theorem of this paper.

THEOREM 1. *Assume that* (2.2) *satisfies* (H1) *and* (H2)(a). *There exists $\omega_0 > 0$ such that if $\omega \in \mathbb{R}$ satisfies* (H2)(b) *and $|\omega| > \omega_0$, then for every $\rho \in [\rho_1, \rho_2]$ we have the following*:

(1) *For μ sufficiently small, equation* (2.3) *induces a well-defined return map $\mathcal{F}_\mu : \Sigma \to \Sigma$.*
(2) *There exists a set $\Delta_{\omega,\rho}$ of values of μ with positive Lebesgue measure such that for every $\mu \in \Delta_{\omega,\rho}$, $\mathcal{F}_\mu$ admits a strange attractor that supports a unique ergodic SRB measure ν. Furthermore, Lebesgue almost every point in Σ is generic with respect to ν.*

Remark 2.1. The set $\Delta_{\omega,\rho}$ has positive lower Lebesgue density at $\mu = 0$, meaning that

$$\varliminf_{\hat{\mu}\to 0} \frac{\mathrm{Leb}(\Delta_{\omega,\rho} \cap [0,\hat{\mu}])}{\hat{\mu}} > 0.$$

Remark 2.2. We have opted to state Theorem 1 in terms of SRB measures. In fact, for $\mu \in \Delta_{\omega,\rho}$, $\mathcal{F}_\mu$ has all of the properties provided by the theory of rank 1 maps (see [48, 52, 53] for details). In particular, ν exhibits exponential decay of correlations for Hölder observables, and $\mathcal{F}_\mu$ has a positive Lyapunov exponent Lebesgue almost everywhere in Σ.

Remark 2.3. As an important condition to be verified, (H2) does not cast doubt on the abundance of the type of strange attractor proved to exist in this paper. By properly adjusting the sign of $h(x, y)$ according to the sign of $u(s) + v(s)$ on ℓ, we can easily achieve $A \neq 0$. Hypothesis (H2)(b) requires that the Fourier spectrum of the function

$$R(s) = (u(s) + v(s))e^{-\int_0^s E(\tau)d\tau}$$

not be identically zero on the frequency range higher than ω_0. Since $R(s)$ decays exponentially as a function of s, the Fourier transform $\hat{R}(\xi)$ is analytic in a strip containing the real ξ-axis by the Paley-Wiener theorem. It follows that $\hat{R}(\xi) = 0$ for at most a discrete set of values of ξ unless $R(s)$ is identically zero.

2.2 Discussion: Unfolding Homoclinic Structures

Theorem 1 describes the dynamics of certain unfoldings of homoclinic orbits. Unfoldings of homoclinic structures (homoclinic bifurcations) have been studied extensively; a rich variety of dynamical phenomena have been identified and investigated. Theorem 1 adds to this body of work. Here we discuss some of the possibilities that Theorem 1 does not cover.

Unfoldings of Homoclinic Tangencies

Let $\{G_\xi\}$ be a C^∞ one-parameter family of surface diffeomorphisms and assume that G_0 has a quadratic homoclinic tangency associated with a hyperbolic stationary point z_0 ($W^s(z_0)$ and $W^u(z_0)$ are tangent at a point). Suppose that $\{G_\xi\}$ generically unfolds the homoclinic tangency and that the eigenvalues ζ_s and ζ_u of $DG_0(z_0)$ satisfy

(1) $0 < \zeta_s < 1 < \zeta_u$,
(2) $\zeta_s \zeta_u < 1$, and
(3) (ζ_s, ζ_u) belongs to the open and dense set of eigenvalue tuples that satisfy the hypotheses of the Sternberg linearization theorem.

In this setting, there exists a set Δ of values of ξ near $\xi = 0$ of positive Lebesgue measure such that for $\xi \in \Delta$, G_ξ admits a strange attractor. Mora and Viana prove this result in [25] by extending the analysis of Benedicks and Carleson [5] to perturbations of the Hénon family (so-called Hénon-like families) and then showing that $\{G_\xi\}$ admits a Hénon-like renormalization. Wang and Young later show that the theory of rank 1 maps applies in this setting [48].

Newhouse has shown that there exist parameter intervals and residual subsets of them such that for a parameter ξ in one of these residual subsets, G_ξ has infinitely many periodic sinks (attracting periodic orbits) [27, 28]. Palis conjectures that the set of values of ξ for which G_ξ has infinitely many periodic sinks is a set of Lebesgue measure zero [33]. Gorodetski and Kaloshin have made progress on this conjecture [12]. By combining the techniques of Mora and Viana with those of Newhouse, Colli proves that there exist parameter intervals converging to $\xi = 0$ and dense subsets of these intervals such that for ξ in one of these dense subsets, G_ξ exhibits infinitely many coexisting Hénon-like strange attractors [10]. The work of Newhouse and of Colli has been generalized to higher dimensions by Palis and Viana [36] and by Leal [18], respectively.

The results described thus far only scratch the surface of the vast body of work on homoclinic and heteroclinic phenomena. For the sake of brevity, we complete this necessarily incomplete discussion by mentioning work on the dominance of hyperbolicity [26, 34] and the work of Palis and Yoccoz [37] on nonuniformly hyperbolic horseshoes. For further reading, we direct the reader to [8, 35] and the references contained therein.

Comparison of Theorem 1 with the Work of Mora and Viana

We can relate the theory of homoclinic tangencies to the setting of this paper (depicted in Figure 2.1) by first fixing a value of $\mu \neq 0$ and then varying ρ to create a nondegenerate homoclinic tangency for a certain value of ρ (using the Melnikov method). There is no compelling reason for this not to work, though we are not aware of any rigorous attempt in the literature. It would then follow that for every fixed $\mu \neq 0$ sufficiently small, there is a positive measure set of values of ρ such that equation (2.3) admits Hénon-like attractors in the sense of Mora and Viana [25]. We emphasize, however, that the Hénon-like attractors obtained in this way are entirely different from the rank 1 attractors we obtain in Theorem 1. The rank 1 attractors obtained in Theorem 1 have entire neighborhoods of the unperturbed homoclinic solution as part of their basins of attraction (they are global). Further, the strength of the expansion associated with the SRB measure supported on the attractor is determined by the forcing frequency ω, while the Hénon-like attractors of Mora and Viana are tiny local sinklike objects obtained by zooming in through renormalization. In fact, the entire purpose of the theory of Wang and Young is to provide a way to prove the existence of large strange attractors obtained through global analysis.

3 A Model of Afraimovich and Shilnikov

In this section we study a model introduced by Afraimovich and Shilnikov in [2]. See also [1]. This simple model allows us to illustrate the geometry of the flow-induced maps and the steps we use to prove the main theorem while working in a setting with relatively low technical complexity. The flow-induced maps of Afraimovich and Shilnikov are derived in Section 3.1, and their geometry is examined in Section 3.2. In Section 3.3 we prove that these flow-induced maps are rank 1 maps in the sense of [48, 52, 53].

3.1 Derivation of Return Maps

We begin by describing an unperturbed system of differential equations. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be C^∞ functions, and let $\alpha, \beta \in \mathbb{R}$ satisfy $0 < \beta < \alpha$. Define

$$\begin{cases} \frac{dx}{dt} = -\alpha x + f(x,y), \\ \frac{dy}{dt} = \beta y + g(x,y). \end{cases} \tag{3.1}$$

We assume that the functions f and g satisfy $f(x,y) = g(x,y) = 0$ for all $(x,y) \in B(\mathbf{0}, 2\varepsilon)$ where $0 < \varepsilon < 1$. This means that equation (3.1) is linear in a neighborhood of $\mathbf{0}$. We also assume that equation (3.1) admits a homoclinic solution $\ell = \{\ell(t) : t \in \mathbb{R}\}$ containing the segments $\{(0,y) : 0 < y < 2\varepsilon\}$ and $\{(x,0) : 0 < x < 2\varepsilon\}$.

Let $\mathbb{S}^1 = [0, 2\pi)$ denote the unit circle, and let $p, q : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}$ be C^∞ functions such that $p = q = 0$ on $B(\mathbf{0}, 2\varepsilon) \times \mathbb{S}^1$. We now introduce the perturbed system

$$\begin{cases} \frac{dx}{dt} = \alpha x + f(x,y) + \mu p(x,y,\theta), \\ \frac{dy}{dt} = \beta y + g(x,y) + \mu q(x,y,\theta), \\ \frac{d\theta}{dt} = \omega. \end{cases} \tag{3.2}$$

Here $\omega \in \mathbb{R}$ is the frequency of the forcing functions and $\mu > 0$ represents the strength of the perturbation. We assume that μ and ε satisfy $0 \le \mu \ll \varepsilon < 1$.

The orbit $\gamma = \{(0,0,\theta) : \theta \in \mathbb{S}^1\}$ is a hyperbolic periodic orbit of equation (3.2) for all μ. For $\mu = 0$, $\Gamma = \ell \times \mathbb{S}^1$ is contained in both the stable manifold and the unstable manifold of γ. We define the Poincaré sections

$$\Sigma^- = \{(x,y,\theta) : 0 \le x \le C_1\mu,\ y = \varepsilon,\ \theta \in \mathbb{S}^1\},$$
$$\Sigma^+ = \{(x,y,\theta) : x = \varepsilon,\ C_2^{-1}\mu \le y \le C_2\mu,\ \theta \in \mathbb{S}^1\},$$

where $\mu \in [0, \mu_0]$, $C_1 > 0$ is such that $C_1\mu_0 \ll \varepsilon$, and C_2 is suitably chosen. We study a situation in which one can define flow-induced maps $\mathcal{M} : \Sigma^- \to \Sigma^+$ and $\mathcal{N} : \Sigma^+ \to \Sigma^-$. The composition $\mathcal{N} \circ \mathcal{M}$ produces a one-parameter family $\{\mathcal{F}_\mu = \mathcal{N} \circ \mathcal{M} : \mu \in [0, \mu_0]\}$ of maps from Σ^- to Σ^-.

The Map $\mathcal{N} : \Sigma^+ \to \Sigma^-$

The flow from Σ^+ to Σ^- is defined by the differential equations

$$\frac{dx}{dt} = -\alpha x, \tag{3.3a}$$
$$\frac{dy}{dt} = \beta y, \tag{3.3b}$$
$$\frac{d\theta}{dt} = \omega. \tag{3.3c}$$

Let $(\varepsilon, \hat{y}, \hat{\theta}) \in \Sigma^+$. Let $T(\hat{y})$ denote the time at which the orbit emanating from $(\varepsilon, \hat{y}, \hat{\theta})$ intersects Σ^-. Write $\mathcal{N}(\varepsilon, \hat{y}, \hat{\theta}) = (x_1, \varepsilon, \theta_1)$. Integrating (3.3b), we have $\varepsilon = e^{\beta T(\hat{y})}\hat{y}$, so $T(\hat{y}) = \log(\varepsilon \hat{y}^{-1})/\beta$. Integrating (3.3a) yields $x_1 = e^{-\alpha T(\hat{y})}\varepsilon = \varepsilon^{1-\alpha/\beta}\hat{y}^{\alpha/\beta}$. The local map $\mathcal{N}$ is therefore given by

$$\begin{cases} x_1 = \varepsilon^{1-\alpha/\beta}\hat{y}^{\alpha/\beta}, \\ \theta_1 = \hat{\theta} + \frac{\omega}{\beta}\log(\varepsilon\hat{y}^{-1}). \end{cases} \tag{3.4}$$

The Map $\mathcal{M} : \Sigma^- \to \Sigma^+$

Let $(x_0, \varepsilon, \theta_0) \in \Sigma^-$. Write $\mathcal{M}(x_0, \varepsilon, \theta_0) = (\varepsilon, \hat{y}, \hat{\theta})$. We assume that for $\mu \in [0, \mu_0]$,

$$\hat{y} = \lambda x_0 + \mu\varphi(x_0, \theta_0), \quad \hat{\theta} = \theta_0 + \xi_1 + \mu\psi(x_0, \theta_0).$$

Here $0 < \lambda < 1$ and $\xi_1 > 0$ are fixed. The functions φ and ψ are C^∞ functions on Σ^-. We assume that $\varphi(x_0, \theta_0) > 0$ for all $(x_0, \theta_0) \in \Sigma^-$. This ensures that the stable and unstable manifolds are pulled apart by the periodic forcing (see Figure 1.3). More precisely, we assume that p and q are such that

$$\psi(x_0, \theta_0) = \xi_2, \quad \varphi(x_0, \theta_0) = B(1 + A\sin\theta_0).$$

Here $\xi_2 \in \mathbb{R}$, $B > 0$, and $0 < A < 1$. The global map $\mathcal{M}$ is therefore given by

$$\begin{cases} \hat{y} = \lambda x_0 + \mu B(1 + A\sin\theta_0), \\ \hat{\theta} = \theta_0 + \xi_1 + \mu\xi_2. \end{cases} \tag{3.5}$$

The Map $\mathcal{F}_\mu = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$

Let $(x_0, \varepsilon, \theta_0) \in \Sigma^-$. Computing $\mathcal{F}_\mu(x_0, \varepsilon, \theta_0) = (x_1, \varepsilon, \theta_1)$ by using (3.4) and (3.5), we have

$$\begin{aligned} x_1 &= \varepsilon^{1-\alpha/\beta}[\lambda x_0 + \mu B(1 + A\sin(\theta_0))]^{\alpha/\beta}, \\ \theta_1 &= \theta_0 + \xi_1 + \mu\xi_2 + \frac{\omega}{\beta}\log\left(\frac{\varepsilon}{\lambda x_0 + \mu B(1 + A\sin(\theta_0))}\right). \end{aligned}$$

Using the spatial rescaling $x \mapsto \mu X$, we obtain

$$X_1 = \varepsilon^{1-\alpha/\beta}\mu^{\alpha/\beta - 1}[\lambda X_0 + B(1 + A\sin(\theta_0))]^{\alpha/\beta}, \tag{3.6a}$$

$$\theta_1 = \theta_0 + \xi_1 + \mu\xi_2 + \frac{\omega}{\beta}\log\left(\frac{\varepsilon\mu^{-1}}{\lambda X_0 + B(1 + A\sin(\theta_0))}\right). \tag{3.6b}$$

Using this formula for $\mathcal{F}_\mu$, Afraimovich, Hsu, and Shilnikov [1, 2] conclude that $\mathcal{F}_\mu$ has a horseshoe for large ω.

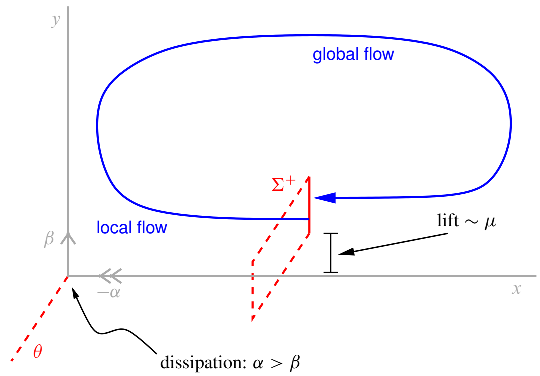


FIGURE 3.1. The flow-induced map on Σ^+.

3.2 Geometry of $\mathcal{F}_\mu$

Dissipation and shear are the two primary mechanisms responsible for the presence of rank 1 chaos in the family $\{\mathcal{F}_\mu\}$. Figure 3.1 illustrates the flow-induced map on the annulus Σ^+.

If the "lift" (μ) is small, then Σ^+ spends a long time near the hyperbolic periodic orbit γ as it flows forward. The flow contracts volume in a neighborhood of $\{(0, 0, \theta) : \theta \in \mathbb{S}^1\}$ because $\alpha > \beta$. Therefore, the flow-induced map on Σ^+ is dissipative if μ is small, and the dissipation becomes stronger as μ decreases.

Shear produces stretch-and-fold geometry in phase space and generates expansion in the singular limit. Figure 3.2 demonstrates the origin and effect of shear on a curve $V \subset \Sigma^-$. Let $0 < S < C_1$ and define $V \subset \Sigma^-$ by $V = \{(S, \theta) : \theta \in \mathbb{S}^1\}$. The global map $\mathcal{M}$ causes the slice V to become curved. We have

$$\mathcal{M}(V) = \{(\lambda S + B(1 + A \sin(\theta - \xi_1 - \mu\xi_2)), \theta) : \theta \in \mathbb{S}^1\}.$$

In particular, the Y-coordinates of the points in $\mathcal{M}(V)$ depend nontrivially on θ. Since the time needed for a point $(Y(\theta), \theta) \in \mathcal{M}(V)$ to reach Σ^- under the local flow is roughly $\log((\mu Y(\theta))^{-1})$, it follows that the local map $\mathcal{N}$ stretches and folds $\mathcal{M}(V)$.

3.3 Theory of Rank 1 Attractors

In this subsection we first introduce admissible rank 1 maps following [48, 52, 53], and we then prove that $\{\mathcal{F}_\mu\}$ is an admissible family of rank 1 maps using the techniques of [49].

Misiurewicz Maps and Admissible 1-D Families

The definition of an admissible family of 1-D maps is rather long and technical. It could therefore present a nontrivial hurdle for the reader. We feel obligated to present this definition for completeness. Readers wishing to skip the material on admissible 1-D families can safely jump to Proposition 3.3, which contains the

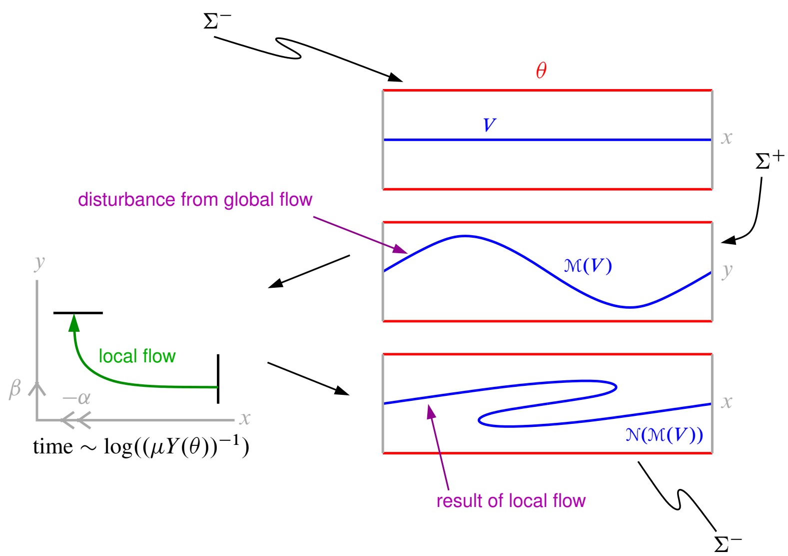


FIGURE 3.2. Stretch-and-fold geometry is produced by the combined effects of the global flow and the hyperbolic periodic orbit.

only result from the 1-D aspect of the theory of rank 1 maps that we need for the results of this paper.

We start with Misiurewicz maps. For $f \in C^2(\mathbb{S}^1, \mathbb{S}^1)$, let $C = C(f) = \{f' = 0\}$ denote the critical set of f, and let C_δ denote the δ-neighborhood of C in $\mathbb{S}^1$. For $x \in \mathbb{S}^1$, let $d(x, C) = \min_{\hat{x} \in C} |x - \hat{x}|$.

DEFINITION 3.1. We say that $f \in C^2(\mathbb{S}^1, \mathbb{S}^1)$ is a *Misiurewicz map* and write $f \in \mathcal{E}$ if the following hold for some $\delta_0 > 0$:

(1) Outside of C_{δ_0}. There exist $\lambda_0 > 0$, $M_0 \in \mathbb{Z}^+$, and $0 < c_0 \le 1$ such that
   (a) for all $n \ge M_0$, if $x, f(x), \ldots, f^{n-1}(x) \notin C_{\delta_0}$, then $|(f^n)'(x)| \ge e^{\lambda_0 n}$;
   (b) if $x, f(x), \ldots, f^{n-1}(x) \notin C_{\delta_0}$ and $f^n(x) \in C_{\delta_0}$ for any n, then

$$|(f^n)'(x)| \ge c_0 e^{\lambda_0 n}.$$

(2) Inside C_{δ_0}.
   (a) We have $f''(x) \ne 0$ for all $x \in C_{\delta_0}$.
   (b) For all $\hat{x} \in C$ and $n > 0$, $d(f^n(\hat{x}), C) \ge \delta_0$.
   (c) For all $x \in C_{\delta_0} \setminus C$, there exists $p_0(x) > 0$ such that $f^j(x) \notin C_{\delta_0}$ for all $j < p_0(x)$ and $|(f^{p_0(x)})'(x)| \ge c_0^{-1} e^{\lambda_0 p_0(x)/3}$.

We remark that Misiurewicz maps are among the simplest maps with nonuniform expansion. The phase space is divided into two regions, C_{δ_0} and $\mathbb{S}^1 \setminus C_{\delta_0}$. Condition (1) in Definition 3.1 says that on $\mathbb{S}^1 \setminus C_{\delta_0}$, f is essentially uniformly expanding. Condition (2c) says that for $x \in C_{\delta_0} \setminus C$, even though $|f'(x)|$ is small, the orbit of x does not return to C_{δ_0} again until its derivative has regained a definite amount of exponential growth. In particular, if n is the first return time of $x \in C_{\delta_0}$ to C_{δ_0}, then $|(f^n)'(x)| \geq c_0^{-1} e^{\lambda_0 n/3}$.

We now define *admissible families* of 1-D maps. Let $F : \mathbb{S}^1 \times [a_1, a_2] \to \mathbb{S}^1$ be a C^2 map. The map F defines a one-parameter family $\{f_a \in C^2(\mathbb{S}^1, \mathbb{S}^1) : a \in [a_1, a_2]\}$ via $f_a(x) = F(x, a)$. We assume that there exists $a^* \in (a_1, a_2)$ such that $f_{a^*} \in \mathcal{E}$. For each $c \in C(f_{a^*})$, there exists a continuation $c(a) \in C(f_a)$ provided a is sufficiently close to a^*.

Let $C(f_{a^*}) = \{c^{(1)}(a^*), \ldots, c^{(q)}(a^*)\}$, where $c^{(i)}(a^*) < c^{(i+1)}(a^*)$ for $1 \leq i \leq q-1$. For $c(a^*) \in C(f_{a^*})$, we define $\beta(a^*) = f_{a^*}(c(a^*))$. For all parameters a sufficiently close to a^*, there exists a unique continuation $\beta(a)$ of $\beta(a^*)$ such that the orbits

$$\{f_{a^*}^n(\beta(a^*)) : n \geq 0\} \quad \text{and} \quad \{f_a^n(\beta(a)) : n \geq 0\}$$

have the same itineraries with respect to the partitions of $\mathbb{S}^1$ induced by $C(f_{a^*})$ and $C(f_a)$. This means that for all $n \geq 0$, $f_{a^*}^n(\beta(a^*)) \in (c^{(j)}(a^*), c^{(j+1)}(a^*))$ if and only if $f_a^n(\beta(a)) \in (c^{(j)}(a), c^{(j+1)}(a))$ (here $c^{(q+1)} = c^{(1)}$). Moreover, the map $a \mapsto \beta(a)$ is differentiable (see proposition 4.1 in [51]).

DEFINITION 3.2. Let $F : \mathbb{S}^1 \times [a_1, a_2] \to \mathbb{S}^1$ be a C^2 map. The associated one-parameter family $\{f_a : a \in [a_1, a_2]\}$ is *admissible* if

(1) there exists $a^* \in (a_1, a_2)$ such that $f_{a^*} \in \mathcal{E}$;
(2) for all $c \in C(f_{a^*})$, we have

$$\xi(c) = \frac{d}{da}(f_a(c(a)) - \beta(a))\bigg|_{a=a^*} \neq 0. \tag{3.7}$$

The next proposition contains all that we need from the 1-D aspect of the theory of rank 1 maps for this paper.

PROPOSITION 3.3 ([22, 49, 50]). *Let $\Psi(\theta) : \mathbb{S}^1 \to \mathbb{R}$ be a C^3 function with nondegenerate critical points, and let $\Phi(\theta, a) : \mathbb{S}^1 \times [a_0, a_1] \to \mathbb{R}$ be such that*

$$\|\Phi(\theta, a)\|_{C^3(\mathbb{S}^1 \times [a_0, a_1])} < \frac{1}{100}.$$

We define a one-parameter family of circle maps $\{f_a : a \in [0, 2\pi]\}$ by

$$f_a(\theta) = \theta + \Phi(\theta, a) + a + \mathcal{K}\Psi(\theta)$$

where $\mathcal{K}$ is a constant. There exists K, determined by Ψ alone, such that if $\mathcal{K} > K$, then $\{f_a\}$ is an admissible family of 1-D maps.

The special case of this proposition in which $\Phi(\theta, a) = 0$ was first proved in [49]. That proof can easily be extended to prove Proposition 3.3. See also proposition 2.1 in [50] and appendix C in [22].

Admissible Families of Rank 1 Maps

We now move to the theory of rank 1 maps (see [48, 52, 53]). We focus on the special case in which the phase space is two-dimensional.

Let $I \subset \mathbb{R}$ be an interval. Let $B_0 \subset \mathbb{R}$ be a set with a limit point at 0 and let $[a_0, a_1] \subset \mathbb{R}$. A two-parameter C^3 family $\{F_{a,b}(X,\theta) : a \in [a_0,a_1],\ b \in B_0\}$ of embeddings of $\Sigma = I \times \mathbb{S}^1$ into Σ is an admissible family of rank 1 maps if the following hold:

(C1) There exists a C^2 function $F_{a,0}(X,\theta)$ of (a, X, θ) such that as $b \to 0$,

$$\|F_{a,b}(X,\theta) - (0, F_{a,0}(X,\theta))\|_{C^3([a_0,a_1]\times\Sigma)} \to 0.$$

(C2) The family $\{f_a(\theta) = F_{a,0}(0,\theta) : a \in [a_0,a_1]\}$ is an admissible family of 1-D maps.

(C3) For all $a \in [a_0, a_1]$ and for every critical point $\widehat{\theta}$ of the 1-D map $f_a(\theta)$, we have

$$\frac{\partial}{\partial X} F_{a,0}(X,\widehat{\theta})\bigg|_{X=0} \neq 0.$$

PROPOSITION 3.4 ([48, 52, 53]). *Let $F_{a,b} : \Sigma \to \Sigma$ be an admissible family of rank* 1 *maps. There exists $\widehat{b} > 0$ such that for all $|b| < \widehat{b}$, there exists a set Δ_b of values of a with positive Lebesgue measure such that for $a \in \Delta_b$, $F_{a,b}$ admits an ergodic SRB measure ν. If we also have $\lambda_0 > \log(10)$, where λ_0 is as in Definition* 3.1*, then ν is the only ergodic SRB measure that $F_{a,b}$ admits on Σ (this is proved in* [49]).

Remark 3.5. See [48, 53] for a complete dynamical profile of $F_{a,b}$ with $a \in \Delta_b$.

More is true if the global distortion bound (C4) holds.

(C4) There exists $C > 0$ such that for all $a \in [a_0, a_1]$, $b \in B_0$, $(X,\theta) \in \Sigma$, and $(X',\theta') \in \Sigma$, we have

$$\left|\frac{\det DF_{a,b}(X,\theta)}{\det DF_{a,b}(X',\theta')}\right| < C.$$

PROPOSITION 3.6 ([48, 52, 53]). *Let $F_{a,b}$ be an admissible family of rank* 1 *maps satisfying* (C4) *and suppose that $\lambda_0 > \log(10)$, where λ_0 is as in Definition* 3.1. *Then for all $|b| < \widehat{b}$ and $a \in \Delta_b$, Lebesgue almost every point in Σ is generic with respect to the unique ergodic SRB measure on Σ.*

$\{F_\mu\}$ an Admissible Family of Rank 1 Maps

We show that $\{\mathcal{F}_\mu\}$ satisfies hypotheses (C1)–(C4). Letting $\mu \to 0$ in (3.6a), we see that $X_1 \to 0$ because $\alpha > \beta$. However, the term $\frac{\omega}{\beta}\log(\mu^{-1}) \pmod{2\pi}$ fails to converge as $\mu \to 0$. The fact that θ_1 is computed modulo 2π allows us to introduce the parameter a and thereby obtain a two-parameter family $\{F_{a,b}\}$ with a well-defined singular limit.

We regard $p = \log(\mu^{-1})$ as the fundamental parameter associated with $\{\mathcal{F}_\mu\}$. Notice that we now have $p \in [\log(\mu_0^{-1}), \infty)$. Think of $\mu = e^{-p}$ as a function of p. Define $\gamma : (0, \mu_0] \to \mathbb{R}$ by

$$\gamma(\mu) = \frac{\omega}{\beta} \log(\mu^{-1}).$$

Let $N \in \mathbb{N}$ satisfy $\frac{\omega}{\beta} \log(\mu_0^{-1}) < N$. Let (μ_n) be the decreasing sequence of values of μ such that $\gamma(\mu_n) = N + 2\pi(n-1)$ for every $n \in \mathbb{N}$. We think of μ as a measure of dissipation and therefore set $b_n = \mu_n$. For $a \in \mathbb{S}^1$ and $n \in \mathbb{N}$, define

$$\begin{aligned} \mu(n, a) &= \gamma^{-1}(\gamma(\mu_n) + a), \\ p(n, a) &= \log(\mu(n, a)^{-1}) = \log(\mu_n^{-1}) + \frac{\beta}{\omega} a. \end{aligned}$$

The map F_{a,b_n} is defined by $F_{a,b_n} = \mathcal{F}_{p(n,a)}$.

The family $\{F_{a,b_n}\}$ has a well-defined singular limit. As $n \to \infty$, F_{a,b_n} converges in the C^3 topology to the map $F_{a,0}$ defined by

$$\begin{aligned} F_{a,0}^{\langle 1 \rangle}(X_0, \theta_0) &= 0, \\ F_{a,0}^{\langle 2 \rangle}(X_0, \theta_0) &= \theta_0 + \xi_1 + \frac{\omega}{\beta} \log(\varepsilon) + a - \frac{\omega}{\beta} \log\left(\lambda X_0 + B(1 + A\sin(\theta_0))\right). \end{aligned}$$

This proves (C1).

Restricting $F_{a,0}^{\langle 2 \rangle}$ to the circle $\{(X_0, \theta_0) : X_0 = 0\}$, we obtain the one-parameter family of circle maps

$$f_a(\theta) = \theta + \xi_1 + \frac{\omega}{\beta} \log(\varepsilon) + a - \frac{\omega}{\beta} \log\big(B(1 + A\sin(\theta))\big).$$

It follows directly from Proposition 3.3 that f_a is an admissible family of 1-D maps provided $\omega\beta^{-1}$ is sufficiently large. This proves (C2). Hypotheses (C3) and (C4) follow from direct computation.

We have shown that the family $\{F_{a,b_n}\}$ is an admissible family of rank 1 maps and therefore Propositions 3.4 and 3.6 apply. We conclude that if $|\omega|$ is sufficiently large, then there exists a set Δ_ω of positive Lebesgue measure such that for $\mu \in \Delta_\omega$, $\mathcal{F}_\mu$ admits a strange attractor on Σ^- with a unique ergodic SRB measure ν, and Lebesgue almost every point on Σ^- is generic with respect to ν. Furthermore, the set Δ_ω has positive lower Lebesgue density at 0, meaning that

$$\varliminf_{r \to 0^+} \frac{\text{Leb}(\Delta_\omega \cap [0, r])}{r} > 0$$

where $\text{Leb}(\cdot)$ denotes Lebesgue measure.

4 Normal Forms around the Homoclinic Loop

4.1 Outline and Setup

In this section we return to the general setting of Section 2 and introduce a sequence of coordinate changes to transform equation (2.3) into certain normal forms. In Section 4.2 we work in a sufficiently small neighborhood U_ε of $(0,0)$ in the xy-plane. In Section 4.3 we work in a small neighborhood around the entire length of the homoclinic loop ℓ outside of $U_{\varepsilon^2/4}$. In Section 4.4 we define the Poincaré sections $\Sigma^\pm$ that we will use to compute the flow-induced maps. Points on $\Sigma^\pm$ are represented by various sets of variables introduced in Sections 4.2 and 4.3. We discuss the issue of coordinate conversion in Section 4.4.

In the rest of this paper, $\alpha, \beta, \rho \in [\rho_1, \rho_2]$ and $\omega > \omega_0$ (it suffices to assume $\omega > 0$) are all regarded as fixed constants. The size of the neighborhood on which all of the coordinate transformations in Section 4.2 are performed is determined by a small number $\varepsilon > 0$. The quantity ε is also regarded as a fixed constant. We regard μ as the only parameter of equation (2.3).

Two Small Scales

The quantities $\mu \ll \varepsilon \ll 1$ represent two small scales of different magnitude. The quantity ε represents the size of a small neighborhood of $(x, y) = (0,0)$ in which the local analysis of Section 4.2 is valid. Define

$$U_\varepsilon = \{(x, y) : x^2 + y^2 < 4\varepsilon^2\} \quad \text{and} \quad \mathcal{U}_\varepsilon = U_\varepsilon \times \mathbb{S}^1.$$

Let L^+ and $-L^-$ be the respective times at which the homoclinic solution $\ell(t)$ enters $U_{\varepsilon/2}$ in the positive and negative directions. The quantities L^+ and L^- are completely determined by ε and ℓ. The parameter μ ($\mu \ll \varepsilon$) controls the magnitude of the time-periodic forcing.

Notation

Quantities that are independent of phase variables, time, and μ are regarded as constants, and K is used to denote a generic constant, the precise value of which is allowed to change from line to line. On occasion, a specific constant is used in different places. We use subscripts to denote such constants as $K_0, K_1, \ldots$. We will also distinguish between constants that depend on ε and those that do not by making such dependencies explicit. A constant that depends on ε is written as $K(\varepsilon)$. A constant written as K is independent of ε.

4.2 Normal Form near the Stationary Point

In this subsection we study equation (2.3) in a sufficiently small neighborhood of $(0,0)$ in the xy-plane. We introduce a sequence of coordinate changes to transform equation (2.3) into a certain normal form. Table 4.1 summarizes the purpose of each coordinate transformation.

TABLE 4.1. Transformations near the stationary point.

Transformation	Purpose
$(x, y) \to (\xi, \eta)$	linearize the flow defined by (2.1) in a neighborhood of $(0, 0)$
$(\xi, \eta) \to (X, Y)$	standardize the location of the hyperbolic periodic orbit
$(X, Y) \to (\mathbf{X}, \mathbf{Y})$	flatten the local invariant manifolds
$(\mathbf{X}, \mathbf{Y}) \to (\mathbb{X}, \mathbb{Y})$	rescale by the factor $1/\mu$

First Coordinate Change $(x, y) \to (\xi, \eta)$

Let (ξ, η) be such that

$$\xi = x + q_1(x, y), \quad \eta = y + q_2(x, y), \tag{4.1}$$

where $q_1(x, y)$ and $q_2(x, y)$ are analytic terms of order at least 2 in x and y. Formula (4.1) defines a near-identity coordinate transformation $(x, y) \to (\xi, \eta)$, the inverse of which we write as

$$x = \xi + Q_1(\xi, \eta), \quad y = \eta + Q_2(\xi, \eta). \tag{4.2}$$

PROPOSITION 4.1. *Assume α and β satisfy the nonresonance condition* (H1)(a). *Then there exists a neighborhood U of* $(0, 0)$*, the size of which is completely determined by equation* (2.1) *and d_1 and d_2 in* (H1)(a)*, such that on U there exists an analytic coordinate transformation* (4.1) *that transforms equation* (2.1) *into the linear system*

$$\frac{d\xi}{dt} = -\alpha\xi, \quad \frac{d\eta}{dt} = \beta\eta.$$

PROOF. See [15]. □

We now use the coordinate transformation of Proposition 4.1 to transform equation (2.3). Observe that by definition $q_1(x, y)$ and $q_2(x, y)$ satisfy

$$(1 + \partial_x q_1(x, y))(-\alpha x + f(x, y)) + \partial_y q_1(x, y)(\beta y + g(x, y)) = -\alpha\xi, \tag{4.3a}$$

$$(1 + \partial_y q_2(x, y))(\beta y + g(x, y)) + \partial_x q_2(x, y)(-\alpha x + f(x, y)) = \beta\eta. \tag{4.3b}$$

We derive the form of (2.3) in terms of ξ and η. We have

$$\begin{aligned}\frac{d\xi}{dt} &= (1 + \partial_x q_1(x, y))\big(-\alpha x + f(x, y) - \mu(\rho h(x, y) + \sin(\theta))\big)\\ &\quad + \partial_y q_1(x, y)\big(\beta y + g(x, y) + \mu(\rho h(x, y) + \sin(\theta))\big)\\ &= -\alpha\xi - \mu(1 + \partial_x q_1(x, y) - \partial_y q_1(x, y))(\rho h(x, y) + \sin(\theta))\end{aligned}$$

where (4.3a) is used for the second equality. Similarly, we have

$$\begin{aligned}\frac{d\eta}{dt} &= (1 + \partial_y q_2(x, y))\big(\beta y + g(x, y) + \mu(\rho h(x, y) + \sin(\theta))\big)\\ &\quad + \partial_x q_2(x, y)\big(-\alpha x + f(x, y) - \mu(\rho h(x, y) + \sin(\theta))\big)\\ &= \beta\eta + \mu(1 + \partial_y q_2(x, y) - \partial_x q_2(x, y))(\rho h(x, y) + \sin(\theta)).\end{aligned}$$

With the functions of x and y written as functions of ξ and η by using (4.2), the form of (2.3) in terms of ξ and η is given by

$$\begin{cases} \frac{d\xi}{dt} = -\alpha\xi - \mu(1 + h_1(\xi,\eta))(\rho H(\xi,\eta) + \sin(\theta)), \\ \frac{d\eta}{dt} = \beta\eta + \mu(1 + h_2(\xi,\eta))(\rho H(\xi,\eta) + \sin(\theta)), \\ \frac{d\theta}{dt} = \omega, \end{cases} \tag{4.4}$$

where $h_1(\xi,\eta) = \partial_x q_1(x,y) - \partial_y q_1(x,y)$ and $h_2(\xi,\eta) = \partial_y q_2(x,y) - \partial_x q_2(x,y)$ are such that $h_1(0,0) = h_2(0,0) = 0$ and $H(\xi,\eta) = h(x,y)$.

Second Coordinate Change: $(\xi,\eta) \to (X,Y)$

With the forcing added, the hyperbolic stationary point $(x,y) = (0,0)$ of equation (2.1) becomes a hyperbolic periodic solution of (2.3) with period $2\pi\omega^{-1}$. We denote this periodic solution in (ξ,η,θ)-coordinates as $\xi = \mu\phi(\theta;\mu)$, $\eta = \mu\psi(\theta;\mu)$.

PROPOSITION 4.2. *For equation* (4.4)*, there exists a unique solution of the form*

$$\xi = \mu\phi(\theta;\mu), \quad \eta = \mu\psi(\theta;\mu), \quad \theta = \omega t,$$

satisfying

$$\phi(\theta;\mu) = \phi(\theta + 2\pi;\mu), \quad \psi(\theta;\mu) = \psi(\theta + 2\pi;\mu).$$

The C^3 norms of the functions $\phi(\theta;\mu)$ and $\psi(\theta;\mu)$, regarded as functions of θ and μ, are bounded by a constant K.

PROOF. Write $\phi = \phi(\theta;\mu)$ and $\psi = \psi(\theta;\mu)$. The functions ϕ and ψ should satisfy

$$\begin{aligned} \omega\frac{d\phi}{d\theta} &= -\alpha\phi - (1 + h_1(\mu\phi,\mu\psi))(\rho H(\mu\phi,\mu\psi) + \sin(\theta)), \\ \omega\frac{d\psi}{d\theta} &= \beta\psi + (1 + h_2(\mu\phi,\mu\psi))(\rho H(\mu\phi,\mu\psi) + \sin(\theta)). \end{aligned} \tag{4.5}$$

From (4.5) it follows that

$$\begin{aligned} \phi(\theta;\mu) = {} & e^{-\alpha\omega^{-1}(\theta-\theta_0)}\phi(\theta_0;\mu) \\ & - \omega^{-1}\int_{\theta_0}^{\theta} e^{\alpha\omega^{-1}(s-\theta)}[1 + h_1(\mu\phi(s;\mu),\mu\psi(s;\mu))] \\ & \qquad \times [\rho H(\mu\phi(s;\mu),\mu\psi(s;\mu)) + \sin(s)]ds, \\ \psi(\theta;\mu) = {} & e^{\beta\omega^{-1}(\theta-\theta_0)}\psi(\theta_0;\mu) \\ & + \omega^{-1}\int_{\theta_0}^{\theta} e^{-\beta\omega^{-1}(s-\theta)}[1 + h_2(\mu\phi(s;\mu),\mu\psi(s;\mu))] \\ & \qquad \times [\rho H(\mu\phi(s;\mu),\mu\psi(s;\mu)) + \sin(s)]ds. \end{aligned}$$

To solve for ϕ and ψ we let $\theta = \theta_0 + 2\pi$ and set $\phi(\theta_0 + 2\pi;\mu) = \phi(\theta_0;\mu)$ and $\psi(\theta_0 + 2\pi;\mu) = \psi(\theta_0;\mu)$, obtaining

$$\begin{aligned}\phi(\theta;\mu) &= \frac{-\omega^{-1}}{1-e^{-2\alpha\omega^{-1}\pi}}\int_0^{2\pi} e^{\alpha\omega^{-1}(s-2\pi)}[1+h_1(\mu\phi(s+\theta;\mu),\mu\psi(s+\theta;\mu))] \\ &\qquad \times [\rho H(\mu\phi(s+\theta;\mu),\mu\psi(s+\theta;\mu)) + \sin(s+\theta)]ds \\ \psi(\theta;\mu) &= \frac{\omega^{-1}}{1-e^{2\beta\omega^{-1}\pi}}\int_0^{2\pi} e^{-\beta\omega^{-1}(s-2\pi)}[1+h_2(\mu\phi(s+\theta;\mu),\mu\psi(s+\theta;\mu))] \\ &\qquad \times [\rho H(\mu\phi(s+\theta;\mu),\mu\psi(s+\theta;\mu)) + \sin(s+\theta)]ds.\end{aligned} \tag{4.6}$$

The existence and uniqueness of $\phi(\theta;\mu)$ and $\psi(\theta;\mu)$ follows directly from applying the contraction mapping theorem to (4.6). The asserted bound on partial derivatives with respect to θ and μ follows from differentiating (4.6) with respect to θ and μ. □

We now introduce new variables (X, Y) by defining

$$X = \xi - \mu\phi(\theta;\mu), \quad Y = \eta - \mu\psi(\theta;\mu). \tag{4.7}$$

We have

$$\begin{aligned}\frac{dX}{dt} &= -\alpha X - \alpha\mu\phi - \mu\omega\frac{d\phi}{d\theta} \\ &\quad - \mu(1+h_1(X+\mu\phi, Y+\mu\psi))(\rho H(X+\mu\phi, Y+\mu\psi) + \sin(\theta)) \\ \frac{dY}{dt} &= \beta Y + \beta\mu\psi - \mu\omega\frac{d\psi}{d\theta} \\ &\quad + \mu(1+h_2(X+\mu\phi, Y+\mu\psi))(\rho H(X+\mu\phi, Y+\mu\psi) + \sin(\theta)).\end{aligned}$$

Using (4.5), the form of (2.3) in terms of X, Y, and θ is given by

$$\begin{cases}\frac{dX}{dt} = -\alpha X + \mu F(X, Y, \theta;\mu), \\ \frac{dY}{dt} = \beta Y + \mu G(X, Y, \theta;\mu), \\ \frac{d\theta}{dt} = \omega,\end{cases} \tag{4.8}$$

where

$$\begin{aligned}F(X,Y,\theta;\mu) &= -[h_1(X+\mu\phi, Y+\mu\psi) - h_1(\mu\phi,\mu\psi)] \\ &\qquad \times (\rho H(X+\mu\phi, Y+\mu\psi) + \sin(\theta)) \\ &\quad - \rho(1+h_1(\mu\phi,\mu\psi))(H(X+\mu\phi, Y+\mu\psi) - H(\mu\phi,\mu\psi)) \\ G(X,Y,\theta;\mu) &= [h_2(X+\mu\phi, Y+\mu\psi) - h_2(\mu\phi,\mu\psi)] \\ &\quad \times (\rho H(X+\mu\phi, Y+\mu\psi) + \sin(\theta)) \\ &\quad + \rho(1+h_2(\mu\phi,\mu\psi))(H(X+\mu\phi, Y+\mu\psi) - H(\mu\phi,\mu\psi))\end{aligned}$$

are such that $F(0,0,\theta;\mu) = G(0,0,\theta;\mu) = 0$. Observe that in the new coordinates (X, Y, θ), the solution $\xi = \mu\phi(\theta;\mu)$, $\eta = \mu\psi(\theta;\mu)$ is represented by

$X = Y = 0$. We remark that on

$$\{(X, Y, \theta; \mu) : \|(X, Y)\| < \varepsilon,\ \theta \in \mathbb{S}^1,\ 0 \le \mu \le \mu_0\},$$

- $F(X, Y, \theta; \mu)$ and $G(X, Y, \theta; \mu)$ are analytic functions bounded by $K\varepsilon$, and
- it follows from Proposition 4.2 that the C^3 norms of both F and G as functions of (X, Y, θ) and μ are bounded by a constant K.

Third Coordinate Change: $(X, Y) \to (\mathbf{X}, \mathbf{Y})$

The periodic solution $(X, Y, \theta) = (0, 0, \omega t)$ of equation (4.8) has a local unstable manifold, which we write as

$$X = \mu W^u(Y, \theta; \mu),$$

and a local stable manifold, which we write as

$$Y = \mu W^s(X, \theta; \mu).$$

PROPOSITION 4.3. *There exists $\varepsilon > 0$ and $\mu_0 = \mu_0(\varepsilon) > 0$ such that $W^u(Y, \theta; \mu)$ and $W^s(X, \theta; \mu)$ are analytically defined on*

$$(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times [0, \mu_0]$$

and satisfy

$$W^u(0, \theta; \mu) = 0, \quad W^s(0, \theta; \mu) = 0.$$

The C^3 norms of $W^u(Y, \theta; \mu)$ and $W^s(X, \theta; \mu)$, regarded as functions of all three of their arguments, are bounded by a constant K.

PROOF. We regard X, Y, θ, and μ in equation (4.8) as complex variables. The existence and smoothness of local stable and unstable manifolds follows from a standard argument based on the contraction mapping theorem. See [15], for instance. □

By definition, $W^u(Y, \theta; \mu)$ satisfies

$$(4.9) \quad -\alpha W^u(Y, \theta; \mu) + F(\mu W^u(Y, \theta; \mu), Y, \theta; \mu) = \omega \partial_\theta W^u(Y, \theta; \mu) + \partial_Y W^u(Y, \theta; \mu)(\beta Y + \mu G(\mu W^u(Y, \theta; \mu), Y, \theta; \mu)).$$

Similarly, $W^s(X, \theta; \mu)$ satisfies

$$(4.10) \quad \beta W^s(X, \theta; \mu) + G(X, \mu W^s(X, \theta; \mu), \theta; \mu) = \omega \partial_\theta W^s(X, \theta; \mu) + \partial_X W^s(X, \theta; \mu)(-\alpha X + \mu F(X, \mu W^s(X, \theta; \mu), \theta; \mu)).$$

Define the new variables $\mathbf{X}$ and $\mathbf{Y}$ by

$$(4.11) \quad \mathbf{X} = X - \mu W^u(Y, \theta; \mu), \quad \mathbf{Y} = Y - \mu W^s(X, \theta; \mu).$$

By using (4.8), (4.9), and (4.10), the form of (2.3) in terms of $(\mathbf{X}, \mathbf{Y}, \theta)$ is given by

$$\begin{cases} \frac{d\mathbf{X}}{dt} = (-\alpha + \mu \mathbf{F}(\mathbf{X}, \mathbf{Y}, \theta; \mu))\mathbf{X}, \\ \frac{d\mathbf{Y}}{dt} = (\beta + \mu \mathbf{G}(\mathbf{X}, \mathbf{Y}, \theta; \mu))\mathbf{Y}, \\ \frac{d\theta}{dt} = \omega. \end{cases} \tag{4.12}$$

where $\mathbf{F}$ and $\mathbf{G}$ are analytic functions of $\mathbf{X}$, $\mathbf{Y}$, θ, and μ defined on $U_\varepsilon \times \mathbb{S}^1 \times [0, \mu_0]$. The C^3 norms of $\mathbf{F}$ and $\mathbf{G}$ are bounded by a constant K. Tracing back to the variables (ξ, η), we have

$$\mathbf{X} = \xi - \mu(\phi(\theta; \mu) + W^u(\eta - \mu\psi(\theta; \mu), \theta; \mu)), \tag{4.13a}$$

$$\mathbf{Y} = \eta - \mu(\psi(\theta; \mu) + W^s(\xi - \mu\phi(\theta; \mu), \theta; \mu)). \tag{4.13b}$$

Fourth Coordinate Change: $(\mathbf{X}, \mathbf{Y}) \to (\mathbb{X}, \mathbb{Y})$

The final coordinate change is a rescaling of $\mathbf{X}$ and $\mathbf{Y}$ by the factor μ^{-1}. Let

$$\mathbb{X} = \mu^{-1}\mathbf{X}, \quad \mathbb{Y} = \mu^{-1}\mathbf{Y}. \tag{4.14}$$

We write equation (4.12) in $\mathbb{X}$ and $\mathbb{Y}$ as

$$\begin{cases} \frac{d\mathbb{X}}{dt} = (-\alpha + \mu \mathbb{F}(\mathbb{X}, \mathbb{Y}, \theta; \mu))\mathbb{X}, \\ \frac{d\mathbb{Y}}{dt} = (\beta + \mu \mathbb{G}(\mathbb{X}, \mathbb{Y}, \theta; \mu))\mathbb{Y}, \\ \frac{d\theta}{dt} = \omega, \end{cases} \tag{4.15}$$

where

$$\mathbb{F}(\mathbb{X}, \mathbb{Y}, \theta; \mu) = \mathbf{F}(\mu\mathbb{X}, \mu\mathbb{Y}, \theta; \mu), \quad \mathbb{G}(\mathbb{X}, \mathbb{Y}, \theta; \mu) = \mathbf{G}(\mu\mathbb{X}, \mu\mathbb{Y}, \theta; \mu),$$

are analytic functions of $\mathbb{X}$, $\mathbb{Y}$, θ, and μ defined on

$$\mathbb{D} = \{(\mathbb{X}, \mathbb{Y}, \theta, \mu) : \mu \in [0, \mu_0],\ (\mathbb{X}, \mathbb{Y}, \theta) \in \mathcal{U}_\varepsilon\}$$

where

$$\mathcal{U}_\varepsilon = \{(\mathbb{X}, \mathbb{Y}, \theta) : \|(\mathbb{X}, \mathbb{Y})\| < 2\varepsilon\mu^{-1},\ \theta \in \mathbb{S}^1\}.$$

Remark 4.4. We remind the reader that all constants represented by K in Section 4.2 are independent of ε and μ.

4.3 A Normal Form around the Homoclinic Loop

In this subsection we derive a normal form for equation (2.3) around the homoclinic loop of equation (2.1) outside of $\mathcal{U}_{\varepsilon^2/4}$. Some elementary estimates are also included.

Derivation of the Normal Form

Let us regard t in $\ell(t) = (a(t), b(t))$ not as time, but rather as a parameter that parametrizes the curve ℓ in (x, y)-space. We replace t by s and write the homoclinic loop as $\ell(s) = (a(s), b(s))$. We have

$$\frac{da(s)}{ds} = -\alpha a(s) + f(a(s), b(s)), \quad \frac{db(s)}{ds} = \beta b(s) + g(a(s), b(s)). \tag{4.16}$$

Define

$$(u(s), v(s)) = \left\| \frac{d}{ds}\,\ell(s) \right\|^{-1} \frac{d}{ds}\,\ell(s).$$

We have

$$\begin{aligned} u(s) &= \frac{-\alpha a(s) + f(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}, \\ v(s) &= \frac{\beta b(s) + g(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}. \end{aligned} \tag{4.17}$$

Let

$$\boldsymbol{e}(s) = (v(s), -u(s)).$$

The vector $\boldsymbol{e}(s)$ is the inward unit normal vector to ℓ at $\ell(s)$. We now introduce a new variable z such that

$$(x, y) = \ell(s) + z\boldsymbol{e}(s).$$

That is,

$$x = x(s, z) = a(s) + v(s)z, \quad y = y(s, z) = b(s) - u(s)z. \tag{4.18}$$

We derive the form of (2.3) in terms of the new variables (s, z) defined by (4.18). Differentiating (4.18), we obtain

$$\begin{aligned} \frac{dx}{dt} &= (-\alpha a(s) + f(a(s), b(s)) + v'(s)z)\frac{ds}{dt} + v(s)\frac{dz}{dt}, \\ \frac{dy}{dt} &= (\beta b(s) + g(a(s), b(s)) - u'(s)z)\frac{ds}{dt} - u(s)\frac{dz}{dt}, \end{aligned} \tag{4.19}$$

where $u'(s) = \frac{du(s)}{ds}$ and $v'(s) = \frac{dv(s)}{ds}$. Denote

$$\begin{aligned} F(s, z) &= -\alpha(a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)), \\ G(s, z) &= \beta(b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)), \\ \mathbb{H}(s, z) &= h(a(s) + zv(s), b(s) - zu(s)). \end{aligned}$$

Using (2.3) and (4.19), we have

$$\begin{aligned} \frac{ds}{dt} &= \frac{v(s)G(s, z) + u(s)F(s, z) + \mu(v(s) - u(s))(\rho\mathbb{H}(s, z) + \sin(\theta))}{\sqrt{F(s, 0)^2 + G(s, 0)^2} + z(u(s)v'(s) - v(s)u'(s))}, \\ \frac{dz}{dt} &= v(s)F(s, z) - u(s)G(s, z) - \mu(u(s) + v(s))(\rho\mathbb{H}(s, z) + \sin(\theta)). \end{aligned}$$

We rewrite these equations as

$$
\begin{aligned}
\frac{ds}{dt} &= 1 + zw_1(s,z,\theta;\mu) + \frac{\mu(v(s)-u(s))(\rho\mathbb{H}(s,0)+\sin(\theta))}{\sqrt{F(s,0)^2+G(s,0)^2}},\\
\frac{dz}{dt} &= E(s)z + z^2 w_2(s,z) - \mu(u(s)+v(s))(\rho\mathbb{H}(s,z)+\sin(\theta)),\\
\frac{d\theta}{dt} &= \omega,
\end{aligned}
\tag{4.20}
$$

where

$$
\begin{aligned}
E(s) &= v^2(s)(-\alpha + \partial_x f(a(s),b(s))) + u^2(s)\big(\beta + \partial_y g(a(s),b(s))\big)\\
&\quad - u(s)v(s)\big(\partial_y f(a(s),b(s)) + \partial_x g(a(s),b(s))\big),\\
\mathbb{H}(s,0) &= h(a(s),b(s)).
\end{aligned}
$$

Equation (4.20) is defined on

$$\{s \in [-2L^-, 2L^+],\ \mu \in [0,\mu_0],\ \theta \in \mathbb{S}^1,\ |z| < K_0(\varepsilon)\mu\},$$

where $K_0(\varepsilon)$ is independent of μ. The C^3 norms of the functions $w_1(s,z,\theta;\mu)$ and $w_2(s,z)$ are bounded by a constant $K(\varepsilon)$.

Finally, we rescale the variable z by letting

$$Z = \mu^{-1}z. \tag{4.21}$$

We arrive at the equations

$$\frac{ds}{dt} = 1 + \mu\widetilde{w}_1(s,Z,\theta;\mu), \tag{4.22a}$$

$$\frac{dZ}{dt} = E(s)Z + \mu\widetilde{w}_2(s,Z,\theta;\mu) - (u(s)+v(s))(\rho\mathbb{H}(s,0)+\sin(\theta)), \tag{4.22b}$$

$$\frac{d\theta}{dt} = \omega, \tag{4.22c}$$

defined on

$$\mathbf{D} = \{(s,Z,\theta;\mu) : s \in [-2L^-, 2L^+],\ |Z| \le K_0(\varepsilon),\ \theta \in \mathbb{S}^1,\ \mu \in [0,\mu_0]\}.$$

We assume that μ_0 is sufficiently small so that

$$\mu \ll \min_{s\in[-2L^-,2L^+]} (F(s,0)^2 + G(s,0)^2).$$

The C^3 norms of the functions $\widetilde{w}_1$ and $\widetilde{w}_2$ are bounded by a constant $K(\varepsilon)$ on $\mathbf{D}$.

System (4.22a)–(4.22c) is the one we need. The function $E(s)$ appears in the integrals A, C, and S in (H2).

Remark 4.5. Observe that all of the generic constants that have appeared thus far in this subsection have the form $K(\varepsilon)$.

Technical Estimates

We adopt the following conventions in comparing the magnitude of two functions $f(s)$ and $g(s)$. We write $f(s) \prec g(s)$ if there exists $K > 0$ independent of s such that $|f(s)| < K|g(s)|$ as $s \to \infty$ (or $-\infty$). We write $f(s) \sim g(s)$ if in addition we have $|f(s)| > K^{-1}|g(s)|$ as $s \to \infty$ (or $-\infty$). We write $f(s) \approx g(s)$ if

$$\frac{f(s)}{g(s)} \to 1$$

as $s \to \infty$ (or $-\infty$).

Recall that $\ell(s) = (a(s), b(s))$ is the homoclinic solution for the hyperbolic stationary point $(0, 0)$ of equation (2.1). The vector $(u(s), v(s))$ is the unit tangent vector to ℓ at $\ell(s)$.

LEMMA 4.6. *As $s \to +\infty$, we have*

(1) $a(s) \sim e^{-\alpha s}$, $a(-s) \prec e^{-2\beta s}$,
(2) $b(s) \prec e^{-2\alpha s}$, $b(-s) \sim e^{-\beta s}$,
(3) $u(s) \approx -1$, $u(-s) \prec e^{-\beta s}$,
(4) $v(s) \prec e^{-\alpha s}$, $v(-s) \approx 1$.

PROOF. We are simply restating the fact that $\ell(s) \to (0, 0)$ with an exponential rate $-\alpha$ in the positive s-direction along the x-axis and with an exponential rate β in the negative s-direction along the y-axis. □

LEMMA 4.7. *Let $E(s)$ be as in* (2.4). *As $L^{\pm} \to +\infty$, we have*

(1) $\int_{-L^-}^{0} (E(s) + \alpha)ds \prec 1$,
(2) $\int_0^{L^+} (E(s) - \beta)ds \prec 1$,
(3) $\int_{-L^-}^{0} E(s)ds \approx -\alpha L^-$,
(4) $\int_0^{L^+} E(s)ds \approx \beta L^+$.

PROOF. Statements (1) and (2) claim that the integrals are convergent as $L^{\pm} \to \infty$. For (1), we observe that by adding α to $E(s)$, we can write $E(s) + \alpha$ as a collection of terms such that each term decays exponentially as $s \to -\infty$ by Lemma 4.6. Similarly, taking β away from $E(s)$, we can write $E(s) - \beta$ as a collection of terms such that each term decays exponentially as $s \to \infty$.

For (3) and (4) we write

$$\int_{-L^-}^{0} E(s)ds = -\alpha L^- + \int_{-L^-}^{0} (E(s) + \alpha)ds,$$
$$\int_0^{L^+} E(s)ds = \beta L^+ + \int_0^{L^+} (E(s) - \beta)ds.$$

Statements (3) and (4) now follow from (1) and (2), respectively. □

LEMMA 4.8. *All of the integrals defined in* (2.5) *are absolutely convergent.*

PROOF. Let us write

$$\begin{aligned} A = &\int_{-\infty}^{-L_0} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau)d\tau}\, ds \\ &+ \int_{-L_0}^{L_0} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau)d\tau}\, ds \\ &+ \int_{L_0}^{\infty} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau)d\tau}\, ds. \end{aligned}$$

We write the first integral as

$$\int_{-\infty}^{-L_0} (u(s) + v(s))h(a(s), b(s))e^{\alpha s} e^{-\int_0^s (E(\tau)+\alpha)d\tau}\, ds$$

and make L_0 sufficiently large so that $|E(\tau) + \alpha| < \alpha/2$ for all $\tau \in (-\infty, -L_0)$. This integral is convergent since the integrand is $< Ke^{\alpha s/2}$ for all $s \in (-\infty, -L_0)$. For the convergence of the third integral, we rewrite it as

$$\int_{L_0}^{\infty} (u(s) + v(s))h(a(s), b(s))e^{-\beta s} e^{-\int_0^s (E(\tau)-\beta)d\tau}\, ds$$

and observe that $|E(\tau)-\beta| < \beta/2$ for $\tau \in [L_0, \infty)$ provided that L_0 is sufficiently large. The proofs for C and S are similar. □

4.4 Poincaré Sections and Conversion of Coordinates

We introduce the Poincaré sections $\Sigma^{\pm}$. Since various sets of phase variables have appeared in Sections 4.2 and 4.3, we also need to know how to explicitly convert coordinates from one set to another on $\Sigma^{\pm}$.

The Poincaré Sections $\Sigma^{\pm}$

Recall that $\{\ell(s) : s \in (-\infty, \infty)\}$ is the homoclinic loop of equation (2.1). Given $\varepsilon > 0$ sufficiently small, let L^+ and $-L^-$ be such that

$$\begin{aligned} \xi(-L^-) &= a(-L^-) + q_1(a(-L^-), b(-L^-)) = 0, \\ \eta(-L^-) &= b(-L^-) + q_2(a(-L^-), b(-L^-)) = \varepsilon, \\ \xi(L^+) &= a(L^+) + q_1(a(L^+), b(L^+)) = \varepsilon, \\ \eta(L^+) &= b(L^+) + q_2(a(L^+), b(L^+)) = 0, \end{aligned} \tag{4.23}$$

where ξ and η are the variables defined through (4.1). Let

$$\hat{K}_0 = \max_{\substack{\theta \in \mathbb{S}^1 \\ \mu \in [0, \mu_0]}} \{|\phi(\theta; \mu)|, |\psi(\theta; \mu)|\}$$

where $\phi(\theta;\mu)$ and $\psi(\theta;\mu)$ are as in Section 4.2. We define two sections in $\mathcal{U}_\varepsilon$, denoted Σ^- and Σ^+, as follows:

$$\Sigma^- = \{(x, y, \theta) : s = -L^-,\ |z| \le (\hat{K}_0 + 1)\mu,\ \theta \in \mathbb{S}^1\}$$

$$\Sigma^+ = \big\{(x, y, \theta) : s = L^+,\ \tfrac{1}{10}(-\rho A)(\hat{K}_0 + 1)e^{(1/2)\beta L^+}\mu \le z \le 10(-\rho A)(\hat{K}_0 + 1)e^{2\beta L^+}\mu,\ \theta \in \mathbb{S}^1\big\}, \tag{4.24}$$

where s and z are as in (4.18). We construct the flow-induced map $\mathcal{F}_\mu$ in two steps.

(1) Starting from Σ^-, the solutions of equation (2.3) move out of $\mathcal{U}_\varepsilon$, following the homoclinic loop of equation (2.1) to eventually hit Σ^+. This defines a flow-induced map from Σ^- to Σ^+, which we denote as $\mathcal{M} : \Sigma^- \to \Sigma^+$. We will prove that $\mathcal{M}(\Sigma^-) \subset \Sigma^+$.

(2) Starting from Σ^+, the solutions of equation (2.3) stay inside of $\mathcal{U}_\varepsilon$, carrying Σ^+ into Σ^-. This map we denote as $\mathcal{N}$.

We define $\mathcal{F}_\mu = \mathcal{N} \circ \mathcal{M}$. Observe that the variables (s, Z, θ) of Section 4.3 are suitable for computing $\mathcal{M}$, and $(\mathbb{X}, \mathbb{Y}, \theta)$ are suitable for computing $\mathcal{N}$. To properly compose $\mathcal{N}$ and $\mathcal{M}$, we need to know how to convert from (s, Z, θ) to $(\mathbb{X}, \mathbb{Y}, \theta)$ on $\Sigma^\pm$ and vice versa.

The New Parameter *p*

As stated earlier, we regard μ as the only parameter of system (2.3). We make a coordinate change on this parameter by letting $p = \ln \mu$, and we regard p, not μ, as our bottom-line parameter. In other words, we regard μ as a shorthand for e^p, and all functions of μ are thought of as functions of p. Observe that $\mu \in (0, \mu_0]$ corresponds to $p \in (-\infty, \log(\mu_0)]$. This is a *very important conceptual point* because by regarding a function $F(\mu)$ of μ as a function of p, we have

$$\partial_p F(\mu) = \mu \partial_\mu F(\mu).$$

Therefore, thinking of $F(\mu)$ as a function of p produces a C^3 norm that is completely different from the one obtained by thinking of $F(\mu)$ as a function of μ.

Notation

In order to apply the theory of rank 1 maps [48, 52, 53], we need to control the C^3 norm of $\mathcal{F}_\mu$. In particular, we must estimate the C^3 norms of certain quantities with respect to various sets of variables on relevant domains. The derivation of the flow-induced maps $\{\mathcal{F}_\mu\}$ involves a composition of maps and multiple coordinate changes. To facilitate the presentation, from this point on we adopt specific conventions for indicating controls on magnitude. For a given constant, we write $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$, or $\mathcal{O}(\mu)$ to indicate that the magnitude of the constant is bounded by K, $K\varepsilon$, or $K(\varepsilon)\mu$, respectively. For a function of a set V of variables on a specific domain, we write $\mathcal{O}_V(1)$, $\mathcal{O}_V(\varepsilon)$, or $\mathcal{O}_V(\mu)$ to indicate that the C^3 norm of the function on the specified domain is bounded by K, $K\varepsilon$, or $K(\varepsilon)\mu$, respectively. We choose to specify the domain in the surrounding text rather than explicitly involving it in the notation. For example, $\mathcal{O}_{\mathbb{X}_0,\mathbb{Y}_0,\theta,\mu}(\varepsilon)$ represents a function of $\mathbb{X}_0$, $\mathbb{Y}_0$, θ, and μ, the C^3 norm of which is bounded above by $K\varepsilon$ on a domain explicitly given in the surrounding text. Similarly, $\mathcal{O}_{Z,\theta,p}(\mu)$ represents a function of Z, θ, and p, the C^3 norm of which is bounded above by $K(\varepsilon)\mu$.

Conversion on $\boldsymbol{\Sigma^-}$

The section Σ^- is defined by $s = -L^-$. A point $q \in \Sigma^-$ is uniquely determined by a pair (Z, θ). First we compute the coordinates $\mathbb{X}$ and $\mathbb{Y}$ for a point given in (Z, θ)-coordinates on Σ^-. Recall that $p = \ln \mu$.

PROPOSITION 4.9. *For $\mu \in (0, \mu_0]$ and $(Z, \theta) \in \Sigma^-$, we have*

$$\mathbb{X} = (1 + \mathcal{O}_{\theta,p}(\varepsilon) + \mu\mathcal{O}_{Z,\theta,p}(1))Z - \mathcal{O}_{\theta,p}(1),$$
$$\mathbb{Y} = \mu^{-1}\varepsilon + \mathcal{O}_{Z,\theta,p}(1).$$

PROOF. By definition, $s = -L^-$ on Σ^-. Let $q \in \Sigma^-$ be represented by (z, θ). Using (4.23), we have

$$\begin{aligned} a(-L^-) &= Q_1(0, \varepsilon) = \mathcal{O}(\varepsilon^2), \\ b(-L^-) &= \varepsilon + Q_2(0, \varepsilon) = \varepsilon + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{4.25}$$

We also have

$$u(-L^-) = \mathcal{O}(\varepsilon), \quad v(-L^-) = 1 - \mathcal{O}(\varepsilon). \tag{4.26}$$

We compute values of $\mathbf{X}$ and $\mathbf{Y}$ for q. Using (4.23) and (4.25),

$$\begin{aligned} \xi &= a(-L^-) + v(-L^-)z + q_1\big(a(-L^-) + v(-L^-)z,\ b(-L^-) - u(-L^-)z\big) \\ &= v(-L^-)z + q_1\big(a(-L^-) + v(-L^-)z,\ b(-L^-) - u(-L^-)z\big) \\ &\quad - q_1(a(-L^-), b(-L^-)) \\ &= (1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \eta &= b(-L^-) - u(-L^-)z + q_2\big(a(-L^-) + v(-L^-)z,\ b(-L^-) - u(-L^-)z\big) \\ &= \varepsilon - u(-L^-)z + q_2\big(a(-L^-) + v(-L^-)z,\ b(-L^-) - u(-L^-)z\big) \\ &\quad - q_2(a(-L^-), b(-L^-)) \\ &= \varepsilon + (\mathcal{O}(\varepsilon) + zh_\eta(z))z. \end{aligned}$$

The functions h_ξ and h_η are analytic on $|z| < (\hat{K}_0 + 1)\mu$, and we have $h_\xi(z) = \mathcal{O}_z(1)$ and $h_\eta(z) = \mathcal{O}_z(1)$. Substituting ξ and η above into (4.13a), we obtain

$$\begin{aligned} \mathbf{X} &= (1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z - \mu\phi(\theta; \mu) \\ &\quad - \mu W^u(\varepsilon - \mu\psi(\theta; \mu) + (\mathcal{O}(\varepsilon) + zh_\eta(z))z, \theta; \mu) \\ &= (1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z - \mu\phi(\theta; \mu) - \mu W^u(\varepsilon - \mu\psi(\theta; \mu), \theta; \mu) \\ &\quad - \mu W^u(\varepsilon - \mu\psi(\theta; \mu) + (\mathcal{O}(\varepsilon) + zh_\eta(z))z, \theta; \mu) \\ &\quad + \mu W^u(\varepsilon - \mu\psi(\theta; \mu), \theta; \mu). \end{aligned}$$

This implies

$$\mathbf{X} = (1 + \mathcal{O}_{\theta,\mu}(\varepsilon) + z\widehat{h}(z, \theta; \mu))z - \mu\mathcal{O}_{\theta,\mu}(1) \tag{4.27}$$

where $\widehat{h}(z, \theta; \mu)$ is analytic in z, θ, and μ and satisfies $\widehat{h} = \mathcal{O}_{z,\theta,\mu}(1)$. Now substitute

$$\mathbf{X} = \mu\mathbb{X}, \quad z = \mu Z,$$

into (4.27) and note that $|Z| < \widehat{K}_0 + 1$. We obtain the claimed formula for $\mathbb{X}$.

For the $\mathbb{Y}$-component, we substitute ξ and η above into (4.13b) to obtain

$$\begin{aligned}\mathbf{Y} = \varepsilon &+ (\mathcal{O}(\varepsilon) + zh_\eta(z))z - \mu\psi(\theta; \mu)\\ &- \mu W^s((1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z - \mu\phi(\theta; \mu), \theta; \mu).\end{aligned}$$

Set $\mathbf{Y} = \mu\mathbb{Y}$ and $z = \mu Z$ and note that $|Z| < \widehat{K}_0 + 1$. We obtain the claimed formula for $\mathbb{Y}$. □

COROLLARY 4.10. *On Σ^-, we have*

$$Z = (1 + \mathcal{O}_{\theta,p}(\varepsilon) + \mu\mathcal{O}_{\mathbb{X},\theta,p}(1))(\mathbb{X} + \mathcal{O}_{\theta,p}(1)).$$

PROOF. We start with (4.27). This equality is invertible and we have

$$z = (1 + \mathcal{O}_{\theta,\mu}(\varepsilon) + \mathbf{W}\widetilde{h}(\mathbf{W}, \theta; \mu))\mathbf{W} \tag{4.28}$$

where

$$\mathbf{W} = \mathbf{X} + \mu\mathcal{O}_{\theta,\mu}(1)$$

and $\widetilde{h}(\mathbf{W}, \theta; \mu)$ is analytic in $\mathbf{W}$, θ, and μ and satisfies $\widetilde{h} = \mathcal{O}_{\mathbf{W},\theta,\mu}(1)$. Writing (4.28) in terms of Z and $\mathbb{X}$, we have

$$Z = (1 + \mathcal{O}_{\theta,p}(\varepsilon) + \mu\mathcal{O}_{\mathbb{X},\theta,p}(1))(\mathbb{X} + \mathcal{O}_{\theta,p}(1)).$$

□

COROLLARY 4.11. *On Σ^-, we have*

$$\mathbb{Y} = \mu^{-1}\varepsilon + \mathcal{O}_{\mathbb{X},\theta,p}(1).$$

PROOF. We first regard $\mathbb{Y}$ as a function of Z, θ, and p using the formula for $\mathbb{Y}$ in Proposition 4.9 and then regard Z as a function of $\mathbb{X}$, θ, and p using Corollary 4.10. □

Remark 4.12. Terms of the form $\mu\mathcal{O}_{\mathbb{X},\theta,p}(1)$ are not equivalent to terms of the form $\mathcal{O}_{\mathbb{X},\theta,p}(\mu)$. A term of the form $\mu\mathcal{O}_{\mathbb{X},\theta,p}(1)$ has C^3 norm bounded above by $K\mu$, while a term of the form $\mathcal{O}_{\mathbb{X},\theta,p}(\mu)$ has C^3 norm bounded above by $K(\varepsilon)\mu$. In estimates in Section 4.4 and 4.4, we always have the former, not the latter.

Conversion on Σ^+

On Σ^+ we need to write $\mathbb{X}$ and $\mathbb{Y}$ in terms of Z.

PROPOSITION 4.13. *On Σ^+ we have*

$$\mathbb{X} = \mu^{-1}\varepsilon + \mathcal{O}_{Z,\theta,p}(1),$$
$$\mathbb{Y} = (1 + \mathcal{O}_{\theta,p}(\varepsilon) + \mu\mathcal{O}_{Z,\theta,p}(1))Z - \mathcal{O}_{\theta,p}(1).$$

PROOF. On Σ^+, $s = L^+$. We have

$$\begin{aligned} a(L^+) &= \varepsilon + Q_1(\varepsilon, 0) = \varepsilon + \mathcal{O}(\varepsilon^2), \\ b(L^+) &= Q_2(\varepsilon, 0) = \mathcal{O}(\varepsilon^2), \end{aligned} \tag{4.29}$$

and

$$u(L^+) = -1 + \mathcal{O}(\varepsilon), \quad v(L^+) = \mathcal{O}(\varepsilon). \tag{4.30}$$

Let $(z, \theta) \in \Sigma^+$. We compute the values of $\mathbf{X}$ and $\mathbf{Y}$ for this point. Using (4.1) and (4.18), we have

$$\begin{aligned} \xi &= a(L^+) + v(L^+)z + q_1(a(L^+) + v(L^+)z,\ b(L^+) - u(L^+)z) \\ &= \varepsilon + \mathcal{O}(\varepsilon)z + q_1(a(L^+) + v(L^+)z,\ b(L^+) - u(L^+)z) \\ &\quad - q_1(a(L^+), b(L^+)) \\ &= \varepsilon + (\mathcal{O}(\varepsilon) + zk_\xi(z))z. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \eta &= b(L^+) - u(L^+)z + q_2(a(L^+) + v(L^+)z,\ b(L^+) - u(L^+)z) \\ &= -u(L^+)z + q_2(a(L^+) + v(L^+)z,\ b(L^+) - u(L^+)z) \\ &\quad - q_2(a(L^+), b(L^+)) \\ &= (1 + \mathcal{O}(\varepsilon) + zk_\eta(z))z. \end{aligned}$$

We now write $\mathbf{X}$ and $\mathbf{Y}$ in terms of z using (4.13a) and (4.13b). The rest of the proof is similar to that of Proposition 4.9. □

COROLLARY 4.14. *If L^+ is sufficiently large, then $\mathbb{Y} > 1$ on Σ^+.*

PROOF. This follows directly from the definition of Σ^+. □

5 Explicit Computation of $\mathcal{M}$ and $\mathcal{N}$

In this section we explicitly compute the flow-induced maps $\mathcal{M} : \Sigma^- \to \Sigma^+$ and $\mathcal{N} : \Sigma^+ \to \Sigma^-$. The map $\mathcal{M} : \Sigma^- \to \Sigma^+$ is computed in Section 5.1. In Section 5.2 we study the time-t map of equation (4.15). The map $\mathcal{N} : \Sigma^+ \to \Sigma^-$ is computed in Section 5.3.

5.1 Computing $\mathcal{M} : \Sigma^- \to \Sigma^+$

Recall that $s = -L^-$ on Σ^-. Let $q_0 = (-L^-, Z_0, \theta_0) \in \Sigma^-$. Let $(s(t), Z(t), \theta(t))$ be the solution of system (4.22a)–(4.22c) initiated at the point $(-L^-, Z_0, \theta_0)$. Let $\widetilde{t}$ be the time such that $s(\widetilde{t}) = L^+$. By definition, $\mathcal{M}(q_0) = (L^+, Z(\widetilde{t}), \theta(\widetilde{t}))$. In this subsection we derive a specific form of $\mathcal{M}$ using $(\mathbb{X}, \theta)$-coordinates to uniquely locate points on Σ^- and (Z, θ)-coordinates to uniquely locate points on Σ^+.

Define

$$K_1(\varepsilon) = -\rho A_L e^{\int_0^{L^+} E(s)ds}$$

where

$$A_L = \int_{-L^-}^{L^+} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau)d\tau}\, ds$$

is obtained by changing the integral bounds of the improper integral A in (2.5) to $-L^-$ and L^+. Also define

$$P_L = e^{\int_{-L^-}^{L^+} E(s)ds}.$$

LEMMA 5.1.

$$P_L \sim \varepsilon^{\alpha/\beta - \beta/\alpha} \ll 1, \quad K_1(\varepsilon) \sim \varepsilon^{-\beta/\alpha} \gg 1.$$

PROOF. Both estimates follow directly from Lemma 4.7 and the fact that

$$\varepsilon \sim e^{-\alpha L^+} \sim e^{-\beta L^-}.$$

□

PROPOSITION 5.2. *Let $(\mathbb{X}_0, \theta_0) \in \Sigma^-$ and write $(\hat{Z}, \hat{\theta}) = \mathcal{M}(\mathbb{X}_0, \theta_0)$. We have*

$$\begin{aligned} \hat{\theta} &= \theta_0 + \omega(L^+ + L^-) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu), \\ \hat{Z} &= K_1(\varepsilon)(1 + c_1 \sin(\theta_0) + c_2 \cos(\theta_0)) \\ &\quad + P_L\big(\mathbb{X}_0 + \mathcal{O}_{\theta_0,p}(1) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\varepsilon) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)\big), \end{aligned} \tag{5.1}$$

where c_1 and c_2 are constants satisfying

$$\frac{1}{4} < \sqrt{c_1^2 + c_2^2} < \frac{1}{2}.$$

PROOF. Using (4.22c), we have

$$\theta(t) = \theta_0 + \omega t.$$

Integrating (4.22a) and (4.22b), for $t \in [-2L^-, 2L^+]$ we have

$$s(t) = -L^- + t + \mathcal{O}_{t,Z_0,\theta_0,p}(\mu).$$

Inverting the last equality, we obtain

$$t(s) = s + L^- + \mathcal{O}_{s,Z_0,\theta_0,p}(\mu).$$

Substituting $\theta(t)$ and $t(s)$ into (4.22b), we obtain

$$\frac{dZ}{ds} = E(s)Z - (u(s) + v(s))(\rho\mathbb{H}(s,0) + \sin(\theta_0 + \omega L^- + \omega s)) + \mathcal{O}_{s,Z_0,\theta_0,p}(\mu). \tag{5.2}$$

Note that in (5.2), (s, Z_0, θ_0, p) is such that $s \in [-2L^-, 2L^+]$, $(Z_0, \theta_0) \in \Sigma^-$, and $p = \log(\mu) \in (-\infty, \log(\mu_0)]$. Using (5.2), we obtain

$$Z(s) = P_s \cdot (Z_0 - \Phi_s(\theta_0) + \mathcal{O}_{s,Z_0,\theta_0,p}(\mu)) \tag{5.3}$$

where

$$P_s = e^{\int_{-L^-}^{s} E(\tau)d\tau},$$
$$\Phi_s(\theta) = \int_{-L^-}^{s} (u(\tau) + v(\tau))(\rho\mathbb{H}(\tau,0) + \sin(\theta + \omega L^- + \omega\tau)) \cdot e^{-\int_{-L^-}^{\tau} E(\hat{\tau})d\hat{\tau}}\, d\tau. \tag{5.4}$$

From (5.3), it follows that

$$\begin{aligned} \hat{\theta} &= \theta_0 + \omega(L^+ + L^-) + \mathcal{O}_{Z_0,\theta_0,p}(\mu), \\ \hat{Z} &= P_L(Z_0 - \Phi_{L^+}(\theta_0) + \mathcal{O}_{Z_0,\theta_0,p}(\mu)). \end{aligned} \tag{5.5}$$

We want to write the right-hand side of (5.5) in $(\mathbb{X}_0, \theta_0)$-coordinates. Using Corollary 4.10, we have

$$\begin{aligned} \hat{\theta} &= \theta_0 + \omega(L^+ + L^-) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu), \\ \hat{Z} &= P_L\big(\mathbb{X}_0 - \Phi_{L^+}(\theta_0) + \mathcal{O}_{\theta_0,p}(1) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\varepsilon) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)\big). \end{aligned} \tag{5.6}$$

Let K_2 be such that

$$|\mathbb{X}_0 + \mathcal{O}_{\theta_0,p}(1) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\varepsilon) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)| < K_2$$

on Σ^-, and observe that by letting

$$K_0(\varepsilon) = \max_{\substack{\theta\in\mathbb{S}^1 \\ s\in[-2L^-,2L^+]}} 2\,|P_s(K_2 - \Phi_s(\theta))|\,, \tag{5.7}$$

we conclude from (5.3) that all solutions of system (4.22a)–(4.22c) initiated inside of Σ^- will stay inside of

$$\{(s, Z, \theta) : s \in [-2L^-, 2L^+],\ |Z| < K_0(\varepsilon)\}$$

before reaching $s = L^+$. To finish the proof of Proposition 5.2, it now suffices for us to prove the following lemma:

LEMMA 5.3. *For $\rho \in [\rho_1, \rho_2]$, we have*

$$-P_L\Phi_{L^+}(\theta) = K_1(\varepsilon)(1 + c_1\sin(\theta) + c_2\cos(\theta))$$

where c_1 and c_2 are constants satisfying

$$\frac{1}{4} < \sqrt{c_1^2 + c_2^2} < \frac{1}{2}.$$

PROOF. Recall that in (5.4), $\mathbb{H}(s, 0) = h(a(s), b(s))$. We have

$$\begin{aligned} P_L\Phi_{L^+}(\theta) &= e^{\int_0^{L^+} E(\tau)d\tau} \cdot \int_{-L^-}^{L^+} (u(s) + v(s))(\rho h(a(s), b(s)) \\ &\qquad + \sin(\theta + \omega L^- + \omega s))e^{-\int_0^s E(\tau)d\tau}\, ds \\ &= e^{\int_0^{L^+} E(\tau)d\tau} \cdot \big(\rho A_L + (C_L \cos(\omega L^-) - S_L \sin(\omega L^-)) \sin(\theta) \\ &\qquad + (S_L \cos(\omega L^-) + C_L \sin(\omega L^-)) \cos(\theta)\big) \end{aligned}$$

where

$$\begin{aligned} A_L &= \int_{-L^-}^{L^+} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau)d\tau}\, ds, \\ C_L &= \int_{-L^-}^{L^+} (u(s) + v(s)) \cos(\omega s)e^{-\int_0^s E(\tau)d\tau}\, ds, \\ S_L &= \int_{-L^-}^{L^+} (u(s) + v(s)) \sin(\omega s)e^{-\int_0^s E(\tau)d\tau}\, ds. \end{aligned}$$

Observe that A, C, and S in (H2) are obtained by letting $L^\pm = \infty$ in A_L, C_L, and S_L. We now write

$$P_L\Phi_{L^+}(\theta) = \rho A_L e^{\int_0^{L^+} E(\tau)d\tau} \cdot (1 + c_1 \sin(\theta) + c_2 \cos(\theta)) \tag{5.8}$$

where

$$c_1 = \frac{(C_L \cos(\omega L^-) - S_L \sin(\omega L^-))}{A_L\rho}, \quad c_2 = \frac{(S_L \cos(\omega L^-) + C_L \sin(\omega L^-))}{A_L\rho}.$$

We have

$$c_1^2 + c_2^2 = \frac{(C_L^2 + S_L^2)}{A_L^2\rho^2}.$$

Using (H2), for $L^\pm$ sufficiently large we have

$$\begin{aligned} |A_L - A| &< \frac{1}{100}|A|, \\ \big|(C_L^2 + S_L^2)^{1/2} - (C^2 + S^2)^{1/2}\big| &< \frac{1}{100}(C^2 + S^2)^{1/2}. \end{aligned}$$

Therefore, for $\rho \in [\rho_1, \rho_2]$, where

$$\rho_1 = -\frac{202}{99}\frac{(C^2 + S^2)^{1/2}}{A}, \quad \rho_2 = -\frac{396}{101}\frac{(C^2 + S^2)^{1/2}}{A}, \tag{5.9}$$

we have

$$\frac{1}{4} < \sqrt{c_1^2 + c_2^2} < \frac{1}{2}.$$

Notice that because of the way in which ρ_1 and ρ_2 are defined, we have $-\rho A_L > 0$. We also have

$$K_1(\varepsilon) = -\rho A_L e^{\int_0^{L^+} E(\tau)d\tau} \sim e^{-\beta/\alpha}$$

from Lemma 5.1. Equation (5.8) for $P_L \Phi_{L^+}(\theta)$ is now in the asserted form. □

By using Lemma 5.3, we can now rewrite (5.6) as (5.1). This finishes the proof of Proposition 5.2. □

Remark 5.4. Observe that in formula (5.1) for $\widehat{Z}$, the term with $K_1(\varepsilon)$ in front dominates the second term because $K_1(\varepsilon) \gg P_L$. The inclusion $\mathcal{M}(\Sigma^-) \subset \Sigma^+$ follows directly from (5.1).

5.2 On the Time-*t* Map of Equation (4.15)

The computation of $\mathcal{N} : \Sigma^+ \to \Sigma^-$ contains two major steps. The first step is to compute the time-t map of equation (4.15) inside $\mathcal{U}_\varepsilon$. This is done in Section 5.2. The second step is to compute the time it takes for a solution of equation (4.15) initiated in Σ^+ to reach Σ^-. This is done in Section 5.3. These computations are technically involved because we need to control the C^3 norm of the map $\mathcal{N}$ on $\Sigma^+ \times (-\infty, \log(\mu_0)]$, where the interval in the product is the domain of the parameter p.

We start with the first step. Let $W(\Sigma^+)$ be a small open neighborhood surrounding Σ^+ in the space $(\mathbb{X}, \mathbb{Y}, \theta)$. In this subsection we let $(\mathbb{X}_0, \mathbb{Y}_0, \theta_0) \in W(\Sigma^+)$ and regard $p = \log(\mu) \in (-\infty, \log(\mu_0)]$ as the parameter of equation (4.15). We study the time-t map of equation (4.15) assuming that up to time t, all solutions initiated from $W(\Sigma^+)$ are completely contained inside $\mathcal{U}_\varepsilon$. Recall that in equation (4.15), $\mathbb{F}(\mathbb{X}, \mathbb{Y}, \theta; \mu)$ and $\mathbb{G}(\mathbb{X}, \mathbb{Y}, \theta; \mu)$ are analytic on

$$\mathbb{D} = \{(\mathbb{X}, \mathbb{Y}, \theta, \mu) : \mu \in [0, \mu_0],\ (\mathbb{X}, \mathbb{Y}, \theta) \in \mathcal{U}_\varepsilon\}$$

where

$$\mathcal{U}_\varepsilon = \{(\mathbb{X}, \mathbb{Y}, \theta) : \|(\mathbb{X}, \mathbb{Y})\| < 2\varepsilon\mu^{-1},\ \theta \in \mathbb{S}^1\}.$$

For $q_0 = (\mathbb{X}_0, \mathbb{Y}_0, \theta_0) \in W(\Sigma^+)$, let

$$q(t, q_0; \mu) = (\mathbb{X}(t, q_0; \mu), \mathbb{Y}(t, q_0; \mu), \theta(t, q_0; \mu))$$

be the solution of equation (4.15) initiated from q_0 at $t = 0$. Using (4.15), we have

$$\begin{aligned} \mathbb{X}(t, q_0; \mu) &= \mathbb{X}_0 e^{\int_0^t (-\alpha + \mu\mathbb{F}(q(s, q_0; \mu); \mu))ds}, \\ \mathbb{Y}(t, q_0; \mu) &= \mathbb{Y}_0 e^{\int_0^t (\beta + \mu\mathbb{G}(q(s, q_0; \mu); \mu))ds}, \\ \theta(t, q_0; \mu) &= \theta_0 + \omega t. \end{aligned} \tag{5.10}$$

We now introduce the functions $U(t, q_0; \mu)$ and $V(t, q_0; \mu)$ and rewrite (5.10) as

$$\begin{aligned} \mathbb{X}(t, q_0; \mu) &= \mathbb{X}_0 e^{(-\alpha + U(t,q_0;\mu))t}, \\ \mathbb{Y}(t, q_0; \mu) &= \mathbb{Y}_0 e^{(\beta + V(t,q_0;\mu))t}, \\ \theta(t, q_0; \mu) &= \theta_0 + \omega t. \end{aligned} \tag{5.11}$$

Using (5.11), we have

$$\begin{aligned} U(t, q_0; \mu) &= t^{-1} \log\left(\frac{\mathbb{X}(t, q_0; \mu)}{\mathbb{X}_0}\right) + \alpha, \\ V(t, q_0; \mu) &= t^{-1} \log\left(\frac{\mathbb{Y}(t, q_0; \mu)}{\mathbb{Y}_0}\right) - \beta. \end{aligned} \tag{5.12}$$

We also have

$$\begin{aligned} U(t, q_0; \mu) &= t^{-1} \int_0^t \mu\mathbb{F}(q(s, q_0; \mu); \mu) ds, \\ V(t, q_0; \mu) &= t^{-1} \int_0^t \mu\mathbb{G}(q(s, q_0; \mu); \mu) ds. \end{aligned} \tag{5.13}$$

In the next proposition we regard $U = U(t, q_0; \mu)$ and $V = V(t, q_0; \mu)$ as functions of t, q_0, and p and write $U = U_{t,q_0,p}$ and $V = V_{t,q_0,p}$, respectively. We define the domain of these two functions as follows: Let

$$\mathbb{D}_{t,q_0,p} = \{q_0 \in W(\Sigma^+),\ p \in (-\infty, \log(\mu_0)],\ t \in [1, T(q_0, p)]\}$$

where the upper bound $T(q_0, p)$ on t is designed to keep the solution inside $\mathcal{U}_\varepsilon$.

PROPOSITION 5.5. *There exists* $K > 0$ *such that*

$$\|U_{t,q_0,p}\|_{C^3(\mathbb{D}_{t,q_0,p})} < K\mu, \quad \|V_{t,q_0,p}\|_{C^3(\mathbb{D}_{t,q_0,p})} < K\mu.$$

Proposition 5.5 is proved in Section 7.1.

Remark 5.6. By combining Proposition 5.5 and (5.11), we can now write the time-t map from $W(\Sigma^+)$ to $\mathcal{U}_\varepsilon$ as

$$\begin{aligned} \mathbb{X}(t, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu) &= \mathbb{X}_0 e^{(-\alpha + \mathcal{O}_{t,\mathbb{X}_0,\mathbb{Y}_0,\theta_0,p}(\mu))t}, \\ \mathbb{Y}(t, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu) &= \mathbb{Y}_0 e^{(\beta + \mathcal{O}_{t,\mathbb{X}_0,\mathbb{Y}_0,\theta_0,p}(\mu))t}, \\ \theta(t, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu) &= \theta_0 + \omega t. \end{aligned} \tag{5.14}$$

5.3 Estimates on $T(Z_0, \theta_0, p)$

For $q_0 = (Z_0, \theta_0) \in \Sigma^+$, let $q(t, q_0; \mu)$ be the solution of equation (4.15) initiated at q_0, and let T be the time at which this solution reaches Σ^-. In this subsection we regard T as a function of Z_0, θ_0, and p and obtain a well-controlled formula for T that is explicit in the variables Z_0, θ_0, and p. Since the images of $\mathcal{M}$ are expressed in (Z, θ)-coordinates through (5.1), we must write the initial conditions for $\mathcal{N}$ in (Z, θ)-coordinates on Σ^+ to facilitate the intended composition of $\mathcal{N}$ and $\mathcal{M}$.

Estimates on $T(Z_0, \theta_0, p)$ are complicated partly because as a function of Z_0 and θ_0, it is implicitly defined through equations written in $(\mathbb{X}, \mathbb{Y}, \theta)$-coordinates on $\Sigma^\pm$. The computational process must therefore involve (5.14) and the coordinate transformations on $\Sigma^\pm$ presented in Sections 4.4 and 4.4. Before presenting the desired quantitative estimates, we explain how to obtain $T(Z_0, \theta_0, p)$ in a conceptual way.

Using (5.11), we obtain

$$\begin{aligned} \mathbb{X}(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu) &= \mathbb{X}_0 e^{(-\alpha + U(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; p))T}, \\ \mathbb{Y}(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu) &= \mathbb{Y}_0 e^{(\beta + V(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; p))T}, \\ \theta(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu) &= \theta_0 + \omega T. \end{aligned} \tag{5.15}$$

In (5.15), $\mathbb{X}_0$ and $\mathbb{Y}_0$ are *not* independent variables. These quantities satisfy

$$\begin{aligned} \mathbb{X}_0 &= \mu^{-1}\varepsilon + \mathcal{O}_{Z_0, \theta_0, p}(1), \\ \mathbb{Y}_0 &= (1 + \mathcal{O}_{\theta_0, p}(\varepsilon) + \mu \mathcal{O}_{Z_0, \theta_0, p}(1)) Z_0 - \mathcal{O}_{\theta_0, p}(1), \end{aligned} \tag{5.16}$$

by Proposition 4.13. We write

$$\begin{aligned} \mathbb{X}(T) &= \mathbb{X}(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu), \\ \mathbb{Y}(T) &= \mathbb{Y}(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; \mu), \\ \theta(T) &= \theta_0 + \omega T. \end{aligned}$$

By definition, $\mathbb{X}(T)$, $\mathbb{Y}(T)$, and $\theta(T)$ are also related through Corollary 4.11. For the benefit of a clear exposition, we write the conclusion of Corollary 4.11 as

$$\mathbb{Y} = \varepsilon\mu^{-1} + \mathbf{f}(\mathbb{X}, \theta; p)$$

where

$$\mathbf{f}(\mathbb{X}, \theta; p) = \mathcal{O}_{\mathbb{X}, \theta, p}(1).$$

We have

$$\mathbb{Y}(T) = \varepsilon\mu^{-1} + \mathbf{f}(\mathbb{X}(T), \theta(T); p). \tag{5.17}$$

We use (5.15) to implicitly define $T(Z_0, \theta_0; p)$. We have

$$\mathbb{Y}(T) = \mathbb{Y}_0 e^{(\beta + V(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; p))T}. \tag{5.18}$$

The right-hand side of (5.18) is relatively simple: we only need to substitute for $\mathbb{X}_0$ and $\mathbb{Y}_0$ using (5.16). The left-hand side of (5.18) is conceptually more complicated. We need to do the following:

(1) Write $\mathbb{Y}(T)$ as a function of $\mathbb{X}(T)$, $\theta(T)$, and p using (5.17).
(2) Substitute for $\mathbb{X}(T)$ and $\theta(T)$ using (5.15), thereby obtaining $\mathbb{Y}(T)$ in terms of T, $\mathbb{X}_0$, $\mathbb{Y}_0$, θ_0, and p.
(3) Use (5.16) to write $\mathbb{X}_0$ and $\mathbb{Y}_0$ in terms of Z_0 and θ_0.

After all of these substitutions are made, we regard (5.18) as the equation that implicitly defines $T(Z_0, \theta_0; p)$. We use this equation as the basis for the computation of $T(Z_0, \theta_0; p)$.

PROPOSITION 5.7. *As a function of Z_0, θ_0, and p, the map T satisfies*

$$\left\| T - \frac{1}{\beta}\log(\mu^{-1}) \right\|_{C^3} < K.$$

Proposition 5.7 is proved in Section 7.2.

5.4 Computing $\mathcal{N} : \Sigma^+ \to \Sigma^-$

We derive a formula for the induced map $\mathcal{N}_p : \Sigma^+ \to \Sigma^-$. For $(Z_0, \theta_0) \in \Sigma^+$, we write $(\mathbb{X}_1, \theta_1) = \mathcal{N}_p(Z_0, \theta_0)$. We start with U and V in (5.15).

LEMMA 5.8. *On $\Sigma^+ \times (-\infty, \log(\mu_0)]$, we have*

$$U(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; p) = \mu\mathcal{O}_{Z_0,\theta_0,p}(1), \quad V(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; p) = \mu\mathcal{O}_{Z_0,\theta_0,p}(1).$$

PROOF. We write U and V as functions of (Z_0, θ_0, p) using Proposition 5.7 for $T(Z_0, \theta_0; p)$ and (5.16) for $\mathbb{X}_0$ and $\mathbb{Y}_0$. This lemma is established by applying the chain rule and using Proposition 5.5, Proposition 5.7, and (5.16). □

PROPOSITION 5.9. *The flow-induced map $\mathcal{N}_p : \Sigma^+ \to \Sigma^-$ is given by*

$$\begin{aligned} \mathbb{X}_1 &= \left(\frac{\mu}{\varepsilon + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)}\right)^{(\tilde{\alpha}/\tilde{\beta})-1} \\ &\quad \times \left([1 + \mathcal{O}_{\theta_0,p}(\varepsilon) + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)]Z_0 - \mathcal{O}_{\theta_0,p}(1)\right)^{\tilde{\alpha}/\tilde{\beta}}, \\ \theta_1 &= \theta_0 + \frac{\omega}{\beta + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)} \\ &\quad \times \log\left(\frac{(\varepsilon + \mu\mathcal{O}_{Z_0,\theta_0,p}(1))\mu^{-1}}{[1 + \mathcal{O}_{\theta_0,p}(\varepsilon) + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)]Z_0 - \mathcal{O}_{\theta_0,p}(1)}\right), \end{aligned} \tag{5.19}$$

where

$$\tilde{\alpha} = \alpha + \mu\mathcal{O}_{Z_0,\theta_0,p}(1), \quad \tilde{\beta} = \beta + \mu\mathcal{O}_{Z_0,\theta_0,p}(1).$$

PROOF. Using (5.17), (5.18), and Lemma 5.8, we have

$$\begin{aligned} T &= \frac{1}{\beta + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)}\log\left(\frac{Y(T)}{Y_0}\right) \\ &= \frac{1}{\beta + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)}\log\left(\frac{(\varepsilon + \mu\mathbf{f}(\mathbb{X}(T), \theta(T); p))\mu^{-1}}{\mathbb{Y}_0}\right). \end{aligned} \tag{5.20}$$

By using Proposition 5.7 and the fact that $\mathbf{f}(\mathbb{X}, \theta; p) = \mathcal{O}_{\mathbb{X},\theta,p}(1)$, we have

$$\mathbf{f}(\mathbb{X}(T), \theta(T); p) = \mathcal{O}_{Z_0,\theta_0,p}(1).$$

Now (5.20) gives

$$T = \frac{1}{\beta + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)} \times \log\left(\frac{\mu^{-1}(\varepsilon + \mu\mathcal{O}_{Z_0,\theta_0,p}(1))}{[1 + \mathcal{O}_{\theta_0,p}(\varepsilon) + \mu\mathcal{O}_{Z_0,\theta_0,p}(1)]Z_0 - \mathcal{O}_{\theta_0,p}(1)}\right). \tag{5.21}$$

Here we use (5.16) for $\mathbb{Y}_0$.

The desired formula for θ_1 now follows from $\theta_1 = \theta_0 + \omega T$. For $\mathbb{X}_1$ we use

$$\mathbb{X}_1 = \mu^{-1}(\varepsilon + \mu\mathcal{O}_{Z_0,\theta_0,p}(1))e^{-(\alpha+\mu\mathcal{O}_{Z_0,\theta_0,p}(1))T}$$

and substitute for T using (5.21). □

6 Proof of Theorem 1

In Section 6.1 we compute $\mathcal{F}_p = \mathcal{N} \circ \mathcal{M}$ by using Propositions 5.2 and 5.9. In Section 6.2 we apply the theory of rank 1 maps to the family $\{\mathcal{F}_p\}$, thereby proving the existence of rank 1 chaos as claimed in Theorem 1.

6.1 The Flow-Induced Map $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$

We regard p as the fundamental parameter of the flow-induced map $\mathcal{F} : \Sigma^- \to \Sigma^-$. For $(\mathbb{X}_0, \theta_0) \in \Sigma^-$, let $(\mathbb{X}_1, \theta_1) = (\mathcal{N} \circ \mathcal{M})(\mathbb{X}_0, \theta_0)$. We compute $\mathcal{F}_p : (\mathbb{X}_0, \theta_0) \mapsto (\mathbb{X}_1, \theta_1)$ by combining (5.1) and (5.19).

PROPOSITION 6.1. *The map $\mathcal{F}_p : \Sigma^- \to \Sigma^-$ is given by*

$$\mathbb{X}_1 = (\mu(\varepsilon + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu))^{-1})^{(\tilde{\alpha}/\tilde{\beta})-1} \times \left(\left(1 + \mathcal{O}^{\partial\mathbb{X}_0\langle\mu\rangle}_{\mathbb{X}_0,\theta_0,p}(\varepsilon) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)\right)\mathcal{Z} - \mathcal{O}^{\partial\mathbb{X}_0\langle\mu\rangle}_{\mathbb{X}_0,\theta_0,p}(1)\right)^{\tilde{\alpha}/\tilde{\beta}}, \tag{6.1a}$$

$$\theta_1 = \theta_0 + \omega(L^+ + L^-) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu) + \frac{\omega}{\beta + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)} \times \log\left(\frac{(\varepsilon + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu))\mu^{-1}}{\left(1 + \mathcal{O}^{\partial\mathbb{X}_0\langle\mu\rangle}_{\mathbb{X}_0,\theta_0,p}(\varepsilon) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)\right)\mathcal{Z} - \mathcal{O}^{\partial\mathbb{X}_0\langle\mu\rangle}_{\mathbb{X}_0,\theta_0,p}(1)}\right), \tag{6.1b}$$

where

$$\mathcal{Z} = K_1(\varepsilon)(1 + c_1\sin(\theta_0) + c_2\cos(\theta_0)) + P_L[\mathbb{X}_0 + \mathcal{O}_{\theta_0,p}(1) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\varepsilon) + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)],$$

$$\tilde{\alpha} = \alpha + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu),$$

$$\tilde{\beta} = \beta + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu),$$

and the superscript $\partial\mathbb{X}_0\langle\mu\rangle$ on a given term indicates that the partial derivative of the term with respect to $\mathbb{X}_0$ is $\mathcal{O}(\mu)$. We also have

$$K_1(\varepsilon) \sim \varepsilon^{-\beta/\alpha}, \quad \frac{1}{4} < \sqrt{c_1^2 + c_2^2} < \frac{1}{2}.$$

PROOF. We first examine the formulas for $\widetilde{\alpha}$ and $\widetilde{\beta}$. The error terms in Proposition 5.9 have the form

$$\mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1),$$

and $\widehat{Z}$ and $\widehat{\theta}$ are given in terms of $\mathbb{X}_0$, θ_0, and p by (5.1). Using (5.1), we see that the C^3 norms of $\widehat{Z}$ and $\widehat{\theta}$ are $< K(\varepsilon)$. It follows from the chain rule that

$$\mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1) = \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu).$$

We follow the same line of reasoning to compute $\mathbb{X}_1$ and θ_1. We replace Z_0 and θ_0 with $\widehat{Z}$ and $\widehat{\theta}$ in (5.19) and then substitute for $\widehat{Z}$ and $\widehat{\theta}$ using (5.1). Using (5.19), we have

$$\begin{aligned}
\mathbb{X}_1 &= \left(\frac{\mu}{\varepsilon + \mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1)}\right)^{(\widetilde{\alpha}/\widetilde{\beta})-1} \\
&\quad \times \left([1 + \mathcal{O}_{\widehat{\theta},p}(\varepsilon) + \mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1)]\widehat{Z} - \mathcal{O}_{\widehat{\theta},p}(1)\right)^{\widetilde{\alpha}/\widetilde{\beta}}, \\
\theta_1 &= \widehat{\theta} + \frac{\omega}{\beta + \mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1)} \\
&\quad \times \log\left(\frac{(\varepsilon + \mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1))\mu^{-1}}{[1 + \mathcal{O}_{\widehat{\theta},p}(\varepsilon) + \mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1)]\widehat{Z} - \mathcal{O}_{\widehat{\theta},p}(1)}\right).
\end{aligned} \tag{6.2}$$

In (6.2), terms of the form $\mu\mathcal{O}_{\widehat{Z},\widehat{\theta},p}(1)$ are rewritten in the form $\mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)$ using (5.1). Terms of the form $\mathcal{O}_{\widehat{\theta},p}(\varepsilon)$ are rewritten in the form $\mathcal{O}^{\partial\mathbb{X}_0\langle\mu\rangle}_{\mathbb{X}_0,\theta_0,p}(\varepsilon)$ because the C^3 norm of $\widehat{\theta}$ is bounded by a constant K independent of ε and because $\partial\widehat{\theta}/\partial\mathbb{X}_0 = \mathcal{O}(\mu)$. Reasoning analogously, terms of the form $\mathcal{O}_{\widehat{\theta},p}(1)$ are rewritten in the form $\mathcal{O}^{\partial\mathbb{X}_0\langle\mu\rangle}_{\mathbb{X}_0,\theta_0,p}(1)$. □

6.2 Proof of Theorem 1

We are finally ready to prove Theorem 1.

The Two-Parameter Family $\{\mathcal{F}_{a,b_n}\}$

We write $\{\mathcal{F}_p\}$ as a two-parameter family $\{\mathcal{F}_{a,b_n}\}$ of 2-D maps. Both a and b_n are derived from $\mu = e^p$ as follows: Let $\mu_0 > 0$ be sufficiently small. Define $\gamma : (0, \mu_0] \to \mathbb{R}$ via $\gamma(\mu) = \frac{\omega}{\beta}\log(\mu^{-1})$. For $n \in \mathbb{Z}^+$ such that $n \geq (2\pi\beta)^{-1}\omega\log(\mu_0^{-1})$, let $\mu_n \in (0, \mu_0]$ be such that $\gamma(\mu_n) = n$. Notice that $\mu_n \to 0$ monotonically. Set $b_n = \mu_n$. For $\mu \in (\mu_{n+1}, \mu_n]$ and $a \in [0, 2\pi) = \mathbb{S}^1$, we define

$$\mu(n, a) = \gamma^{-1}(\gamma(\mu_n) + a) = \mu_n e^{-(\beta/\omega)a}$$

and

$$p(n,a) = \log(\mu(n,a)) = \log(\mu_n) - \frac{\beta}{\omega}\, a.$$

Define

$$\mathcal{F}_{a,b_n} = \mathcal{F}_{p(n,a)}.$$

Verification of (C1)–(C4)

We prove Theorem 1 by applying Propositions 3.4 and 3.6. We verify (C1)–(C4) for $\{\mathcal{F}_{a,b_n}\}$. Proposition 6.2 establishes (C1).

PROPOSITION 6.2. *We have*

$$\|\mathcal{F}_{a,b_n}(\mathbb{X},\theta) - (0,\mathcal{F}_{a,0}(\mathbb{X},\theta))\|_{C^3(\Sigma^-\times[0,2\pi))} \to 0 \tag{6.3}$$

as $b_n \to 0$*, where*

$$\begin{aligned}
&\mathcal{F}_{a,0}(\mathbb{X},\theta)\\
&\quad = \theta + \omega(L^+ + L^-) + a + \frac{\omega}{\beta}\log(\varepsilon K_1(\varepsilon)^{-1})\\
&\qquad - \frac{\omega}{\beta}\log\Bigg[(1+\mathcal{O}_{\theta,p}(\varepsilon))\Bigg(1 + c_1\sin(\theta) + c_2\cos(\theta)\\
&\qquad\qquad + \frac{P_L}{K_1(\varepsilon)}(\mathbb{X} + \mathcal{O}_{\theta,p}(1) + \mathcal{O}_{\mathbb{X},\theta,p}(\varepsilon))\Bigg)\\
&\qquad\qquad - K_1(\varepsilon)^{-1}\mathcal{O}_{\theta,p}(1)\Bigg].
\end{aligned} \tag{6.4}$$

PROOF. The only problematic term in (6.1b) has the form

$$\frac{\omega}{\beta + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)}\log(\mu^{-1}),$$

which we write as

$$\frac{\omega}{\beta}\log(\mu^{-1}) + \frac{\omega\cdot\mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu)}{\beta(\beta + \mathcal{O}_{\mathbb{X}_0,\theta_0,p}(\mu))}\log(\mu^{-1}).$$

Observe that the C^3 norm of the second term $\to 0$ as $b_n \to 0$, and the first term may be computed modulo 2π and is therefore equal to a. Viewing μ as a function of a, the C^3 norm of $\mathbb{X}_1$ is bounded by

$$K(\varepsilon)\mu^{(\tilde{\alpha}/\tilde{\beta})-1}$$

and therefore decays to 0 as $b_n \to 0$ provided that (H1)(b) holds. □

For (C2) we apply Proposition 3.3 to the family of circle maps

$$\begin{aligned}\mathcal{F}_{a,0}(0,\theta)&\\ &= \theta + \omega(L^+ + L^-) + a + \frac{\omega}{\beta}\log(\varepsilon K_1(\varepsilon)^{-1})\\ &\quad - \frac{\omega}{\beta}\log\Bigg[(1+\mathcal{O}_{\theta,p}(\varepsilon))\bigg(1 + c_1\sin(\theta) + c_2\cos(\theta)\\ &\qquad + \frac{P_L}{K_1(\varepsilon)}(\mathcal{O}_{\theta,p}(1) + \mathcal{O}_{\mathbb{X},\theta,p}(\varepsilon))\bigg)\\ &\qquad - K_1(\varepsilon)^{-1}\mathcal{O}_{\theta,p}(1)\Bigg].\end{aligned} \tag{6.5}$$

To apply Proposition 3.3 to the family $\{\mathcal{F}_{a,0}(0,\theta)\}$, we set

$$\mathcal{K} = \frac{\omega}{\beta}, \quad \Psi(\theta) = -\log(1 + c_1\sin(\theta) + c_2\cos(\theta)),$$
$$\Phi(\theta, a) = \mathcal{F}_{a,0}(0,\theta) - \gamma - \theta - a - \mathcal{K}\Psi(\theta),$$

where

$$\gamma = \omega(L^+ + L^-) + \frac{\omega}{\beta}\log(\varepsilon K_1(\varepsilon)^{-1}).$$

The assumption on the C^3 norm of Φ is satisfied if ε is sufficiently small.

Hypothesis (C3) follows directly from (6.4). Hypothesis (C4) follows from a direct computation using (6.2). Finally, to apply Proposition 3.6 we need to verify that $\lambda_0 > \log(10)$. This follows if ω is sufficiently large. The proof of Theorem 1 is complete.

7 Computational Proofs

7.1 Proof of Proposition 5.5

Let $\mathbb{F} = \mathbb{F}(\mathbb{X}, \mathbb{Y}, \theta; \mu)$ and $\mathbb{G} = \mathbb{G}(\mathbb{X}, \mathbb{Y}, \theta; \mu)$ be as in equation (4.15). For a combination $\mathbb{Z} = \mathbb{X}^{d_1}\mathbb{Y}^{d_2}\mu^{d_3}$ of powers of the variables $\mathbb{X}$, $\mathbb{Y}$, and μ, let $\partial^k_{\mathbb{Z}}$ denote the corresponding partial derivative operator, where $k = d_1 + d_2 + d_3$ is the order. There exists $K_3 > 0$ such that for every $\mathbb{Z}$ of order ≤ 3 and $0 \leq i \leq 3$, we have

$$\left|\partial^k_{\mathbb{Z}}(\partial^i_{\theta^i}\mathbb{F}\cdot\mathbb{Z})\right| < K_3, \quad \left|\partial^k_{\mathbb{Z}}(\partial^i_{\theta^i}\mathbb{G}\cdot\mathbb{Z})\right| < K_3, \tag{7.1}$$

on $\mathbb{D}_{t,q_0,p}$. This is because the C^3 norms of $\mathbf{F}(\mathbf{X}, \mathbf{Y}, \theta; \mu)$ and $\mathbf{G}(\mathbf{X}, \mathbf{Y}, \theta; \mu)$ are bounded on $\mathcal{U}_\varepsilon \times [0, \mu_0)$ and because we have $\mathbb{F}(\mathbb{X}, \mathbb{Y}, \theta; \mu) = \mathbf{F}(\mu\mathbb{X}, \mu\mathbb{Y}, \theta; \mu)$ and $\mathbb{G}(\mathbb{X}, \mathbb{Y}, \theta; \mu) = \mathbf{G}(\mu\mathbb{X}, \mu\mathbb{Y}, \theta; \mu)$.

C^0 Estimates

Using (7.1) with $i = k = 0$ and (5.13), we have

$$\|U\|_{C^0(\mathbb{D}_{t,q_0,p})} < K_3\mu, \quad \|V\|_{C^0(\mathbb{D}_{t,q_0,p})} < K_3\mu. \tag{7.2}$$

C^1 Estimates

We now estimate the first derivatives.

On $\partial_{\mathbb{Y}_0} U$ *and* $\partial_{\mathbb{Y}_0} V$. Using $\theta(t) = \theta_0 + \omega t$, we have $\partial_{\mathbb{Y}_0}\theta = 0$. Using (5.13), we have

$$\partial_{\mathbb{Y}_0} U = \mu t^{-1} \int_0^t (\partial_{\mathbb{X}}\mathbb{F} \cdot \partial_{\mathbb{Y}_0}\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{F} \cdot \partial_{\mathbb{Y}_0}\mathbb{Y}) ds, \tag{7.3a}$$

$$\partial_{\mathbb{Y}_0} V = \mu t^{-1} \int_0^t (\partial_{\mathbb{X}}\mathbb{G} \cdot \partial_{\mathbb{Y}_0}\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{G} \cdot \partial_{\mathbb{Y}_0}\mathbb{Y}) ds. \tag{7.3b}$$

To make these formulas useful, we need to write $\partial_{\mathbb{Y}_0}\mathbb{X}$ and $\partial_{\mathbb{Y}_0}\mathbb{Y}$ in terms of $\partial_{\mathbb{Y}_0} U$ and $\partial_{\mathbb{Y}_0} V$. For this purpose we use (5.12). We have

$$\partial_{\mathbb{Y}_0}\mathbb{X} = t\mathbb{X}\partial_{\mathbb{Y}_0} U, \quad \partial_{\mathbb{Y}_0}\mathbb{Y} = t\mathbb{Y}\partial_{\mathbb{Y}_0} V + \frac{\mathbb{Y}}{\mathbb{Y}_0}. \tag{7.4}$$

Combining (7.3a), (7.3b), and (7.4), we obtain

$$\begin{aligned} \partial_{\mathbb{Y}_0} U &= \mu t^{-1} \int_0^t (\partial_{\mathbb{X}}\mathbb{F} \cdot \mathbb{X} \cdot s\partial_{\mathbb{Y}_0} U + \partial_{\mathbb{Y}}\mathbb{F} \cdot \mathbb{Y} \cdot s\partial_{\mathbb{Y}_0} V) ds \\ &\quad + \mu t^{-1} \int_0^t \partial_{\mathbb{Y}}\mathbb{F} \cdot \frac{\mathbb{Y}}{\mathbb{Y}_0}\, ds, \\ \partial_{\mathbb{Y}_0} V &= \mu t^{-1} \int_0^t (\partial_{\mathbb{X}}\mathbb{G} \cdot \mathbb{X} \cdot s\partial_{\mathbb{Y}_0} U + \partial_{\mathbb{Y}}\mathbb{G} \cdot \mathbb{Y} \cdot s\partial_{\mathbb{Y}_0} V) ds \\ &\quad + \mu t^{-1} \int_0^t \partial_{\mathbb{Y}}\mathbb{G} \cdot \frac{\mathbb{Y}}{\mathbb{Y}_0}\, ds. \end{aligned} \tag{7.5}$$

Using (7.1), we have

$$|\partial_{\mathbb{X}}\mathbb{F} \cdot \mathbb{X}| < K_3, \quad |\partial_{\mathbb{X}}\mathbb{G} \cdot \mathbb{X}| < K_3, \quad |\partial_{\mathbb{Y}}\mathbb{F} \cdot \mathbb{Y}| < K_3, \quad |\partial_{\mathbb{Y}}\mathbb{G} \cdot \mathbb{Y}| < K_3.$$

Using (7.5), we have

$$\begin{aligned} |\partial_{\mathbb{Y}_0} U| &\le K\mu t^{-1} \int_0^t (|s\partial_{\mathbb{Y}_0} U| + |s\partial_{\mathbb{Y}_0} V|) ds + K\mu, \\ |\partial_{\mathbb{Y}_0} V| &\le K\mu t^{-1} \int_0^t (|s\partial_{\mathbb{Y}_0} U| + |s\partial_{\mathbb{Y}_0} V|) ds + K\mu, \end{aligned} \tag{7.6}$$

from which it follows that

$$|\partial_{\mathbb{Y}_0} U| < K\mu, \quad |\partial_{\mathbb{Y}_0} V| < K\mu.$$

On $\partial_{\mathbb{X}_0} U$ *and* $\partial_{\mathbb{X}_0} V$. Mimic the proof above.

On $\partial_{\theta_0}U$ and $\partial_{\theta_0}V$. We follow similar lines of computation. Since $\partial_{\theta_0}\theta = 1$, we have

$$\partial_{\theta_0}U = \mu t^{-1}\int_0^t (\partial_{\mathbb{X}}\mathbb{F}\cdot\partial_{\theta_0}\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{F}\cdot\partial_{\theta_0}\mathbb{Y} + \partial_\theta\mathbb{F})ds,$$
$$\partial_{\theta_0}V = \mu t^{-1}\int_0^t (\partial_{\mathbb{X}}\mathbb{G}\cdot\partial_{\theta_0}\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{G}\cdot\partial_{\theta_0}\mathbb{Y} + \partial_\theta\mathbb{G})ds.$$

Analogously to (7.4), we have

$$\partial_{\theta_0}\mathbb{X} = t\mathbb{X}\partial_{\theta_0}U, \quad \partial_{\theta_0}\mathbb{Y} = t\mathbb{Y}\partial_{\theta_0}V.$$

Arguing as above, we conclude that

$$|\partial_{\theta_0}U| < K\mu, \quad |\partial_{\theta_0}V| < K\mu.$$

On $\partial_p U$ and $\partial_p V$. We follow similar lines of computation. Note that we have

$$\partial_p\mu = \mu, \quad \partial_p\mathbb{F} = \mu\partial_\mu\mathbb{F},$$

and so on. Starting with (5.13), we have

$$\begin{aligned}\partial_p U &= \mu t^{-1}\int_0^t \mathbb{F}\,ds + \mu t^{-1}\int_0^t (\partial_{\mathbb{X}}\mathbb{F}\cdot\partial_p\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{F}\cdot\partial_p\mathbb{Y} + \mu\partial_\mu\mathbb{F})ds,\\ \partial_p V &= \mu t^{-1}\int_0^t \mathbb{G}\,ds + \mu t^{-1}\int_0^t (\partial_{\mathbb{X}}\mathbb{G}\cdot\partial_p\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{G}\cdot\partial_p\mathbb{Y} + \mu\partial_\mu\mathbb{G})ds,\end{aligned} \tag{7.7}$$

and using (5.12) we have

$$\partial_p\mathbb{X} = t\mathbb{X}\partial_p U, \quad \partial_p\mathbb{Y} = t\mathbb{Y}\partial_p V. \tag{7.8}$$

Now argue as above.

On $\partial_t U$ and $\partial_t V$. The partial derivatives of U and V with respect to t are easier to estimate because when differentiating with respect to t using (5.13), no derivatives are involved on the right-hand side, so the estimates on $\partial_t U$ and $\partial_t V$ are obtained directly from C^0 estimates. We have

$$|\partial_t U| < K\mu, \quad |\partial_t V| < K\mu.$$

This completes the desired estimates on the first derivatives.

C^2 Estimates

We now move to the second derivatives. We estimate $\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U$ and $\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}V$ first. Using (7.3a), we have

$$\begin{aligned}\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U = \mu t^{-1}\int_0^t &\big(\partial^2_{\mathbb{X}\mathbb{X}}\mathbb{F}\cdot(\partial_{\mathbb{Y}_0}\mathbb{X})^2 + 2\partial^2_{\mathbb{X}\mathbb{Y}}\mathbb{F}\cdot(\partial_{\mathbb{Y}_0}\mathbb{X})(\partial_{\mathbb{Y}_0}\mathbb{Y})\\ &+ \partial_{\mathbb{Y}\mathbb{Y}}(\mathbb{F}\cdot\partial_{\mathbb{Y}_0}\mathbb{Y})^2\big)ds\\ &+ \mu t^{-1}\int_0^t \big(\partial_{\mathbb{X}}\mathbb{F}\cdot\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}\mathbb{X} + \partial_{\mathbb{Y}}\mathbb{F}\cdot\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}\mathbb{Y}\big)ds.\end{aligned}$$

Using (7.4), we have

$$\begin{aligned}\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}\mathbb{X} &= t\partial_{\mathbb{Y}_0}\mathbb{X}\cdot\partial_{\mathbb{Y}_0}U + t\mathbb{X}\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U,\\ \partial^2_{\mathbb{Y}_0\mathbb{Y}_0}\mathbb{Y} &= t\partial_{\mathbb{Y}_0}\mathbb{Y}\cdot\partial_{\mathbb{Y}_0}V + t\mathbb{Y}\cdot\partial_{\mathbb{Y}_0\mathbb{Y}_0}V + \frac{\partial_{\mathbb{Y}_0}\mathbb{Y}}{\mathbb{Y}_0} - \frac{\mathbb{Y}}{\mathbb{Y}_0^2}.\end{aligned} \tag{7.9}$$

Therefore, $\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U$ is given by

$$\begin{aligned}\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U = \mu t^{-1}\int_0^t &\big(\partial^2_{\mathbb{X}\mathbb{X}}\mathbb{F}\cdot(\partial_{\mathbb{Y}_0}\mathbb{X})^2 + 2\partial^2_{\mathbb{X}\mathbb{Y}}\mathbb{F}\cdot(\partial_{\mathbb{Y}_0}\mathbb{X})(\partial_{\mathbb{Y}_0}\mathbb{Y})\\ &\quad + \partial_{\mathbb{Y}\mathbb{Y}}(\mathbb{F}\cdot\partial_{\mathbb{Y}_0}\mathbb{Y})^2\big)ds\\ &+ \mu t^{-1}\int_0^t(\partial_{\mathbb{X}}\mathbb{F}\cdot\partial_{\mathbb{Y}_0}\mathbb{X}\cdot s\partial_{\mathbb{Y}_0}U + \partial_{\mathbb{Y}}\mathbb{F}\cdot\partial_{\mathbb{Y}_0}\mathbb{Y}\cdot s\partial_{\mathbb{Y}_0}V)ds\\ &+ \mu t^{-1}\int_0^t\partial_{\mathbb{Y}}\mathbb{F}\cdot\left(\frac{\partial_{\mathbb{Y}_0}\mathbb{Y}}{\mathbb{Y}_0} - \frac{\mathbb{Y}}{\mathbb{Y}_0^2}\right)ds\\ &+ \mu t^{-1}\int_0^t\big(\partial_{\mathbb{X}}\mathbb{F}\cdot\mathbb{X}\cdot s\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U + \partial_{\mathbb{Y}}\mathbb{F}\cdot\mathbb{Y}\cdot s\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}V\big)ds.\end{aligned} \tag{7.10}$$

To estimate the first three integrals in (7.10), we use (7.4 for $\partial_{\mathbb{Y}_0}\mathbb{X}$ and $\partial_{\mathbb{Y}_0}\mathbb{Y}$. Using the first derivative estimates and using (7.1) repeatedly, we bound these integrals by $K\mu$. Note that we also need $\mathbb{Y}_0 > 1$ (see Corollary 4.14) for the third integral. Together with an analogous formula for $\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}V$ in which we replace $\mathbb{F}$ with $\mathbb{G}$, we conclude that

$$|\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}U| < K\mu, \quad |\partial^2_{\mathbb{Y}_0\mathbb{Y}_0}V| < K\mu.$$

All other second derivatives are estimated similarly. Here we skip the details to avoid repetitive computations.

C^3 Estimates

Third derivatives are estimated in the same spirit. Since the formulas for a given third derivative depend on previous computations of relevant second derivatives, here we estimate $\partial^3_{\mathbb{Y}_0\mathbb{Y}_0 p}U$ and $\partial^3_{\mathbb{Y}_0\mathbb{Y}_0 p}V$ as a representative example. Of all of the third derivatives, these are the most tedious to compute.

To compute $\partial^3_{\mathbb{Y}_0\mathbb{Y}_0 p}U$ we apply ∂_p to (7.10). The explicit factor μ written in front of all integrals generates a collection of terms that is identical to the right-hand side of (7.10). We showed when estimating second derivatives that the size of each of these terms is bounded by $K\mu$.

The remaining terms are produced by applying ∂_p to the functions inside of the integrals in (7.10). The terms produced from the first three integrals are estimated using the C^2 estimates. Estimate (7.1) is used repeatedly. It is critically important that potentially problematic terms in the form of powers of $\mathbb{Y}$ and $\mathbb{X}$, introduced by using the likes of (7.4), (7.8), and (7.9), are always matched perfectly with corresponding partial derivatives with respect to $\mathbb{F}$ or $\mathbb{G}$. Applying ∂_p to the fourth integral, we obtain an integral term of the form

$$(\mathrm{I}) = \mu t^{-1} \int_0^t (\partial_{\mathbb{X}} \mathbb{F} \cdot \mathbb{X} \cdot s \partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} U + \partial_{\mathbb{Y}} \mathbb{F} \cdot \mathbb{Y} \cdot s \partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} V) ds$$

and a collection of other terms that can be treated the same way as the terms produced by differentiating the first three integrals. We have

$$|(\mathrm{I})| \le K \mu t^{-1} \int_0^t \left(\left| s \partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} U \right| + \left| s \partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} V \right| \right) ds.$$

Combining this analysis with analogous estimates for $|\partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} V|$, we obtain

$$\left| \partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} U \right| < K \mu, \quad \left| \partial^3_{\mathbb{Y}_0 \mathbb{Y}_0 p} V \right| < K \mu.$$

This completes the proof of Proposition 5.5.

7.2 Proof of Proposition 5.7

The proof of this proposition is lengthy because of the complicated composition process explained earlier in Section 5.3.

C^0 Estimates

We first establish a C^0 control on T.

LEMMA 7.1. *There exist constants $K_4 < K_5$ independent of ε such that for all $q_0 \in \Sigma^+$, we have $K_4 \log(\mu^{-1}) < T(q_0; \mu) < K_5 \log(\mu^{-1})$.*

PROOF. Using

$$\mathbb{Y}(T) = \mathbb{Y}_0 e^{(\beta + V(T))T},$$

we obtain

$$T = \frac{1}{\beta + V(T)} \log \left(\frac{\mathbb{Y}(T)}{\mathbb{Y}_0} \right).$$

Since $(\mathbb{X}(T), \mathbb{Y}(T), \theta(T))$ is on Σ^-, Proposition 4.9 implies that

$$\mathbb{Y}(T) \approx \mu^{-1} \varepsilon,$$

and the desired estimates follow from $|V(T)| < K\mu$ and $1 < \mathbb{Y}_0 < K(\varepsilon)$. □

LEMMA 7.2. *We have $\mu^{-1} e^{-\alpha T} < 1$.*

PROOF. We substitute

$$T = \frac{1}{\beta + V(T)} \log \left(\frac{\mathbb{Y}(T)}{\mathbb{Y}_0} \right)$$

into (5.15) to obtain

$$\mathbb{X}(T) = \left(\frac{\mathbb{Y}_0}{\mathbb{Y}(T)} \right)^{\frac{\alpha - U(T)}{\beta + V(T)}} \mathbb{X}_0.$$

We then use $\mathbb{Y}(T) \approx \varepsilon\mu^{-1}$, $\mathbb{X}_0 \approx \varepsilon\mu^{-1}$, $|U(T)| < K\mu$, $|V(T)| < K\mu$, and $\alpha > \beta$ to conclude that $\mathbb{X}(T) \ll \varepsilon$. We have

$$\frac{1}{10}\varepsilon\mu^{-1}e^{-\alpha T} < \mathbb{X}_0 e^{(-\alpha+U(T))T} = \mathbb{X}(T) \ll \varepsilon.$$

For the first inequality, we use $\mathbb{X}_0 \approx \varepsilon\mu^{-1}$ and $|U(T)T| < K\mu\log(\mu^{-1}) \ll 1$. This proves the lemma. □

C^1 Estimates

We present C^1 estimates with respect to (Z_0, θ_0, p), where $(Z_0, \theta_0) \in \Sigma^+$ and $p \in (-\infty, \log(\mu_0))$.

LEMMA 7.3. *There exist constants K_7 and K_8 independent of ε such that*

$$\|\mathbb{X}(T)\|_{C^1} < K_7 + K_8\|T\|_{C^1}, \quad \|\theta(T)\|_{C^1} < K_7 + K_8\|T\|_{C^1}.$$

PROOF. The bound on $\theta(T)$ is trivial because $\theta(T) = \theta_0 + \omega T$. For $\mathbb{X}(T)$, we have

$$\begin{aligned}\mathbb{X}(T) &= \mathbb{X}_0 e^{(-\alpha+U(T))T} \\ &= \varepsilon\mu^{-1}e^{(-\alpha+U(T,\mathbb{X}_0,\mathbb{Y}_0,\theta_0;p))T} + \mathcal{O}_{Z_0,\theta_0,p}(1)e^{(-\alpha+U(T,\mathbb{X}_0,\mathbb{Y}_0,\theta_0;p))T}.\end{aligned}$$

Notice that for the second equality, (5.16) is used for $\mathbb{X}_0$. We regard $\mathbb{X}_0$ and $\mathbb{Y}_0$ as functions of Z_0, θ_0, and p defined by (5.16). The desired estimate follows from using Proposition 5.5 for U and (5.16) for $\mathbb{X}_0$ and $\mathbb{Y}_0$. We also use Lemma 7.2. □

LEMMA 7.4. *We have*

$$\left\| T - \frac{1}{\beta}\log(\mu^{-1}) \right\|_{C^1} < K.$$

PROOF. Using (5.17), we write (5.18) as

$$\mu^{-1}(\varepsilon + \mu\mathbf{f}(\mathbb{X}(T), \theta(T); p)) = \mathbb{Y}_0 e^{(\beta+V(T))T}.$$

Solving for T, we obtain

$$\begin{aligned}T - \frac{1}{\beta}\log(\mu^{-1}) = &-\frac{V(T)}{\beta(\beta+V(T))}\log(\mu^{-1}) - \frac{1}{\beta+V(T)}\log(\mathbb{Y}_0) \\ &+ \frac{1}{\beta+V(T)}\log(\varepsilon + \mu\mathbf{f}(\mathbb{X}(T), \theta(T); p)).\end{aligned} \tag{7.11}$$

In (7.11), $V(T) = V(T, \mathbb{X}_0, \mathbb{Y}_0, \theta_0; p)$, and $\mathbb{X}_0$ and $\mathbb{Y}_0$ are written in terms of Z_0, θ_0, and p using (5.16). Using Proposition 5.5, we have

$$\left\| T - \frac{1}{\beta}\log(\mu^{-1}) \right\|_{C^0} < K.$$

First derivatives of T are estimated by directly differentiating (7.11). We estimate $\partial_{Z_0}T$ as a representative example. Differentiating (7.11), we have

$$\partial_{Z_0}T = (\mathrm{I}) + (\mathrm{II})\partial_{Z_0}T,$$

where (I) is a collection of terms that do not depend on $\partial_{Z_0} T$ and (II) is a function of Z_0, θ_0, and p. Using Proposition 5.5 for $V(T)$, (5.16) for $\mathbb{X}_0$ and $\mathbb{Y}_0$, and Lemma 7.3 for $\partial_{Z_0}\mathbb{X}(T)$ and $\partial_{Z_0}\theta(T)$, we have $|(\mathrm{I})| < K$ and $|(\mathrm{II})| \ll 1$. □

Higher Derivative Estimates

With the first derivatives controlled by Lemmas 7.3 and 7.4, we estimate the second derivatives by first proving a version of Lemma 7.3 and then proving a version of Lemma 7.4 for the C^2 norms. We then do the same for the C^3 norms. This completes the proof of Proposition 5.7.

8 Application to a Duffing Equation

In this section we apply Theorem 1 to a periodically perturbed Duffing equation.

8.1 A Periodically Perturbed Duffing Equation

We start with the second-order equation

$$\frac{d^2q}{dt^2} - q + q^3 = 0. \tag{8.1}$$

We define $p = \frac{dq}{dt}$ and write (8.1) as the first-order system

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = q - q^3. \tag{8.2}$$

Using the simple linear change of coordinates

$$x = \frac{1}{2}(q - p), \quad y = \frac{1}{2}(q + p),$$

we write equation (8.2) in (x, y) as

$$\frac{dx}{dt} = -x + \frac{1}{2}(x + y)^3, \quad \frac{dy}{dt} = y - \frac{1}{2}(x + y)^3. \tag{8.3}$$

Notice that $(x, y) = (0, 0)$ is a saddle that is not dissipative. The curve

$$a(t) = \frac{2\sqrt{2}e^{3t}}{(1 + e^{2t})^2}, \quad b(t) = \frac{2\sqrt{2}e^{t}}{(1 + e^{2t})^2},$$

is a homoclinic solution such that $(a(0), b(0)) = (\sqrt{2}/2, \sqrt{2}/2)$ and

$$\lim_{t\to\pm\infty} (a(t), b(t)) = (0, 0).$$

Define

$$\ell = \{\ell(t) = (a(t), b(t)) : t \in \mathbb{R}\}.$$

We construct systems with rank 1 attractors in two steps. First, we add certain autonomous perturbations to (8.3), obtaining

$$\begin{aligned} \frac{dx}{dt} &= -x + \frac{1}{2}(x + y)^3 + \gamma(y + x)^2(y - x), \\ \frac{dy}{dt} &= y - \lambda y - \frac{1}{2}(x + y)^3 - \gamma(y + x)^2(y - x). \end{aligned} \tag{8.4}$$

Here $\lambda, \gamma \in \mathbb{R}$ are parameters. Notice that for $\lambda \in (0,1)$, $(x,y) = (0,0)$ is a dissipative saddle. The following proposition provides a set of values of (λ, γ) for which (8.4) admits a homoclinic solution.

PROPOSITION 8.1 ([16]). *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, there exists γ_λ satisfying $|\gamma_\lambda| < 10\lambda$ such that the following hold for $\gamma = \gamma_\lambda$:*

(1) *System (8.4) has a homoclinic solution for $(x,y) = (0,0)$, which we denote as*

$$\ell_\lambda = \{\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t)) : t \in \mathbb{R}\}.$$

(2) *For any given $L > 0$, there exists $K_0(L)$ independent of λ such that for all $t \in [-L, L]$,*

$$\|\ell_\lambda(t) - \ell_0(t)\| < K_0(L)\lambda$$

where $\ell_0(t) = \ell(t) = (a(t), b(t))$.

Now let $\lambda \in (0, \lambda_0)$ and let $\gamma = \gamma_\lambda$. We add time-periodic forcing to (8.4), obtaining

$$\begin{aligned} \frac{dx}{dt} &= -x + \frac{1}{2}(x+y)^3 + \gamma_\lambda (y+x)^2(y-x) \\ &\quad - \mu(\rho(y+x)^2(y-x) + \sin(\omega t)), \\ \frac{dy}{dt} &= (1-\lambda)y - \frac{1}{2}(x+y)^3 - \gamma_\lambda(y+x)^2(y-x) \\ &\quad + \mu(\rho(y+x)^2(y-x) + \sin(\omega t)). \end{aligned} \tag{8.5}$$

We will apply Theorem 1 to system (8.5). Before doing so, we show that (8.5) is equivalent to (1.3). To this end, $q = x + y$ and (8.5) imply

$$\frac{d^2q}{dt^2} = \beta_1 q - \beta_2 q^3 - (a - bq^2)\frac{dq}{dt} + (1-\lambda)\mu \sin(\omega t) \tag{8.6}$$

where

$$\beta_1 = (1-\lambda), \quad \beta_2 = \left[1 - \frac{1}{2}\lambda - (\gamma_\lambda(1-\lambda) + \mu\rho(1-\lambda))\frac{\lambda}{2-\lambda}\right],$$
$$a = \lambda, \quad b = -(\gamma_\lambda(1-\lambda) + \mu\rho(1-\lambda))\left(1 - \frac{\lambda}{2-\lambda}\right).$$

Rescaling t and q by

$$\tilde{t} = t\sqrt{\beta_1}, \quad \tilde{q} = q\sqrt{\frac{\beta_2}{\beta_1}},$$

equation (8.6) becomes

$$\frac{d^2q}{dt^2} - q + q^3 + (\tilde{a} - \tilde{b}q^2)\frac{dq}{dt} = \tilde{\mu}\sin(\omega t), \tag{8.7}$$

where

$$\widetilde{a} = \frac{a}{\beta_1^{1/2}}, \quad \widetilde{b} = \frac{b\beta_1^{1/2}}{\beta_2}, \quad \widetilde{\mu} = \mu\frac{(1-\lambda)}{\beta_1}\sqrt{\frac{\beta_2}{\beta_1}}, \quad \widetilde{\omega} = \frac{\omega}{\beta_1^{1/2}}.$$

8.2 On the Unperturbed System

Let A_0, C_0, and S_0 denote the integrals A, C, and S, respectively, associated with the homoclinic loop ℓ of (8.3). These integrals are defined by (2.5). We compute A_0, C_0, and S_0 in this subsection. Recall that

$$\ell(t) = (a(t), b(t)) = \left(\frac{2\sqrt{2}e^{3t}}{(1+e^{2t})^2}, \frac{2\sqrt{2}e^{t}}{(1+e^{2t})^2}\right) \tag{8.8}$$

is a homoclinic solution of (8.3). The unit tangent vector

$$(u(t), v(t)) = \left\| \frac{d}{dt}\,\ell(t) \right\|^{-1} \frac{d}{dt}\,\ell(t)$$

is given by

$$\begin{aligned} u(t) &= \frac{-(e^{2t}-3)}{\sqrt{(e^{2t}-3)^2 + (e^{-2t}-3)^2}}, \\ v(t) &= \frac{e^{-2t}-3}{\sqrt{(e^{2t}-3)^2 + (e^{-2t}-3)^2}}. \end{aligned} \tag{8.9}$$

We write (8.3) as

$$\frac{dx}{dt} = -x + f(x,y), \quad \frac{dy}{dt} = y + g(x,y), \tag{8.10}$$

where

$$f(x,y) = \frac{1}{2}(x+y)^3, \quad g(x,y) = -\frac{1}{2}(x+y)^3.$$

The quantity E defined by (2.4) is given by

$$\begin{aligned} E(t) = {} & v^2(t)\big(-1 + \partial_x f(a(t),b(t))\big) + u^2(t)\big(1 + \partial_y g(a(t),b(t))\big) \\ & - u(t)v(t)\big(\partial_y f(a(t),b(t)) + \partial_x g(a(t),b(t))\big). \end{aligned}$$

Using (8.8) and (8.9), we have

$$E(t) = -\frac{(e^{-2t}-3)^2 - (e^{2t}-3)^2}{(e^{2t}-3)^2 + (e^{-2t}-3)^2}\left(1 - \frac{12e^{2t}}{(1+e^{2t})^2}\right). \tag{8.11}$$

Define

$$V(s) = -\int_0^s E(\tau)\,d\tau. \tag{8.12}$$

LEMMA 8.2. *For $s \in (-\infty, \infty)$, we have*

$$V(s) = \frac{1}{2}\log\left(\frac{8e^{2s}((1-3e^{2s})^2 + e^{4s}(e^{2s}-3)^2)}{(e^{2s}+1)^6}\right).$$

PROOF. Let $x = e^{2t}$. We have

$$V(s) = \frac{1}{2}\int_1^{e^{2s}} \left(\frac{1}{x}\right)\left(\frac{(1-3x)^2 - x^2(x-3)^2}{(1-3x)^2 + x^2(x-3)^2}\right)\left(1 - \frac{12x}{(1+x)^2}\right)dx.$$

Observe that

$$(1-3x)^2 + x^2(x-3)^2 = ((x-a)^2 + a^2)((x-b)^2 + b^2) \tag{8.13}$$

where $a, b > 0$ satisfy $a + b = 3$ and $ab = \frac{1}{2}$. We have

$$\begin{aligned} V(s) &= \frac{1}{2}\int_1^{e^{2s}} \frac{(-x^4 + 6x^3 - 6x + 1)(x^2 - 10x + 1)}{x(x+1)^2((x-a)^2 + a^2)((x-b)^2 + b^2)}\, dx \\ &= \frac{1}{2}\int_1^{e^{2s}} \left(\frac{1}{x} - \frac{6}{1+x} + \frac{4x^3 - 18x^2 + 36x - 6}{((x-a)^2 + a^2)((x-b)^2 + b^2)}\right)dx \\ &= \frac{1}{2}\int_1^{e^{2s}} \left(\frac{1}{x} - \frac{6}{1+x} + \frac{2(x-a)}{(x-a)^2 + a^2} + \frac{2(x-b)}{(x-b)^2 + b^2}\right)dx. \end{aligned}$$

We then have

$$\begin{aligned} V(s) &= \frac{1}{2}\log\left(\frac{8e^{2s}((e^{2s} - a)^2 + a^2)((e^{2s} - b)^2 + b^2)}{(e^{2s} + 1)^6}\right) \\ &= \frac{1}{2}\log\left(\frac{8e^{2s}((1 - 3e^{2s})^2 + e^{4s}(e^{2s} - 3)^2)}{(e^{2s} + 1)^6}\right). \end{aligned}$$

□

Using Lemma 8.2, we obtain

$$\begin{aligned} A_0 &= \int_{-\infty}^{\infty} (u(s) + v(s))(b(s) + (a(s))^2(b(s) - a(s))e^{-\int_0^s E(\tau)d\tau}\, ds = \frac{16}{15}, \\ &\int_{-\infty}^{\infty} u(s)b(s)e^{-\int_0^s E(\tau)d\tau}\, ds = -\int_{-\infty}^{\infty} v(s)a(s)e^{-\int_0^s E(\tau)d\tau}\, ds = \frac{2}{3}, \\ &\int_{-\infty}^{\infty} v(s)b(s)e^{-\int_0^s E(\tau)d\tau}\, ds = \int_{-\infty}^{\infty} u(s)a(s)e^{-\int_0^s E(\tau)d\tau}\, ds = 0. \end{aligned} \tag{8.14}$$

We now compute C_0 and S_0. The equalities

$$a(-t) = b(t), \quad b(-t) = a(t), \quad u(-t) = -v(t), \quad v(-t) = -u(t),$$

imply that the functions E and $u + v$ are odd. Therefore, we have

$$C_0 = 0, \quad S_0 = 2\int_{-\infty}^{\infty} u(s)\sin(\omega s)e^{-\int_0^s E(\tau)d\tau}\, ds.$$

We compute S_0 using the residue theorem. Define

$$\begin{aligned} \xi_c(\omega) &= \int_{-\infty}^{\infty} u(s)\cos(\omega s)e^{-\int_0^s E(\tau)d\tau}\, ds, \\ \xi_s(\omega) &= \int_{-\infty}^{\infty} u(s)\sin(\omega s)e^{-\int_0^s E(\tau)d\tau}\, ds. \end{aligned} \tag{8.15}$$

LEMMA 8.3. *We have*

$$\xi_c(\omega) = \frac{\sqrt{2}\pi\omega^2}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}, \quad \xi_s(\omega) = -\left(\frac{\sqrt{2}\pi\omega}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}\right).$$

Using Lemma 8.3, we obtain

$$S_0 = 2\xi(\omega) = -\left(2\frac{\sqrt{2}\pi\omega}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}\right).$$

PROOF. Using Lemma 8.2, we have

$$\xi_c(\omega) = -2\sqrt{2}I_c, \quad \xi_s(\omega) = -2\sqrt{2}I_s$$

where

$$I_c = \int_{-\infty}^{\infty} \frac{e^{3s}(e^{2s}-3)}{(e^{2s}+1)^3} \cdot \cos(\omega s)ds, \quad I_s = \int_{-\infty}^{\infty} \frac{e^{3s}(e^{2s}-3)}{(e^{2s}+1)^3} \cdot \sin(\omega s)ds.$$

Let $I = I_c + iI_s$. We have

$$I = \int_{-\infty}^{\infty} \frac{e^{3s}(e^{2s}-3)}{(e^{2s}+1)^3} \cdot e^{i\omega s}\, ds.$$

We evaluate I as follows. On the $z = x + iy$ plane, let ζ_1 be the real axis, oriented from left to right, and let ζ_2 be the line $\{x + \pi i : x \in \mathbb{R}\}$ oriented from right to left. Define

$$f(z) = \frac{e^{3z}(e^{2z}-3)}{(e^{2z}+1)^3} \cdot e^{i\omega z}.$$

We have

$$\int_{\ell_1} f(z)dz = I, \quad \int_{\ell_2} f(z)dz = e^{-\omega\pi} I.$$

So by the residue theorem, we have

$$(1 + e^{-\omega\pi})I = 2\pi i \operatorname{Res}\left(f, \frac{\pi i}{2}\right).$$

Setting $t = z - \pi i/2$, we write $f(z)$ as

$$\begin{aligned} f(z) &= -ie^{-\frac{1}{2}\omega\pi}\, \frac{e^{3t}(e^{2t}+3)}{(e^{2t}-1)^3} \cdot e^{i\omega t} \\ &= -\frac{i}{8}\, e^{-\frac{1}{2}\omega\pi}\, \frac{e^{(5+i\omega)t} + 3e^{(3+i\omega)t}}{t^3(1 + t + \frac{2}{3}t^2 + \mathcal{O}(t^3))^3}. \end{aligned}$$

We have

$$e^{(5+i\omega)t} + 3e^{(3+i\omega)t} = 4 + (14 + 4i\omega)t + (26 + 14i\omega - 2\omega^2)t^2 + \mathcal{O}(t^3)$$

and

$$\left(1 + t + \frac{2}{3}t^2\right)^{-3} = 1 - 3t + 4t^2 + \mathcal{O}(t^3).$$

Consequently,

$$\operatorname{Res}\left(f, \frac{\pi i}{2}\right) = -\frac{i}{4}\, e^{-\frac{1}{2}\omega\pi}(i\omega - \omega^2),$$

from which it follows that

$$I = \frac{\pi e^{-\frac{1}{2}\omega\pi}}{2(1+e^{-\omega\pi})}(i\omega - \omega^2).$$

In conclusion, we have

$$\xi_c(\omega) = \frac{\sqrt{2}\pi\omega^2}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}, \quad \xi_s(\omega) = -\left(\frac{\sqrt{2}\pi\omega}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}\right).$$

□

8.3 Application of Theorem 1

We apply Theorem 1 to system (8.5). Let λ_0 satisfy Proposition 8.1, and let $\Delta \subset (0, \lambda_0)$ be such that for $\lambda \in \Delta$ we have

$$|n - (1-\lambda)m| > \lambda_0^2 (n+m)^{-2} \tag{8.16}$$

for all $n, m \in \mathbb{Z}^+$ such that $m + n > 0$. Let $\lambda \in \Delta$ and $\gamma = \gamma_\lambda$ be fixed in system (8.4). We write (8.4) as

$$\frac{dx}{dt} = -x + f_\lambda(x,y), \quad \frac{dy}{dt} = (1-\lambda)y + g_\lambda(x,y), \tag{8.17}$$

where

$$f_\lambda(x,y) = -g_\lambda(x,y) = \frac{1}{2}(x+y)^3 + \gamma_\lambda(y+x)^2(y-x).$$

System (8.17) is naturally in the form assumed for the unperturbed equation (2.1) in Section 2 with

$$\alpha = 1, \quad \beta = 1 - \lambda.$$

The point $(x, y) = (0, 0)$ is a *dissipative homoclinic saddle* for (8.17) and the homoclinic solution $\ell_\lambda = (a_\lambda(t), b_\lambda(t))$ is that of Proposition 8.1(1). System (8.5), which we write as

$$\begin{aligned} \frac{dx}{dt} &= -x + f_\lambda(x,y) - \mu(\rho(y+x)^2(y-x) + \sin(\omega t)), \\ \frac{dy}{dt} &= (1-\lambda)y + g_\lambda(x,y) + \mu(\rho(y+x)^2(y-x) + \sin(\omega t)), \end{aligned} \tag{8.18}$$

assumes the form of (2.2) with

$$h(x,y) = (y+x)^2(y-x).$$

We regard (8.17) as the unperturbed system and (8.18) as the forced equation. Let

$$(u_\lambda(t), v_\lambda(t)) = \left\| \frac{d}{dt}\, \ell_\lambda(t) \right\|^{-1} \frac{d}{dt}\, \ell_\lambda(t).$$

Writing $E(t)$ from (2.4) and A, C, and S from (2.5) as $E_\lambda(t)$, A_λ, C_λ, and S_λ, respectively, we have

$$
\begin{aligned}
E_\lambda(t) = {} & v_\lambda^2(t)(-1 + \partial_x f_\lambda(a_\lambda(t), b_\lambda(t))) \\
& + u_\lambda^2(t)(1 - \lambda + \partial_y g_\lambda(a_\lambda(t), b_\lambda(t))) \\
& - u_\lambda(t) v_\lambda(t)\big(\partial_y f_\lambda(a_\lambda(t), b_\lambda(t)) + \partial_x g_\lambda(a_\lambda(t), b_\lambda(t))\big),
\end{aligned} \tag{8.19}
$$

and

$$
\begin{aligned}
A_\lambda &= \int_{-\infty}^{\infty} (u_\lambda(s) + v_\lambda(s)) h(a_\lambda(s), b_\lambda(s)) e^{-\int_0^s E_\lambda(\tau) d\tau}\, ds, \\
C_\lambda &= \int_{-\infty}^{\infty} (u_\lambda(s) + v_\lambda(s)) \cos(\omega s) e^{-\int_0^s E_\lambda(\tau) d\tau}\, ds, \\
S_\lambda &= \int_{-\infty}^{\infty} (u_\lambda(s) + v_\lambda(s)) \sin(\omega s) e^{-\int_0^s E_\lambda(\tau) d\tau}\, ds.
\end{aligned} \tag{8.20}
$$

We let $A_{\lambda,L}$, $C_{\lambda,L}$, and $S_{\lambda,L}$ be the integrals obtained by changing the integral bounds from $\pm\infty$ to $\pm L$ in (8.20).

LEMMA 8.4. *There exist λ_0 sufficiently small and L_0 sufficiently large such that for any given $L > L_0$, we have*

$$|A_\lambda - A_{\lambda,L}| < 10^3 e^{-\frac{1}{2}L}, \quad |C_\lambda - C_{\lambda,L}| < 10^3 e^{-\frac{1}{2}L}, \quad |S_\lambda - S_{\lambda,L}| < 10^3 e^{-\frac{1}{2}L},$$

for all $\lambda \in [0, \lambda_0)$.

PROOF. Observe that for $t \in (-\infty, -L)$, we have $E_\lambda(t) < -\frac{1}{2}$ and for $t \in (L, \infty)$, $E_\lambda(t) > \frac{1}{2}$ provided that L is sufficiently large and λ sufficiently small. □

LEMMA 8.5. *For any given $L > 0$, there exists $K_1(L)$ independent of λ such that*

$$|A_{\lambda,L} - A_{0,L}| < K_1(L)\lambda, \quad |C_{\lambda,L} - C_{0,L}| < K_1(L)\lambda,$$
$$|S_{\lambda,L} - S_{0,L}| < K_1(L)\lambda.$$

PROOF. This lemma follows from Proposition 8.1(2). □

Verification of Hypotheses of Theorem 1

Let $\lambda \in \Delta$. Hypothesis (H1)(a) follows from (8.16), and (H1)(b) holds because $\alpha = 1$, $\beta = 1 - \lambda$. For (H2) we use Lemmas 8.3, 8.4, and 8.5 together with (8.14). To prove $A_\lambda \neq 0$, we first choose $L > L_0$ sufficiently large so that

$$10^3 e^{-\frac{1}{2}L} < 10^{-3} A.$$

Note that $A = \frac{16}{15}$ from (8.14). It then follows from Lemma 8.4 that

$$|A_\lambda - A_{\lambda,L}| < 10^{-3} A, \quad |A_0 - A_{0,L}| < 10^{-3} A.$$

Next we let λ_0 be sufficiently small so that

$$|A_{\lambda,L} - A_{0,L}| < K_1(L)\lambda_0 < 10^{-3}A$$

where $K_1(L)$ is as in Lemma 8.5. We obtain

$$A_\lambda \geq A_0 - |A_0 - A_{0,L}| - |A_{\lambda,L} - A_{0,L}| - |A_\lambda - A_{\lambda,L}| > 1.$$

Hypothesis (H2)(b) is proved in a similar manner using S_0 and S_λ.

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