Area Preserving Maps: Normal Form

1. Formal Series and Normal Forms

A. *Formal Series*

Let

$$f(x,y) = \sum_{k=1}^{\infty} f_k(x,y)$$

where x, y are two variables and

$$f_k(x,y) = \sum_{j=0}^{k} f_j^{(k)} x^j y^{k-j}$$

is a collection of terms of degree k in x and y. Let us for the moment allow the coefficients to be arbitrary complex numbers.

In this part of the lecture we disregard the issue of convergence. We call $f(x,y)$ a formal series, indicating that it is an algebraic object, not a function defined on numerical domains. It is, however, easy to make sense of many mathematical operations on the set of all formal series:

(i) Let $f(x,y), g(x,y)$ be two formal series, $f+g$ and $f\cdot g$ are trivially so. Also $\frac{1}{1-f(x,y)}$ is naturally defined as a formal series. (Note that it is not so for $\frac{1}{f(x,y)}$)

(ii) Let $f(x,y), \phi(\xi,\eta), \psi(\xi,\eta)$ be formal series, then the composition $f(\phi(\xi,\eta),\psi(\xi,\eta))$ are also trivially so.

(iii) Similarly are the partial derivatives $\partial_x f(x,y)$, $\partial_y f(x,y)$ for a formal series $f(x,y)$. Further more, for a formal mapping, DT, the matrix of derivatives, and $\det(DT)$, the determinant of DT, is well-defined as well.

(iv) We call $T : (x, y) \to (f(x, y), g(x, y))$ a formal mapping. We say that $S : (x, y) \to (F(x, y), G(x, y))$ is an inverse of T (denote as $S = T^{-1}$) if

$$f(F(x, y), G(x, y)) = x; \qquad g(F(x, y), G(x, y)) = y.$$

(v) The collection of all (real or complex valued) analytic functions of two variables $f(x, y)$ satisfying $f(0, 0) = 0$ is a subset of the collection of all formal series.

(vi) Let T, S be two given formal mappings. We say T and S are formally conjugated if there exists a formal mapping H such that

$$T \circ H = H \circ S.$$

B *Questions of Study*

(a) When can we formally conjugate a formal mapping T to its linear part?

Remark: In the case of one variable, T is formally conjugated to its linear part if the linear coefficient is not a root of unity. We will see a similar condition for the formal series of two variables.

(b) If T is not conjugated to its linear part, then what is the next best we can do?

Remark: We try, by using conjugating mappings, to remove from T the terms that *do not effect* the dynamical structure of T. If we can conjugate T to its linear part, then we are more or less showing that the higher order terms in T does not effect the dynamical structure of T. If T can not be conjugated to its linear part, then the higher order terms (at least some) make differences. In this case we would like to identify these terms. We will use conjugating mapping to remove as many terms as we could. non-removable terms are the things that make the difference.

Take for example, formal series of one variables as an example: if λ is a root of unity, say $\lambda^k = 1$, then T in general is not formally conjugated to $S : z \to \lambda z$. However, we can use a conjugating series to inductively remove as much as we could. From the previous lecture we see that the non-removable terms are in the form of $(z^{k+1})^n$. So use a formal conjugating series we can conjugate T to

$$S : z \to \lambda z + \sum_{n=1}^{\infty} a_n (z^k)^n.$$

S a **Normal Form** for T.

We can now present (b) in more precise terms: In the case (a) fails to be true for T, what are the *Normal Forms* for T? (i.e. identify the terms we can not remove by a conjugating mapping).

C. *A non-resonance condition*

Let $T : (x, y) \to (f(x, y), g(x, y))$ be a formal mapping and

$$f(x, y) = ax + by + \textstyle\sum_{k=1}^{\infty} f_k$$

$$g(x, y) = cx + dy + \textstyle\sum_{k=1}^{\infty} g_k.$$

Let λ, μ be the eigenvalues of the linear part for T. We say that λ, μ are not resonant if $\lambda \neq \mu$, and $\lambda^i \mu^j \neq \lambda, \mu$ for all $p, q \geq 0$, $p + q > 1$.

Claim 1.1: If λ and μ are not resonant, then T is formally conjugate to its linear part.

Proof: First we turn the linear part diagonal by using a linear transformation (because $\lambda \neq \mu$). So we can assume without loss of generality that

$$f = \lambda x + \sum_{k>1} f_k, \quad g = \mu y + \sum_{k>1} g_k.$$

Let us now introduce the formal coordinate change $C : (\xi, \eta) \to (x, y)$ by

$$x = \xi + \sum_{k>1} \phi_k(\xi, \eta), \quad y = \eta + \sum_{k>1} \psi_k(\xi, \eta)$$

where

$$\phi_k(\xi, \eta) = \sum_{j=0}^{k} a_j^{(k)} \xi^j \eta^{k-j}, \quad \psi_k(\xi, \eta) = \sum_{j=0}^{k} b_j^{(k)} \xi^j \eta^{k-j}.$$

We need to inductively determine $a_j^{(k)}$, $b_j^{(k)}$ so that

$$C \circ S = T \circ C$$

where S is such that $(\xi, \eta) \to (\lambda\xi, \mu\eta)$. This implies that for all k we must have

$$\phi_k(\lambda\xi, \mu\eta) = \lambda\phi_k(\xi, \eta) + \cdots, \quad \psi_k(\lambda\xi, \mu\eta) = \mu\psi_k(\xi, \eta) + \cdots$$

where the dotted terms are degree k of coefficients in $f_j^{(n)}, g_j^{(n)}, n \leq k$, and $a_j^{(n)}, b_j^{(n)}$, $n < k$. Write these two equations explicitly in the powers of ξ and η, we obtain

$$\sum_{j=0}^{k} (\lambda^j \mu^{k-j} - \lambda) a_j^{(k)} \xi^j \eta^{k-j} = \cdots ,$$

$$\sum_{j=0}^{k} (\lambda^j \mu^{k-j} - \mu) b_j^{(k)} \xi^j \eta^{k-j} = \cdots .$$

From the non-resonance assumption, we can solve these equations to uniquely determine all $a_j^{(k)}$ and $b_j^{(k)}$. $\square$

Remarks: (i) This Claim gives a complete answer to Question (a).

(ii) A answer for Question (b) is also attained from the the same computation. If the non-resonance condition is violated, say $\lambda^{n_0}\mu^{m_0} = \lambda$, then by formal coordinate change, the terms $\xi^{n_0}\eta^{m_0}$, $(\xi^{n_0}\eta^{m_0})^{m_0}\eta_{n_0}$ and so on can not be removed from the first equation. Note that there might be more than one resonance relations so the general criterion on the non-removable terms depends sensitively on λ and μ.

(iii) To get a balanced equation, we need to replace the linear mapping S in $CS = TC$ by an S with non-removable terms, resulting a **normal form** that is not linear for T (We will see concrete examples with more details soon).

(iv) If the case non-resonance condition is violated, normal forms for T are **not unique**. At any point we face a term of resonance, the coefficients of the resonance terms in conjugating function can be taken arbitrarily. One natural way to make the choice unique is to simply set all the arbitrary coefficients to zero. We can also uniquely determine these coefficients in other ways.

2. Area Preserving Mappings

A *Normal Forms*

Let $T : (x, y) \to (f(x, y), g(x, y))$ be as in the above. We say that T is **area-preserving** if

$$\partial_x f \cdot \partial_y g - \partial_y f \cdot \partial_x g = 1.$$

– If T is area-preserving, then $\lambda \cdot \mu = 1$. So the non-resonance condition is always violated for area-preserving maps.

– Let us also assume that λ *is not a root of unity*. In this case, it is easy to determine the terms involved in normal forms. We can construct formal conjugating function C such that $C \circ S = T \circ C$, and $S : (\xi, \eta) \to (\xi_1, \eta_1)$ is in the form of

$$\xi_1 = u(\xi\eta)\xi, \quad \eta_1 = v(\xi\eta)\eta,$$

where

$$u = \lambda + \sum_{n=1}^{\infty} \alpha_{2n}(\xi\eta)^n, \qquad v = \mu + \sum_{n=1}^{\infty} \beta_{2n}(\xi\eta)^n.$$

(If λ is a root of unity, then we can not assume this normal form. Homework: to determine the appropriate normal form if λ is a root of unity.)

– In this case $C \circ S = T \circ C$ is written explicitly as

$$\begin{aligned} \phi_{2n+1}(\lambda\xi, \mu\eta) + \alpha_{2n}(\xi\eta)^n\xi &= \lambda\phi_{2n+1}(\lambda, \mu) + \cdots \\ \psi_{2n+1}(\lambda\xi, \mu\eta) + \beta_{2n}(\xi\eta)^n\eta &= \lambda\phi_{2n+1}(\lambda, \mu) + \cdots . \end{aligned}$$

Observe that, the resonance terms on the right are now balanced by the terms of α_{2n} and β_{2n}.

– Let us now detail two ways to uniquely determine a normal form for T. For the first we simply set all arbitrary coefficients of ϕ_k and ψ_k to zero in the inductive process. This is equivalent to

(N1) $\partial_\xi \phi - 1$ and $\partial_\eta \psi - 1$ contains no terms in the form of $(\xi\eta)^n$.

The second way is to impose

(N2) $\partial_\xi \phi - \partial_\eta \psi$ and $\partial_\xi \phi \partial_\eta \psi - \partial_\eta \phi \partial_\xi \psi - 1$ contains not terms in the form of $(\xi\eta)^n$.

Remarks on (N2): (i) That (N2) uniquely determines a conjugating mapping and a normal form is left as a homework.

(ii) The reason for (N2) is as follows: area preservation is an important property for T we strongly desire to preserve in the course of the coordinate changes, i.e., we desire that $S := C^{-1}TC$ is also area preserving. One way to achieve that is to make C area preserving, i.e. to insist on

$$\partial_\xi \phi \partial_\eta \psi - \partial_\eta \phi \partial_\xi \psi = 1.$$

However, this equality is possibly too much to ask: we are potentially imposing self-contradicting conditions. By (N2) we try as hard as we could in making C area preserving.

(iii) At this moment, it is not clear if any of the normal forms obtained from T is area preserving.

B. *The Area preserving character*

Claim 2.1: Let $S = C^{-1}TC$ where C is determined by (N1) and $S = C^{-1}TC$ where C is determined by (N2). Then both S are area-preserving.

Proof for (N2): Denote $\omega = \xi\eta$ and recall that $S : (\xi, \eta) \to (\xi_1, \eta_1)$ is such that

$$\xi_1 = u(\omega)\xi, \qquad \eta_1 = v(\omega)\eta.$$

We have

$$\det(DS) = \begin{vmatrix} u + u_\omega \omega & u_\omega \xi^2 \\ v_\omega \eta^2 & v + v_\omega \omega \end{vmatrix} = (uv\omega)_\omega = 1 + \cdots .$$

We also let $\tau(\xi, \eta) = \det(DC)$. Then from $C \circ S = T \circ C$ we obtain

$$\tau(u\xi, v\eta)(1 + \cdots) = \tau(\xi, \eta) \qquad (1)$$

where $\cdots$ are terms exclusively in ω (Here we used $\det(DT) = 1$).

We are now ready to show $\tau(\xi, \eta) = 1$. If not, let τ_k be the terms of lowest degree for $\tau - 1$ (The constant terms for τ is one by definition). We have

$$\tau_k(u\xi, v\eta) = \tau_k(\xi, \eta).$$

Assume $\tau_k = \sum_{j=1}^{k} \gamma_j \xi^j \eta^{k-j}$. We must have

$$\gamma_j(\lambda^j \mu^{k-j} - 1) = 0.$$

Note that we have assumed $\gamma_j = 0$ for $j = k/2$ by (N2). For $j \neq k/2$ we must have $\gamma_j = 0$ because

$$\lambda^j \mu^{k-j} - 1 = \mu^{k-2j} - 1 \neq 0.$$

Proof for (N1) We make the following observations

– Let C_1 be determined by (N1) and C_2 be determined by (N2) Then $C_2^{-1} C_1$ transfers one normal form to another. However, any coordinate change that turns one normal form into another must have the form

$$(\xi, \eta) \to (\xi \Phi(\xi\eta), \eta \Psi(\xi\eta)) \tag{2}$$

(We leave the proof of this claim as a homework).

– Let $S : (\xi, \eta) \to (\xi_1, \eta_1)$ be such that

$$\xi_1 = u(\xi\eta)\xi, \qquad \eta_1 = v(\xi\eta)\eta.$$

Then S is area preserving if and only if $uv = 1$ (or $\xi_1\eta_1 = \xi\eta$). This is because

$$\det(DS) = (uv\omega)_\omega.$$

– For $S_1 : (\xi, \eta) \to (\xi_1, \eta_1)$ and $S_2 : (\hat{\xi}, \hat{\eta}) \to (\hat{\xi}_1, \hat{\eta}_1)$ we have

$$\xi_1\eta_1 = \hat{\xi}_1\hat{\eta}_1\Phi(\hat{\xi}_1\hat{\eta}_1)\Psi(\hat{\xi}_1\hat{\eta}_1) = \hat{\xi}\hat{\eta}\Phi(\hat{\xi}, \hat{\eta})\Psi(\hat{\xi}\hat{\eta}) = \xi\eta$$

Where the first and the last equalities are by (2) and the middle one by $\hat{\xi}_1\hat{\eta}_1 = \hat{\xi}\hat{\eta}$ (Here we use the fact that S_2 is area-preserving). $\square$

Remark: Observe in the above we did not use (N1) in determining the normal form. So we actually proved that any normal form for T obtained by the inductive process detailed above is area-preserving.

C. *Hyperbolic case:* $|\lambda| > 1 > |\mu|$

The discussions in the above is for the study of the mappings $T : (x, y) \to (f(x, y), g(x, y))$ where $f(x, y), g(x, y)$ are analytic functions around $(0, 0)$ satisfying $f(0, 0) = g(0, 0) = 0$.

Let $T : (x, y) \to (f(x, y), (x, y))$ be an area preserving map that is real analytic around $(x, y) = (0, 0)$. Let λ, μ be the eigenvalues of the matrix $DT_{(0,0)}$, and C, S be the formal coordinate change and the normal form in the above satisfying (N1).

Claim 2.2: Under the assumption that $|\lambda| > 1 > |\mu|$, the formal series constructed above (ϕ, ψ for C and u, v for C) are convergent in a small neighborhood of $(0, 0)$.

Proof: This is another example of the majorant argument.

– For $F(x, y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$. We write

$$F(x, y) = \sum_{n=-\infty}^{\infty} F_n(x, y)$$

where

$$F_n(x, y) = \sum_{i,j \geq 0; i-j=n} a_{ij} x^i y^j.$$

For instance,

$F_0(x, y) = a_{00} + a_{11} xy + \cdots + a_{nn} x^n y^n + \cdots$;

$$F_1(x,y) = a_{10}x + a_{21}x^2y + \cdots + a_{n+1n}x^{n+1}y^n + \cdots ;$$

$$F_{-1}(x,y) = a_{01}y + a_{12}xy^2 + \cdots + a_{nn+1}x^ny^{n+1} + \cdots$$

and so on. Observe that

$$F_n(ux, u^{-1}y) = u^n F_n(x,y).$$

– WLOG, let us assume that the linear part of T is already in diagonal form. Let

$$\widehat{f}(x,y) := f(x,y) - \lambda x; \quad \widehat{g}(x,y) := g(x,y) - \mu y;$$

$$G(x,y) = \frac{c(x+y)^2}{1 - c(x+y)}.$$

Then there exists $c > 0$ such that

$$\widehat{f}(x,y), \widehat{g}(x,y) \rhd G(x,y).$$

(This is easy to see as follows. For $f = \sum a_{ij}x^iy^j$ let $|f| = \sum |a_{ij}|x^iy^j$. We have

$$f(x,y) \rhd |f|(x,y) \rhd |f|(x+y, x+y)$$

By the assumption that f is convergent around $(x,y) = (0,0)$, so are $|f|(x,y)$ and $|f|(x+y,x+y)$. The last one is $\rhd G(x,y)$ for some $c > 0$.)

– Let C be $(\xi,\eta) \to (\phi,\psi)$ and S be $(\xi,\eta) \to (u\xi, v\eta)$. We know that $u = v^{-1}$. $C \circ S = T \circ C$ is written as

$$\begin{aligned} \phi(u\xi, u^{-1}\eta) - \lambda\phi(\xi,\eta) &= \widehat{f}(\phi,\psi) \\ \psi(u\xi, u^{-1}\eta) - \mu\psi(\xi,\eta) &= \widehat{g}(\phi,\psi) \end{aligned}$$

from which we obtain

$$(u^n - \lambda)\phi_n = (\hat{f}(\phi, \psi))_n; \quad (u^n - \mu)\psi_n = (\hat{g}(\phi, \psi))_n \quad (3)$$

for $n = \pm 1, \pm 2, \cdots$.

– Condition (N1) is now written as

$$\phi_1 = \xi, \qquad \psi_{-1} = \eta$$

so we have

$$(u - \lambda)\xi = (\hat{f}(\phi, \psi))_1, \quad (v - \mu)\eta = (\hat{g}(\phi, \psi))_{-1}. \qquad (4)$$

– It is straight forward to check that there exists $c_1 > 0$ such that for all $n \neq 1$

$$(u^n - \lambda)^{-1} \rhd \frac{c_1}{1 - c_1|v - \mu|}.$$

Similarly, we have for $n \neq -1$,

$$(u^n - \mu)^{-1} \rhd \frac{c_1}{1 - c_1|v - \mu|}$$

where $|v - \mu|$ is obtained by change all coefficients of the expansion of $v - \mu$ to their absolute values (These are left as homework). This is combined with (3) to give

$$|\phi| - \xi, \ |\psi| - \eta \rhd \frac{c_1}{1 - c_1|v - \mu|} G(|\phi|, |\psi|).$$

We also have from (4) that

$$\eta|v - \mu| \rhd G(|\phi|, |\psi|).$$

– Let

$$W(\xi) = \frac{1}{\xi}(|\phi(\xi,\xi)| - \xi + |\psi(\xi,\xi)| - \xi) + |v(\xi,\xi) - \mu|.$$

We obtain

$$\xi W \triangleright \frac{c_2}{1 - c_1 W} G(\xi(W+1), \xi(W+1)).$$

We can now easily confirm the convergence of $W(\xi)$ because everything in the last displayed line is explicit in W. Convergence of W implies convergence for ϕ, ψ and v. $\square$

D. *Real elliptic case:* $|\lambda| = |\mu| = 1$

We are interested in the case when x, y are two real variables and $f(x,y), g(x,y)$ are real valued analytic functions defined on a small neighborhood of $(x,y) = (0,0) \in \mathbb{R}^2$. In this case, T is area preserving means that the area U and $T(U)$ are the same.

Under the assumptions that both f and g are real analytic and T is area preserving, we have the following three possibilities for λ and μ:

(i) (Parabolic case) λ and μ are roots of unity.

The case $\lambda^k = 1$, $k \neq 1$ can be reduced to the cases $\lambda = \mu = 1$ by considering T^k instead of T. This is a *highly degenerate* case in which the dynamical structure of the linear part of the mapping (every point in phase space is a fixed point) does not give any indication of the dynamical structure of the mappings with higher order terms.

(ii) (Hyperbolic case) $\lambda \neq \mu$ and both are real.

We have $|\lambda| > 1 > |\mu|$ from $\lambda \cdot \mu = 1$. In this case the formal changing of coordinates and the normal form derived above are with real coefficients, and the majorant argument assures the convergence of both. So restricted to the real domain, there is a real analytic coordinate change that transfers T to a real analytic normal form S. We conclude that T analytically conjugate to its normal forms, which is not exactly its linear part but with a similar dynamical structure.

(iii) (Elliptic case) λ and μ are both complex.

From $\lambda\mu = 1$ (T is area preserving) and $\mu = \bar{\lambda}$ (both λ and μ are eigenvalues of a real matrix), we have $|\lambda| = 1$. Write λ as $e^{2\pi\alpha i}$. λ is not a root of unity implies that α is irrational. Formal coordinate changes and normal forms are derived above.

Note that, however, in turning the linear part of T into diagonal form, we moved out the domain of real numbers so both the formal coordinate change and the normal form are derived in complex domain. This is not exactly what we want: a *real coordinate change* that gives a *real normal form*.

3. Real normal form in elliptic case

Recall that $T : (x, y) \to (x_1, y_1)$ where

$$\begin{aligned} x_1 &= f(x,y) = ax + by + \sum_{k>1} f_k(x,y) \\ y_1 &= g(x,y) = cx + dy + \sum_{k>1} g_k(x,y) \end{aligned}$$

where

$$f_k = \sum_{j=0}^{k} f_j^{(k)} x^j y^{k-j}, \quad g_k = \sum_{j=0}^{k} g_j^{(k)} x^j y^{k-j}.$$

Here we assume that *all coefficients are real*.

- Let (p_1, p_2) be the complex eigenvector for λ, then $(\bar{p}_1, \bar{p}_2)$ is the complex eigenvector for μ under the assumption that $\mu = \bar{\lambda}$. To make the linear part diagonal, we introduce complex coordinate change $(x, y) \to (X, Y)$ by

$$x = p_1 X + \bar{p}_1 Y, \quad y = p_2 X + \bar{p}_2 Y.$$

T is then conjugated to $S : (X, Y) \to (X_1, Y_1)$ where

$$X_1 = p(X, Y), \quad Y_1 = q(X, Y).$$

Claim 3.1: Under the assumption that $f(x, y)$ and $g(x, y)$ are *real* analytic, we have

$$q(X, Y) = \bar{p}(Y, X)$$

where $\bar{p}(\cdot, \cdot)$ is obtain by change all coefficients to their conjugates.

The proof of this claim is left as a homework.

• We now introduce coordinate change (again in complex coefficients) to conjugate S to its normal form. Let

$$X = \phi(\xi, \eta), \qquad Y = \psi(\xi, \eta)$$

be the coordinate change and

$$\xi_1 = u(\xi\eta)\xi, \qquad \eta_1 = v(\xi\eta)\eta.$$

be the normal form. u, v, ϕ and ψ are obtain from

$$\begin{aligned} \phi(u\xi, v\eta) &= p(\phi(\xi,\eta), \psi(\xi,\eta)) \qquad (5) \\ \psi(u\xi, v\eta) &= q(\phi(\xi,\eta), \psi(\xi,\eta)) \qquad (6) \end{aligned}$$

Note that u, v, ϕ and ψ are *uniquely* determined if we further asking

$$\partial_\xi \phi(\xi,\eta) - 1, \partial_\eta \psi(\xi,\eta) - 1 \qquad (7)$$

contains no terms in the power of $\xi\eta$ alone.

Claim 3.2: We have $v(\xi\eta) = \bar{u}(\eta\xi)$, $\psi(\xi,\eta) = \bar{\phi}(\eta,\xi)$.

Proof: It suffices to check that if $u(\xi\eta), v(\xi\eta)$, $\phi(\xi,\eta)$ and $\psi(\xi,\eta)$ are solutions for (5)-(7), then $\hat{u} := \bar{v}(\eta\xi), \hat{v} := \bar{u}(\eta\xi), \hat{\phi} := \bar{\psi}(\eta,\xi)$ and $\hat{\psi} := \bar{\phi}(\eta,\xi)$ is also a set of solution for (5)-(7). Our claim then follows from the uniqueness of the solutions.

(5) for $\hat{u}, \hat{v}, \hat{\phi}$ and $\hat{\psi}$ is written as

$$\hat{\phi}(\hat{u}\xi, \hat{v}\eta) = p(\hat{\phi}, \hat{\psi}),$$

which we write in $u(\xi\eta), v(\xi\eta), \phi(\xi,\eta)$ and $\psi(\xi,\eta)$ as

$$\bar{\psi}(\bar{u}(\eta\xi)\eta, \bar{v}(\eta\xi)\xi) = p(\bar{\psi}(\eta,\xi), \bar{\phi}(\eta,\xi))$$

To see the last equality hold we take conjugation on both side to obtain

$$\psi(u(\eta\xi)\eta, v(\eta\xi)\xi) = \bar{p}(\psi(\eta,\xi), \phi(\eta,\xi))$$

By using Claim 3.1 we obtain

$$\psi(u(\eta\xi)\eta, v(\eta\xi)\xi) = q(\phi(\eta,\xi), \psi(\eta,\xi)). \qquad (8)$$

Switching ξ to η and η to ξ in the last equality we obtain

$$\psi(u(\xi\eta)\xi, v(\xi\eta)\eta) = q(\phi(\xi,\eta), \psi(\xi,\eta)). \qquad (9)$$

(9) is identical to (6) for u, v, ϕ and ψ (Note that (8) and (9) are equivalent in the sense that if one hold then so does the other).

Similarly (6) for $\hat{u}, \hat{v}, \hat{\phi}$ and $\hat{\psi}$ is equivalent to (5) for u, v, ϕ and ψ. (7) for $\hat{u}, \hat{v}, \hat{\phi}$ and $\hat{\psi}$ is a triviality.

- If we only consider real pairs for (x, y), then $X = \bar{Y}$. Let $X_R = Re(X), X_I = Im(X)$, then from (x, y) to (X_R, X_I) is a real coordinate transform. Also from Claim 2 we have $\xi = \bar{\eta}$ if $X = \bar{Y}$ ($X = \phi(\xi, \eta)$ so $Y = \bar{X} = \psi(\bar{\eta}, \bar{\xi})$. But $Y = \psi(\xi, \eta)$ so $\xi = \bar{\eta}$.)

– Let us write

$$\xi = \xi_R + i\xi_I; \qquad X = X_R + iX_I.$$

Then $(X_R, X_I) \to (\xi_R, \xi_I)$ is a real coordinate change defined by

$$X_R = \frac{1}{2}(\phi(\xi_R + i\xi_I, \xi_R - i\xi_I) + \bar{\phi}(\xi_R - i\xi_I, \xi_R + i\xi_I));$$

$$X_I = \frac{1}{2i}(\phi(\xi_R + i\xi_I, \xi_R - i\xi_I) - \bar{\phi}(\xi_R - i\xi_I, \xi_R + i\xi_I)).$$

– Let us write

$$u(\xi\eta) = \sum_{k \geq 0} \alpha_k (\xi\eta)^k.$$

We obtain

$u(\xi, \eta) = A + iB; \quad v(\xi, \eta) = A - iB, \quad A^2 + B^2 = 1$

where

$$A = \sum_{k \geq 0} Re(\alpha_k)(\xi_R^2 + \xi_I^2)^k;$$

$$B = \sum_{k \geq 0} Im(\alpha_k)(\xi_R^2 + \xi_I^2)^k.$$

Hence for the elliptic case, we have the following *REAL* normal form

$$(\xi_R)_1 = A\xi_R - B\xi_I, \quad (\xi_I)_1 = A\xi_I + B\xi_R.$$

– Use now the polar coordinates (r, θ) for (ξ_R, ξ_I), we obtain finally

$$r_1 = r; \quad \theta_1 = \theta + 2\pi\alpha + \sum_{k>1} \gamma_k r^{2k}.$$

This is a normal form for T in elliptic case (From the rectangular form to the polar form is left as an exercise).

Remarks on convergence

(a) If the formal series constructed above converge, then the dynamical structure for T conjugate to that of the normal form, which is relatively simple: Every circle centered at $(0, 0)$ is invariant. In addition, the dynamics induced on each of these circles is simply a rigid rotation.

(b) Unlike the hyperbolic case, the formal series and the normal form constructed above **do not** converge in general. For a long time in history, the dynamical structure of an area-preserving mapping around an elliptic fixed point is a staggering mathematical mystery that prevented real progresses.

(c) Let us present the problem in more precise terms: Let us assume that $k_0 > 0$ is the smallest integer such that $\gamma_{k_0} \neq 0$. Then we clearly (following the process of constructing normal form) can find a real analytic change of coordinates that conjugate T to

$$\begin{aligned} r_1 &= r + \mathcal{O}(r^{2k_0+1}) \\ \theta_1 &= \theta + 2\pi\alpha + \gamma_{k_0} r^{2k_0} + \mathcal{O}(r^{2k_0+1}). \end{aligned}$$

Question: Without the higher order $\mathcal{O}(\cdot)$-terms, the dynamical structure is easy. What is the effect of the higher order perturbations on such relatively clean structure?

Answer to this question is in the next lecture, for which the normal form above is only a motivational first step.