

# Area Preserving Maps: KAM Theory

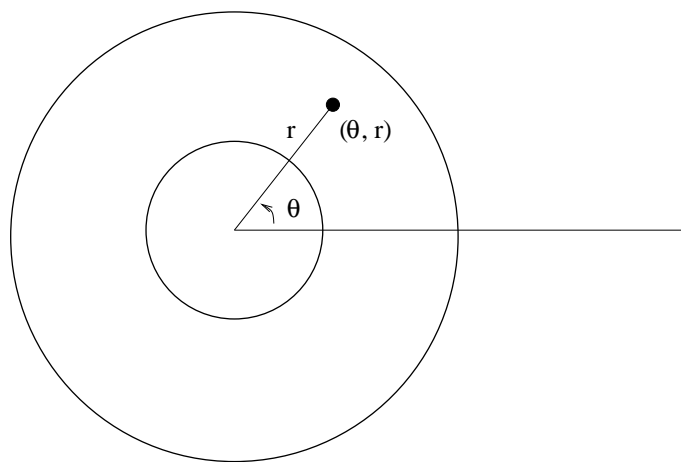
Let  $T : (\theta, r) \rightarrow (\theta_1, r_1)$  be such that

$$\theta_1 = \theta + r + f(\theta, r),$$

$$r_1 = r + g(\theta, r)$$

where  $f(\theta, r), g(\theta, r)$ , real analytic on  $\mathbb{R}^1 \times [a_0, b_0]$ , are such that

$$f(\theta + 2\pi, r) = f(\theta, r), \quad g(\theta + 2\pi, r) = g(\theta, r).$$



$T$  represents a real analytic mapping on  $A := S^1 \times [a_0, b_0]$ .  $(\theta, r)$  are the polar coordinates representing the annulus  $A$ .

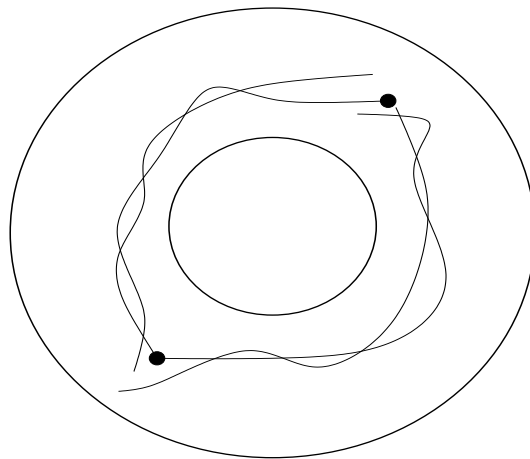
We call  $T$  integrable if  $f = g = 0$ . The dynamical structure of the integrable system is easy to understand: that every circle of constant radius is invariant, and the dynamics induced on a given invariant circle is a rigid rotation with the radius as its rotation number.

**Question:** When small perturbations  $f(\theta, r)$  and  $g(\theta, r)$  are added to the integrable system, what happens to individual invariant circles?

**Remark:** Let us make a connection between the answer to this question and the convergence of the formal series constructed in the previous lecture. If these series converge, then the orbit structure of the perturbed system is identical to that of an integrable system, and all invariant circles under small perturbation are only deformed.

This is, however, not the case in general. Invariant circles of rational rotation number are extremely venerable: perturbations in general create hyperbolic periodic

orbit and transversal intersections of stable and unstable manifolds. The formal series constructed in previous lecture simply do not converge.



For a long period of time, progress was on hold until late 1950's when Kolmogorov observed that the invariant circles of *irrational rotation numbers* are much more robust than those of rational rotation numbers: majority of them will persist under small perturbations. Kolmogorov also introduced an iteration technique to overcome a major technical hurdle, which we have used in proving the existence of Siegel's disk in previous lectures.

In early 1960's, two detailed proofs for Kolmogorov's theorem occurred independently, one by Arnold, and the other by Moser, both with substantial technical improvements. (Arnold on non-degenerate case and on applications to celestial mechanics, Moser with the removal

of real analytic condition). This theorem is thereafter commonly referred to as the KAM theory.

**Standing Assumption on  $T$ :** For any given simple closed curve  $\gamma$  around  $r = 0$  in  $A$ , we have  $T(\gamma) \cap \gamma \neq \emptyset$ .

We remark that the standing assumption is fulfilled if  $T$  is area-preserving.

### Notations:

(a) We call  $\omega \in [a_0, b_0]$  a Diophantine rotation number of type  $(c_0, \mu)$  for  $c_0, \mu > 0$  if

$$\left| \frac{\omega}{2\pi} q - p \right| > \frac{c_0}{q^\mu}$$

for all  $q, p \in \mathbb{Z}^+$ .

(b) Let  $u(\xi), v(\xi)$  be periodic functions of period  $2\pi$  and assume that  $\gamma : \xi \rightarrow (\xi + u(\xi), v(\xi))$  is a simple closed curve in  $A$ . We say that  $\gamma$  is

an  $\varepsilon$ -deformation of the circle of radius  $\omega$  for  $\varepsilon, \omega > 0$  if

$$|u(\xi)| + |v(\xi) - \omega| < \varepsilon.$$

(c) We say that  $\gamma$  in (b) is an invariant curve if for every  $\xi$  there exists  $\xi_1$  such that  $T(\gamma(\xi)) = \gamma(\xi_1)$ .

(d) Assume that  $\gamma$  above is invariant under  $T$ . We say that the dynamics induced by  $T$  on  $\gamma$  is a rigid rotation of angle  $\omega$  if

$$\xi_1 = \xi + \omega.$$

**Theorem (KAM Theory)** Let  $\omega$  be a Diophantine number of type  $(c_0, \mu)$  and  $\varepsilon > 0$ . Then there exists  $\delta > 0$  depending on  $c_0, \mu$  and  $\varepsilon$ , such that for any  $T$  satisfying the standing assumption and

$$|f(\theta, r)| + |g(\theta, r)| < \delta,$$

$T$  admits an invariant curve  $\gamma$  that is an  $\varepsilon$  deformation of the circle of radius  $r$ . Furthermore, the dynamics induced by  $T$  on  $\gamma$  is a rigid rotation of angle  $\omega$ .

## Strategy of Proof

### A. *Conjugating to integrable system*

Let us first set up a false goal: we want to find a coordinate change  $C : (\xi, \eta) \rightarrow (\theta, r)$  on  $A$ , written as

$$\theta = \xi + u(\xi, \eta), \quad r = \eta + v(\xi, \eta)$$

so that  $C$  conjugate  $T$  to the unperturbed part of  $T$ , i.e.,  $C$  is such that  $S = C^{-1}TC$  assume that form

$$\xi_1 = \xi + \eta, \quad \eta_1 = \eta.$$

If  $C$  exists, then  $CS = TC$ . Write in concrete terms we must have

$$u(\xi + \eta, \eta) = u(\xi, \eta) + v(\xi, \eta) + f(\xi + u, \eta + v)$$

$$v(\xi + \eta, \eta) = v(\xi, \eta) + g(\xi + u, \eta + v).$$

It is often impossible to solve equations of this kind because of the convoluted involvement of  $u$  and  $v$  in the perturbational terms  $f$  and  $g$ .

We try to imitating our earlier proof on Siegel's theorem: instead of finding the conjugating mapping  $C$  in one step, we drop  $u$  and  $v$  in  $f$  and  $g$  to form a new set of equations

$$u(\xi + \eta, \eta) = u(\xi, \eta) + v(\xi, \eta) + f(\xi, \eta)$$

$$v(\xi + \eta, \eta) = v(\xi, \eta) + g(\xi, \eta).$$

We hope that

(a) We can solve this new set of equations, and effectively control the properties of functions  $u$  and  $v$  obtained.

(b) Though the coordinate change  $C$  so obtained is not going to conjugate  $T$  to an integrable system, it transfers  $T$  into a new  $S_1 := C^{-1}TC$  that is much closer to the integrable system than  $T$  was.

(c) We then apply the same on  $S_1$  to get  $S_2$ , much closer to the integrable system than  $S_1$  and so on.

(d) We then establish the convergence of the sequence of coordinate changes to get an exact conjugation.

**Remarks:**

(1) This process was followed closely and the details worked out in a different setting (Siegel's disk).

(2) As we have already pointed out that it does not work in current situation.

## B. Working around $r = \omega$

The plan outlined above does not work because the conjugation  $C$  we hope to construct does not exist on  $A$  in general. Perturbations create hyperbolic periodic orbit and transversal intersections of stable and unstable manifolds. Hence we need to adjust our goal according to the observation made by Kolmogorov: That instead of working on  $A$ , we work only around the circle of radius  $\omega$ .

Observe that trying to find  $C$  in a fixed neighborhood of  $r = \omega$  is equally bad: such small neighborhood always contain circles of rational rotation number. To deal with this problem we *shrink the size of the neighborhood* around  $r = \omega$  in the iterative process. To be more precise, we construct  $C_n$  (The conjugation) on  $U_n$  (a small neighborhood around  $r = \omega$  at step

$n$ ), then we will shrink to  $U_{n+1} \subset U_n$  to construct  $C_{n+1}$  on  $U_{n+1}$ . So at the end we will be able to construct a conjugation between  $T$  and the integrable system, but this conjugation only works on exactly the circle of radius  $\omega$  for  $(\xi, \eta)$ . We will end up with a coordinate change that conjugating the dynamics of the integrable system to  $T$  on a curve instead of any two dimensional region in  $A$ . This is exactly what we hope to achieve.

In summary, we will work following the construction of conjugations outlined, we will however, working around  $r = \omega$ , and shrink the size of the domain of the desired conjugation as we go along to obtain, at the end, the invariant curve of rotation number  $\omega$  as claimed.

### **C.** *A cautions point: issues on average*

In principle, we prove the existence of the indicated invariant curve following exactly the

same outlines as in our previous proof on the existence of Siegel's disk. We caution, however, that this current case is technically more involved. Let us now illustrate the main reason for the technical complications.

Let us look back at the equations for  $u$  and  $v$  we intend to solve in one iteration. They are

$$\begin{aligned}u(\xi + \eta, \eta) &= u(\xi, \eta) + v(\xi, \eta) + f(\xi, \eta) \\v(\xi + \eta, \eta) &= v(\xi, \eta) + g(\xi, \eta).\end{aligned}$$

Note that  $\xi$  is an angular variable, so  $u$  and  $v$  have to be of period  $2\pi$  in  $\xi$ . Integrating on the second equation with respect to  $\xi$  over a period of  $2\pi$ , we conclude that

$$g^* := \frac{1}{2\pi} \int_0^{2\pi} g(\xi, \eta) d\xi = 0.$$

an equality that is not assumed to hold. So these equations need to be properly adjusted to define  $C$ , causing technical complications.

## 2. Conjugation in one iteration

To facilitate the writing we assume that  $\omega$  is a Diophantine number of type  $(1, 2)$ . To take advantage of the assumption that  $f(\theta, r)$  and  $g(\theta, r)$  are real analytic, we regard  $\theta$  and  $r$  as complex variables. Let

$$D_\rho = \{|\operatorname{Im}(\theta)| < \rho\}; \quad D_\sigma = \{|r - \omega| < \sigma\}$$

and  $D_{\rho, \sigma} = D_\rho \times D_\sigma$ .  $f(\theta, r), g(\theta, r)$  are real analytic on  $D_{\rho, \sigma}$  and

$$|f(\theta, r)| + |g(\theta, r)| < \delta.$$

We now introduce new coordinates  $(\xi, \eta)$ .  $C : (\xi, \eta) \rightarrow (\theta, r)$  is written as

$$\theta = \xi + u(\xi, \eta), \quad r = \eta + v(\xi, \eta).$$

Let  $S = C^{-1}TC$ ,  $S$  is then written as

$$\xi_1 = \xi + \eta + \phi(\xi, \eta), \quad \eta_1 = \eta + \psi(\xi, \eta).$$

We wish to find  $u(\xi, \eta), v(\xi, \eta)$  such that

$$|\phi(\xi, \eta)| + |\psi(\xi, \eta)| \ll \delta.$$

The following are the two main technical issues

(1) For the success of the indicated iteration scheme, we need to make the domain on which  $C$  and  $S$  are defined smaller. Let us assume that  $S$  is defined on  $D_{\hat{\rho}, \hat{\sigma}}$  where

$$D_{\hat{\rho}, \hat{\sigma}} = \{Im(\xi) < \hat{\rho}\} \times \{|\eta - \omega| < \hat{\sigma}\}.$$

for some  $\hat{\rho} < \rho, \hat{\sigma} < \sigma$ .

(2) A natural way to define  $C$  was discussed above. We indicated that the equations obtained by dropping  $u, v$  inside  $f, g$  are not necessarily solvable.

We now introduce the following modifications to define  $C$ . Let

$$u(\xi + \omega, \eta) = u(\xi, \eta) + v(\xi, \eta) + f(\xi, \eta) \quad (1)$$

$$v(\xi + \omega, \eta) = v(\xi, \eta) + g(\xi, \eta) - g^*(\eta) \quad (2)$$

where

$$g^*(\eta) = \frac{1}{2\pi} \int_0^{2\pi} g(\xi, \eta) d\xi.$$

**Main Lemma** Let  $\rho, \sigma, \hat{\rho}, \hat{\sigma} < 1$  be as in the above. Under the assumption that

$$\delta < K^{-1}(\rho - \hat{\rho})^7 \sigma$$

for a constant  $K^{-1}$  that will be determined later in the proof. We have

(a) Let  $C$  be the coordinate change determined by (1) and (2) and  $S := C^{-1}TC$ . Then  $S$  is well-defined on  $D_{\hat{\rho}, \hat{\sigma}}$ .

(b) For  $u, v$  so obtained we have

$$\begin{aligned} |u| + |v| &< (\rho - \hat{\rho})^{-6} \delta; \\ |\partial_\xi u| + |\partial_\xi v| &< (\rho - \hat{\rho})^{-7} \delta; \\ |\partial_\eta u| + |\partial_\eta v| &< (\rho - \hat{\rho})^{-6} \frac{\delta}{\sigma}. \end{aligned}$$

(c) For  $\phi, \psi$  we have

$$|\phi| + |\psi| < K \left\{ (\rho - \hat{\rho})^{-7} \left( \frac{\delta^2}{\sigma} + \sigma\delta \right) + \left( \frac{\hat{\sigma}}{\sigma} \right)^2 \delta \right\}.$$

**Remark:** (b) estimates the size of the deformation, which we claim is in the same magnitude of the size of the perturbation. (c) estimate  $|\phi| + |\psi|$ , which we hope is going to be  $\ll \delta$ . By directly looking into (c), it is not clear that we have achieved what we hoped for. That we in fact do will become clear later.

**Proof:** Let us skip the proof of (a), which is the trivial part of the three.

The estimates on derivatives in (b) follows easily from the first estimate and the Cauchy formula. To prove the first estimate in (b) let us study the following preparatory question for the moment. Let  $h(x)$  be real analytic on

$\{|Im(x)| < \rho\}$  satisfying  $h(x+2\pi) = h(x)$ ,  $|h(x)| < \delta$ . We want to find  $w(x)$  satisfying

$$w(x + \omega) - w(x) = h(x) - h^*.$$

To find a specific solution we write

$$h - h^* = \sum_{k \neq 0} h_k e^{-kxi}$$

and

$$w = \sum_{k \neq 0} w_k e^{-kxi}.$$

to obtain

$$w_k = \frac{h_k}{e^{k\omega i} - 1}.$$

**Sublemma 1:** For  $\hat{\rho} < \rho$  and on  $\{Im(x) < \hat{\rho}\}$ , we have

$$|w| < \delta(\rho - \hat{\rho})^{-3}.$$

**Proof:** First we show that for  $k \neq 0$ ,

$$|h_k| < \delta e^{-|k|\rho}.$$

This is because

$$h_k = \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-kxi} dx.$$

Evaluate this integral along the path

$$\gamma := \{x : \operatorname{Im}(x) = -\rho, \operatorname{Re}(x) 0 \leq \leq 2\pi\}$$

we obtain

$$|h_k| < \delta e^{-k\rho}$$

for  $k > 0$ . For  $k < 0$  we use the path  $\operatorname{Im}(x) = \rho$ .

Since  $\omega$  is a Diophantine number of type  $(1, 2)$ , we have

$$|e^{ik\omega} - 1| < \frac{1}{|k|^{-2}}.$$

So we have for  $\operatorname{Im}(x) < \hat{\rho}$ ,

$$|w| < \delta \sum_{k>1} k^{-2} e^{-k(\rho-\hat{\rho})} < K(\rho - \hat{\rho})^{-3}.$$

Let us write  $\mathcal{L}h = w$ .  $w(x) := \mathcal{L}h + w^*$  is the general solution where  $w^*$  is an arbitrary constant.

We are now ready to regard  $\xi$  as  $x$  in the above to solve the equations

$$\begin{aligned}u(\xi + \omega, \eta) &= u(\xi, \eta) + v(\xi, \eta) + f(\xi, \eta) \\v(\xi + \omega, \eta) &= v(\xi, \eta) + g(\xi, \eta) - g^*(\eta)\end{aligned}$$

for  $u$  and  $v$ . In fact we solve the second equation first to obtain

$$v(x) = v^* + \mathcal{L}g,$$

then we set  $v^* = -f^*$  to solve the first equation to conclude

$$u = \mathcal{L}f + \mathcal{L}^2g, \quad v = -f^* + \mathcal{L}g.$$

The first estimate in (b) follows directly from the last explicit solution and Sublemma 1.

**We now work on (c).**

First let us estimate  $|\phi| + |\psi|$  under the assumption that

$$g^* = \frac{1}{2\pi} \int_0^{2\pi} g(\xi, \eta) = 0.$$

This estimate is rather long, but nevertheless straight forward.

Substituting  $\theta = \xi + u(\xi, \eta)$ ,  $r = \eta + v(\xi, \eta)$  in to  $T$  we obtain

$$\begin{aligned}\xi_1 + u(\xi_1, \eta_1) &= \xi + \eta + u(\xi, \eta) + v(\xi, \eta) + \\ &\quad f(\xi + u(\xi, \eta), \eta + v(\xi, \eta)) \\ \eta_1 + v(\xi_1, \eta_1) &= \eta + v(\xi, \eta) + \\ &\quad g(\xi + u(\xi, \eta), \eta + v(\xi, \eta))\end{aligned}$$

Let  $S := C^{-1}TC$ . We write  $S$  as

$$\begin{aligned}\xi_1 &= \xi + \eta + \phi(\xi, \eta) \\ \eta_1 &= \eta + \psi(\xi, \eta).\end{aligned}$$

Substituting into the previous equations we obtain

$$\begin{aligned}\phi &= u(\xi, \eta) + v(\xi, \eta) - u(\xi + \eta + \phi, \eta + \psi) + \\ &\quad f(\xi + u(\xi, \eta), \eta + v(\xi, \eta)) \\ \psi &= v(\xi, \eta) - v(\xi + \eta + \phi, \eta + \psi) + \\ &\quad g(\xi + u(\xi, \eta), \eta + v(\xi, \eta)).\end{aligned}$$

Recall that  $u(\xi, \eta), v(\xi, \eta)$  are determined by

$$\begin{aligned}u(\xi + \omega, \eta) &= u(\xi, \eta) + v(\xi, \eta) + f(\xi, \eta) \\ v(\xi + \omega, \eta) &= v(\xi, \eta) + g(\xi, \eta).\end{aligned}$$

We obtain

$$\begin{aligned}
 |\phi| &\leq |f(\xi + u(\xi, \eta), \eta + v(\xi, \eta)) - f(\xi, \eta)| + \\
 &\quad |u(\xi + \omega, \eta) - u(\xi + \eta + \phi, \eta + \psi)| \\
 &\leq \frac{\delta}{\sigma}(|u| + |v|) + (|\partial_\xi u| + |\partial_\eta u|)(|\phi| + |\psi|) + \\
 &\quad |\partial_\xi u||\eta - \omega|
 \end{aligned}$$

where  $\sigma$  is divided in the first term because we used the mean value theorem, and to estimate  $f'$  we used Cauchy estimates. Similarly, we have

$$\begin{aligned}
 |\psi| &\leq |g(\xi + u(\xi, \eta), \eta + v(\xi, \eta)) - g(\xi, \eta)| + \\
 &\quad |v(\xi + \omega, \eta) - v(\xi + \eta + \phi, \eta + \psi)| \\
 &\leq \frac{\delta}{\sigma}(|u| + |v|) + (|\partial_\xi v| + |\partial_\eta v|)(|\phi| + |\psi|) + \\
 &\quad |\partial_\xi v||\eta - \omega|.
 \end{aligned}$$

These two are then combined with (b) to give us

$$|\phi| + |\psi| < (\rho - \hat{\rho})^{-7} \left( \frac{\delta^2}{\sigma} + \sigma\delta \right).$$

We remark that end result obtained here are neater than what was detailed in (c). This is because we assume that  $g^* = 0$ .

Let us now remove the assumption that  $g^*(\eta) = 0$ . For  $g^*(\eta) \neq 0$  we let

$$h(\eta) = g^*(\omega) + \partial_{\eta}g^*(\omega)(\eta - \omega).$$

By the mean value theorem we have

$$|g^*(\eta) - h(\eta)| < \frac{\delta}{\sigma^2}(\hat{\sigma})^2.$$

Following the same computation above we obtain

$$\begin{aligned} |\phi| + |\psi - h| &< (\rho - \hat{\rho})^{-7} \left( \frac{\delta^2}{\sigma} + \sigma\delta \right) + \\ &\quad (\rho - \hat{\rho})^{-6} \frac{\delta}{\sigma} |h| + |g^*(\eta) - h(\eta)| \\ &\leq (\rho - \hat{\rho})^{-7} \left( \frac{\delta^2}{\sigma} + \sigma\delta \right) + \left( \frac{\hat{\sigma}}{\sigma} \right)^2 \delta. \end{aligned}$$

For the last inequality we use our earlier estimate on  $|g^* - h|$  and  $|h| < \delta$ .

Let  $\mathcal{Q}$  be the quantity on the right hand, we rewrite what we have obtained so far as

$$|\phi| + |\psi - h| < \mathcal{Q}.$$

Note that this is not exactly what we claimed in (c). (c) reads as  $|\phi| + |\psi| < Q$ . Simply combining what we obtained and  $|h| < \delta$  will only give  $|\phi| + |\psi| < K\delta$ . This leads to the conclusion that  $S$  is not in any way closer to the integrable system than  $T$ . If this is so, then  $C$  and  $S$  we constructed are useless.

On the other hand, it is a good thing that we are having trouble here. This is because so far in our argument the standing assumption for  $T$  has never been used. We can easily construct  $T$  violating the standing assumption, for which KAM theory is false.

To finish, we now use the standing assumption for  $T$  to argue that  $|h| < 4Q$ .

– For every real constant  $\eta_0$ , let  $\gamma := \{(\xi, \eta_0), 0 < \xi \leq 2\pi\}$ . By the standing assumption for  $T$ , we have  $S(\gamma) \cap \gamma \neq \emptyset$ . So there exists  $(\xi_0, \eta_0)$  such that  $(\xi_1, \eta_1) \in \gamma$  so  $\eta_1 = \eta_0$ . Note that  $\eta_1 = \eta_0$  implies  $\psi(\xi_0, \eta_0) = 0$  so

$$|h(\eta_0)| < Q$$

for all real  $\eta$ .

– Set  $\eta_0 = \omega$  we conclude  $|h(\omega)| = |g^*(\omega)| < Q$ .

– Set  $\eta_0 = \omega + \hat{\sigma}$  we conclude

$$|g^*(\omega) + \partial_\eta(\omega)\hat{\sigma}| < Q.$$

This implies

$$|\partial_\eta(\omega)|\hat{\sigma} \leq 2Q.$$

– Finally we have for all  $\eta$  satisfying  $|\eta - \omega| < \hat{\sigma}$ ,

$$|h(\eta)| \leq |g^*(\omega)| + |\partial_\eta(\omega)|\hat{\sigma} < 3Q.$$

We finish the proof of (c) of the Main Lemma.

### 3. Proof of the Theorem

We now construct a sequence of coordinate change  $\{C_n\}$  and a sequence of mappings  $\{S_n\}$  as follows. Let  $S_0 = T$ , we obtain a coordinate change  $C_1$  by solving (1)-(2) for  $S_0$ . Then we

let  $S_1 = C_1^{-1}S_0C_1$  and solve (1)-(2) for  $S_1$  to construct a coordinate change  $C_2$ . We then let  $S_2 = C_2^{-1}S_1C_2$ . At step  $n$  we solve (1)-(2) for  $S_{n-1}$  to obtain a coordinate change  $C_n$ , and let  $S_n = C_n^{-1}S_{n-1}C_n$ .

The Main Lemma is needed in each step to make sure that the objects constructed properly converge to define the invariant curve we claim to exist.

(1) It is up to us to chose  $\rho_n, \sigma_n$  ( $\rho, \sigma$  at step  $n$ ). However, since we want to end up with a function that is real analytic in  $\xi$ ,  $\rho_n$  can not decline to zero (We have observed the same thing in constructing Siegel's disk). One way to meet this requirement is to let

$$\rho_n - \rho_{n+1} = \frac{1}{2^n}\rho_0.$$

Let us simply make the last line as the definition for  $\{\rho_n\}$

(2) Let us now define  $\{\delta_n\}$  for all  $n > 0$  by letting  $\delta_{n+1} = (\delta_n)^{\frac{5}{4}}$ . We also define  $\{\sigma_n\}$  for all  $n$  by letting  $\sigma_n = (\delta_n)^{\frac{2}{3}}$ .

(3) Let us assume that  $S_n$  is such that  $|\phi_n| + |\psi_n| < \delta_n$  on  $D_{\rho_n, \sigma_n}$ . We let  $\hat{\rho} = \rho_{n+1}, \hat{\sigma} = \sigma_{n+1}$  to construct  $C_{n+1}, S_{n+1}$ . The Main Lemma is ready to apply because we clearly have

$$\delta_n < K^{-1}(\rho_n - \rho_{n+1})^7 \sigma_n$$

for all  $n$  because  $K^n d_n \rightarrow 0$  for  $K = 2^{100}$ .

(4) as a result, we obtain

$$\begin{aligned} |\phi_{n+1}| + |\psi_{n+1}| &< K \left\{ 2^{7n} \left( (\delta_n)^{\frac{4}{3}} + (\delta_n)^{\frac{5}{3}} \right) + \delta_{n+1} \delta_n^{\frac{1}{3}} \right\} \\ &< \delta_{n+1}. \end{aligned}$$

(5) This last estimate, in turn, allow us to repeat (3)-(4) for the next step. This will inductively carry us through all  $n$ , constructing

$\{S_n\}$  and  $\{C_n\}$  for which the result of the Main Lemma hold.

We are now ready to prove the existence of the invariant curve claimed. Let  $V_n = C_1 \circ C_2 \circ \cdots \circ C_n$ . We write  $V_n : (\xi, \eta) \rightarrow (\theta, r)$  as

$$\theta = \xi + p_n, \quad r = \eta + q_n.$$

We have inductively

$$p_n = u_n + \cdots + u_1, \quad q_n = v_n + \cdots + v_1$$

By the Main Lemma we have on  $D_{\rho_n, \sigma_n}$ . As  $n \rightarrow \infty$ , this region goes to  $\{(\xi, \omega), \text{Im}(\xi) < \frac{1}{2}r_0\}$ . Not only  $p_n(\xi, \omega)$  and  $q_n(\xi, \omega)$  converge, they converge faster than any geometric sequence. By making  $|f| + |g| < \delta_0$  sufficiently small, we can further making the magnitude of these limit  $\varepsilon$ -small for any pre-assigned  $\varepsilon > 0$ .

Let

$$u(\xi) = \lim_{n \rightarrow \infty} p_n(\xi, \omega), \quad v(\xi) = \lim_{n \rightarrow \infty} q_n(\xi, \omega) + \omega.$$

It is easy to check that

$$\theta = \xi + u(\xi), \quad r = v(\xi)$$

is the invariant curve claimed by the KAM theory (Left as an Homework).

We have concluded the Theorem (KAM theory).  $\square$

**Remark:** (a) We assume that  $\omega$  is Diophantine number of type  $(1, 2)$  to avoid the distractive parameters. The same proof works for all Diophantine numbers.

(b) For the area preserving maps around an elliptic fixed point, the theorem we proved here does not directly apply. To study the dynamical structure, we first compute the normal form to write the mapping around the origin as

$$\begin{aligned} r_1 &= r + \mathcal{O}(r^{2k_0+2}) \\ \theta_1 &= \theta + 2\pi\alpha + \gamma_{k_0} r^{2k_0} + \mathcal{O}(r^{2k_0+2}). \end{aligned}$$

We then re-scale by using  $r \rightarrow \varepsilon r$  to turn this mapping into

$$\begin{aligned}r_1 &= r + \mathcal{O}(\varepsilon^{2k_0+1}) \\ \theta_1 &= \theta + 2\pi\alpha + \gamma_{k_0}\varepsilon^{2k_0}r^{2k_0} + \mathcal{O}(\varepsilon^{2k_0+2}).\end{aligned}$$

Last we introduce the coordinate change  $r^{2k_0} \rightarrow r$  to re-write this mapping as

$$\begin{aligned}r_1 &= r + \mathcal{O}(\varepsilon^{2k_0+1}) \\ \theta_1 &= \theta + 2\pi\alpha + \gamma_{k_0}\varepsilon^{2k_0}r + \mathcal{O}(\varepsilon^{2k_0+2}).\end{aligned}$$

Note that this last mapping we obtain is not necessarily area preserving, but it surely satisfying the standing assumption in this lecture (simple closed curve and their image intersect).

We can now re-write the corresponding statements on the existence of invariant curves and copy the same proof. We did not work directly on this case because we wanted to avoid unnecessarily distracting parameters  $\varepsilon$  in our presentation.