Introductory Remarks

1. Objects of Study

Phase Space M: A geometric object (e.g. sphere, tori, open set in R^n).

A System T: A mapping $M \to M$.

Orbits: $x \to T(x) \to T(T(x)) \to \cdots$.

Ex1: $M = R^1$, $T : R^1 \mapsto R^1$ defined by

$$T(x) = 1 - 2x^2.$$

$x = 0.1$, $x_1 = 0.98$, $x_2 = -0.9208$, $\cdots$.

Ex2: $M = S^1$, $T(\theta) = \theta + \pi$.

$\theta = 1$, $\theta_1 = 1 + \pi$, $\theta_2 = 1$, $\cdots$.

Ex3: Example: $M = R^2, T : (x, y) \to (x_1, y_1)$

$$T : \left\{ \begin{array}{rcl} x_1 & = & 2x \\ y_1 & = & \frac{1}{2}y \end{array} \right.$$

$$z = (1,1), z_1 = (2, \tfrac{1}{2}), z_2 = (2^2, \tfrac{1}{2^2}), \cdots.$$

Ex4: $M = R^2$, $\quad T : (r,\theta) \to (r_1, \theta_1)$

$$T : \begin{cases} r_1 & = & r \\ \theta_1 & = & \theta + r \end{cases}$$

$$z = (1,0), z_1 = (1,1), \cdots, z_n = (1, n \mod (2\pi)), \cdots.$$

– For $T : M \to M$, and $x_0 \in M$ given, the orbit started from x_0 is denoted as $\{x_n\}_{n=0}^{\infty}$.

– If T^{-1} exists, we say that T is invertible. We then are able to talk about backward orbit from x_0: $x_{-1} = T^{-1}x_0$ and so on.

– Ex1 in the above is not invertible. Ex2, Ex3 and Ex4 are invertible.

2. Fundamental Questions

Q1: Behave of individual orbits.

– Fixed points: $T(x) = x$.

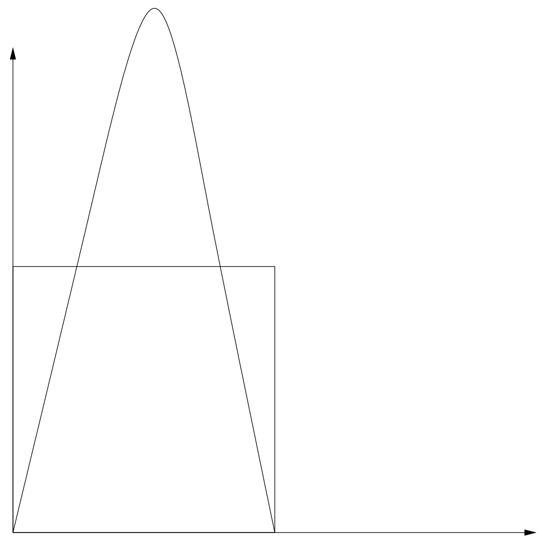
Ex: $T(x) = \mu x(1 - x)$

$$x = \mu x(1 - x) \rightarrow x_1 = 0, \quad x_2 = 1 - \mu^{-1}.$$

– Periodic orbits: $T^n(x) = x$.

The smallest n is the period.

Ex: $T(x) = 7x(1 - x)$



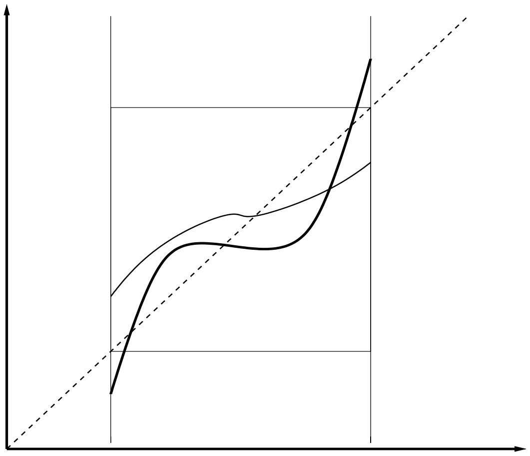
Claim: Periodic orbit of all period exists.

Proof: Let $I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, 1]$. $\forall n > 0$ given, $\exists$ an interval I, such that

(i) $T^i(I) \subset I_1, i = 0, 1, \cdots, n-2$;

(ii) $T^{n-1}I \subset I_2$, $T^n I = I_1$.

Fact: Let $T : I \to R$ be continuous such that $T(I) \subset I$ (or $T(I) \supset I$). The T has a fixed point in I.



Note that the period of the orbit constructed is n by design.

– Transitive orbit: closure($\{x_i\}_{i=0}^{\infty}$)= M.

Ex: $M = S^1$, $T(\theta) = \theta + r$, $\frac{r}{2\pi} \notin \mathbb{Q}$.

Claim: Every orbit of $T(\theta)$ is dense in S^1.

Proof: For $\theta_0 \in S^1$, we have two possibilities:

(a) There exist $n_1 \neq n_2$ such that $T^{n_1}(\theta_0) = T^{n_2}(\theta_0)$.

It follows that $|n_1 - n_2|r = 2\pi m$. So $\frac{r}{2\pi} \in \{\mathbb{Q}\}$.

(b) $\theta_i \neq \theta_j$ for all $i \neq j$.

In this case $\{\theta_i\}_{i=0}^{\infty}$ has at least one accumulation point: for any $\varepsilon > 0$ given, there exists n_1, n_2, such that $d(\theta_{n_1}, \theta_{n_2}) < \varepsilon$. It follows that $r_n := d(\theta_{|n_2-n_1|}, \theta_0) < \varepsilon$.

Note that $T^{|n_2-n_1|}\theta = \theta + r_n$. $\{T^{m|n_2-n_1|}\theta_0\}_{m=1}^{\infty}$ is ε-dense in S^1.

– Non-wondering points: $x \in M$ is a non-wondering point if for any open set U such that $x \in U$, there exists $i > 0$ such that $T^i U \cap U \neq \emptyset$. Otherwise x is a wondering point.

- Measure preserving maps:

  For $S \subset M$, let $T^{-1}S = \{x \in M, T(x) \in S\}$.

  T is measure preserving if $m(T^{-1}(S)) = m(S)$.

  Ex: $M = S^1$, $T(\theta) = 3\theta$ is measure preserving.

  Ex: If $T : U \to U$ is invertible and $det|DT| = 1$, then

  T is measure preserving.

- Poincare's recurrence thorem

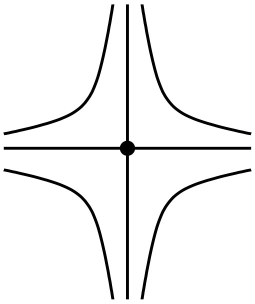
Theorem Assume that $T : M \to M$ is invertible and measure preserving, and $m(U) < \infty$. Then all point in M is non-wondering.

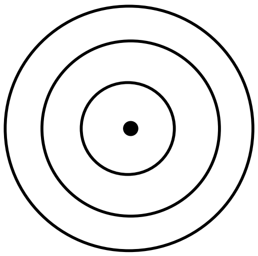
Proof: If $T^iU \cap T^jU = \emptyset$ for all $i \neq j$, then $m(\cup T^iU) = \sum m(T^iU) = \infty$, contradicting to $m(M) < \infty$. So there exists $n_1 \neq n_2$ such that $T^{n_1}U \cap T^{n_2}U \neq \emptyset \to T^{|n_1-n_2|}U \cap U \neq \emptyset$.

Q2: Organizations of Orbits.

– Phase portrait: (orbit structure)

Example A: $T : \left\{ \begin{array}{lcl} x_1 & = & 2x \\ y_1 & = & \frac{1}{2}y \end{array} \right.$





Example B: $T : \left\{ \begin{array}{lcr} r_1 & = & r \\ \theta_1 & = & \theta + r \end{array} \right.$

– Local Stability:

Let x_0 be a fixed point. An open neighborhood of x_0 is denoted as $U(x_0)$, $V(x_0)$, etc.

(a) x_0 is **stable** if for every $U(x_0)$ given, there exists an $V(x_0)$ such that $T^i V(x_0) \subset U(x_0)$ for all $i \geq 1$.

(b) x_0 is **asymptotically stable** (a sink, or an attracting fixed point) if there exists $U(x_0)$, such that for all $x \in U(x_0)$, $T^i x \to x_0$ as $i \to \infty$.

$\{x \in M : T^i x \to x_0\}$ is the **attracting basin** for x_0.

(c) Let $x_0 \in M$ be such that $T^n x_0 = x_0$, then x_0 is a fixed point of T^n. (a) and (b) apply to define respectively **stable** and **asymptotically stable** periodic orbits.

Example B is Stable but not asymptotically stable.

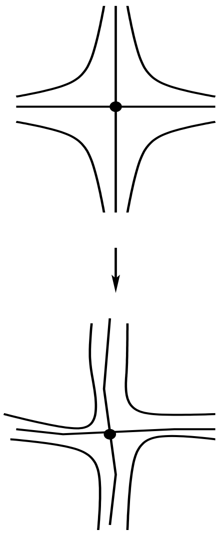
Example A is not stable.

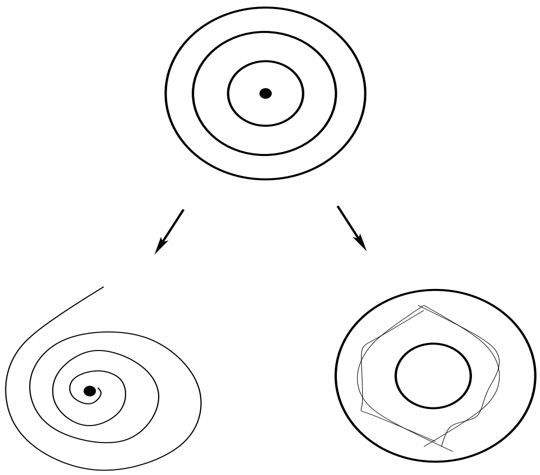
No asymptotically stable fixed point for measure preserving maps.

$T(x) = \frac{1}{2}x$ has a unique fixed point $x_0 = 0$ that is asymptotically stable. Its attracting basin is $\mathbb{R}$.

– Structure stability: Does the orbit structure of a given T change under small perturbation?

Hyperbolic structure and Elliptic structure





3. **Source of Inspirations**:

Differential Equation ⇒ Dynamical Systems

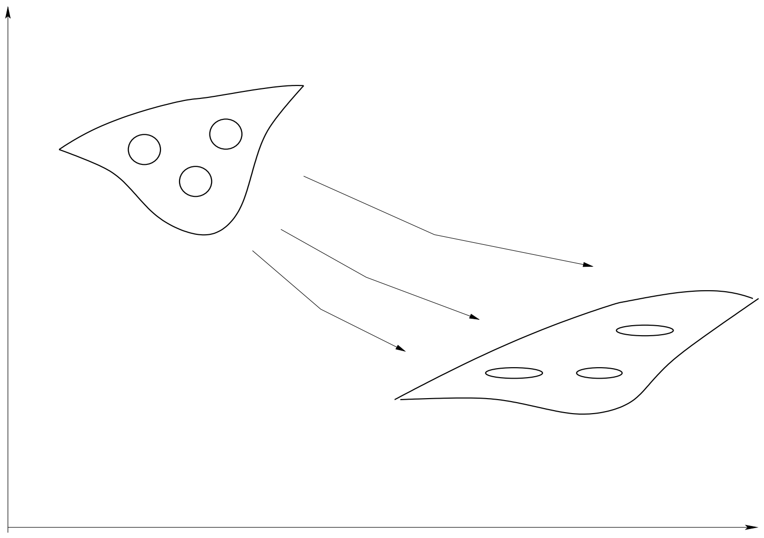
$$\frac{dx}{dt} = f(x,t) \qquad x \in \mathbb{R}^n.$$

(a) Autonomous Equations: $f(x,t) = f(x)$

Solution: $x = x(t, x_0)$;

Map: $T(x_0) = x(1, x_0)$; $T^n(x_0) = x(n, x_0)$.

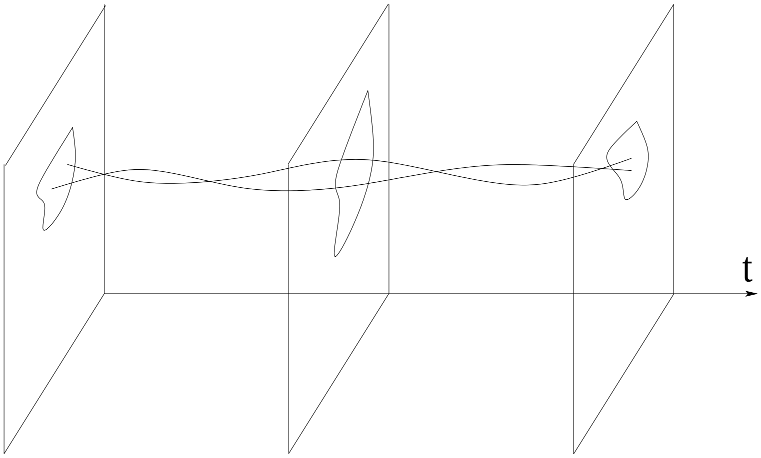
Arnold's cat: $U \to TU$.



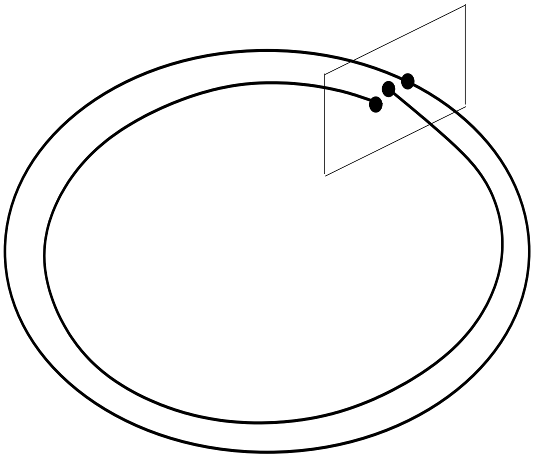
(b) Non-auto Equations: $f(x,t) = f(x,t+T)$

Solution: $x(t,x_0)$;

Map: $T(x_0) = x(T,x_0)$; $T^n(x_0,T) = x(nT,x_0)$.



(c) Poincáre section.



Homework

1. Find T^{-1} for Ex. 2-4 in the first two pages.

2. Let x_0 be a periodic point of period n in $[0, 1]$ for $T(x) = 7x(1 - x)$. Is x_0 stable? Why?

3. Let $T(\theta) = 3\theta$ is defined on S^1. Prove that T is measure preserving.

4. Let $x \in \mathbb{R}^n$, and $A = (a_{ij})_{n\times n}$ be a constant matrix. Find the time-1 map of the equation $\frac{dx}{dt} = Ax$.

5. For the given set of differential equations

$$\begin{aligned}\frac{dx}{dt} &= x + y - x(x^2 + y^2)^{\frac{1}{2}}\\ \frac{dy}{dt} &= -x + y - y(x^2 + y^2)^{\frac{1}{2}}\end{aligned}$$

(a) Find all periodic solutions of this equation.

(b) Let S be the y-axis. Find the Poincare map induced by this equation around the indicated periodic solutions.

6. Find the time-T map of the following set of differential equations

$$\begin{aligned}\frac{dr}{dt} &= -r + \sin\theta \sum_{n=-\infty}^{\infty} \delta(t - nT)\\ \frac{d\theta}{dt} &= 1 + r\end{aligned}$$

where $\delta(t)$ is the standard δ-function, T is a fixed constant.