1D Dynamics: Periodic Orbits

Maps of study: \( T(x) : \mathbb{R} \to \mathbb{R} \).

\( T(x) \) is as smooth as we need along the way.

1. Graph of \( T(x) \) and orbits

- Fixed points: Intersection of the graph \( y = T(x) \) and \( y = x \).

- A given orbit: Trace the graph.
2. Stability of a fixed point

**Claim:** Let \( x_0 \) be a fixed point. \( x_0 \) is asymptotically stable if \( |T'(x_0)| < 1 \). It is unstable if \( |T'(x_0)| > 1 \).

**Proof:** If \( |T'(x_0)| < 1 \), then by continuity there exist \( I(x_0) \) (an interval contains \( x_0 \)), such that \( |T'(x)| < \lambda < 1 \) for all \( x \in I(x_0) \). By mean value theorem then,

\[
|T(x) - x_0| = |T(x) - T(x_0)| < \lambda |x - x_0|
\]

for all \( x \in I(x_0) \). This implies \( |T^n(x) - x_0| < \lambda^n |x - x_0| \to 0 \). The other half is similar.

*A demonstration using graph*
3. \(|T'(x_0)| = 1\): Degenerate case

**Ex:** \(T(x) = x + x^3\): unstable at \(x = 0\).

**Proof:** \(T'(x) = 1 + 3x^2 > 1\) around \(x = 0\). So \(|T(x) - 0| > x\) for all \(x \neq 0\).

**Ex:** \(T(x) = x - x^3\): asymptotically stable at \(x = 0\).

**Proof:** \(T'(x) = 1 - 3x^2 < 1\) around \(x = 0\). So \(|T(x) - 0| < |x - 0|\). Starting from, say, \(x \neq 0\), \(\{x_n\}\) is a increasing sequence. So \(x_n \to \hat{x}\). \(\hat{x}\) must be a fixed point. So \(\hat{x} = 0\) and \(x_n \to 0\).

**Ex:** \(T(x) = x^3 \sin \frac{1}{x} + x\): Stable but not asymptotically stable at \(x = 0\).

**Proof:** Fixed points of this \(T(x)\) is defined by

\[
x^3 \sin \frac{1}{x} = 0.
\]

This is a case in which \(T(x)\) has infinitely many fixed points accumulating at \(x = 0\).
4. Existence of periodic orbits

Claim: Let $T : \mathbb{R} \to \mathbb{R}$. If $T$ has a periodic orbit of period three, then it has periodic orbit of all periods.

Proof: Assume $a < b < c$ is such that $f(a) = b$, $f(b) = c$ and $f(c) = a$. Let $I_0 = [a, b]$, $I_1 = [b, c]$, we have $T(I_0) \supset I_1$ and $f(I_1) \supset I_0 \cup I_1$.

Basic observation: For any interval $A$ such that $T^i(A) \supset I_1$, there are two sub-intervals $A^0, A^1 \subset A$, such that $T^{i+1}(A^0) = I_0$, $T^{i+1}(A^1) = I_1$.

Let $n$ be fix, we will be able to find an sub-interval $A$ in $I_1$ such that

(a) $T^i(A) \subset I_1$ for all $i < n - 1$;

(b) $T^{n-1}(A) = I_0$.

Since $T^n(A) = T(I_0) \supset I_1 \supset A$. $T^n$ has a fixed point, which is a periodic orbit of $T$ of period $n$. The period of this orbit can not be less than $n$ by design.
5. Sarkovskii’s Theorem

Sarkovskii order

\[ 3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \cdots \triangleright 2^m \cdot 3 \triangleright 2^m \cdot 5 \triangleright \cdots \triangleright 2^n \triangleright 2^{n-1} \triangleright \cdots \triangleright 2 \triangleright 1. \]

Remark: We can always write an integer \( n \) in the form \( n = p2^m \) where \( p \geq 1 \) is odd and \( m \geq 0 \) be positive.

Theorem: Assume that \( T : \mathbb{R} \to \mathbb{R} \) is continuous. If \( T \) has a periodic orbit of period \( n \), then for all \( n' \) such that \( n \triangleright n' \), \( T \) has a periodic orbit of period \( n' \).

- The previous claim (period three implies all period) is a special case of this claim.

- If a 1D map has only finitely many periodic solutions, their period has to be multiples of 2.

- This claim is true only for interval maps (not even true for maps from \( S^1 \) to \( S^1 \)).
**Case 1:** Let us first consider the case that $T$ has a periodic orbit of *odd* period.

**A crucial observation:**

- Let $n$ be the smallest odd period $T$ allows (assume $n$ exists). List this periodic orbit as

$$x_1 < x_2 < \cdots < x_n$$
on $\mathbb{R}$.

- Let $r$ be the largest integer such that $T(x_r) > x_r$. Then

(a) $r = \frac{1}{2}(n - 1)$; and  

(b) $T$ on $\{x_i\}$ is as one of the following two pictures:
Proof for Case 1:

All period > $n$:  $I_1 \rightarrow I_2 \rightarrow \cdots I_{n-1} \rightarrow I_1 \rightarrow \cdots I_1$.

Other even period < $n$: Starting from $I_{n-1}$ go to other vertices then come back.
Proof of the observation:

Let $n$ be the smallest odd period $T$ admits, and $x_1 < x_2 \cdots < x_n$ be a periodic orbit of period $n$. Let $I_i = [x_i, x_{i+1}]$.

Let $I_r := [x_r, x_{r+1}]$ where $r$ is the largest such that $T(x_r) \geq x_{r+1}$.

**Claim 1:** There exists $i \neq r$ such that $f(I_i) \supset I_r$.

**proof:** Denote $L = \{x_1, \cdots, x_r\}$, $R = \{x_{r+1}, \cdots, x_n\}$. We observe that

- $T$ does not map $L$ into $L$ or $R$ into $R$. Otherwise $\{x_1, \cdots, x_n\}$ is not one single periodic orbit.

- $T$ do not map $L$ completely to $R$ and $R$ completely to $L$ because $R$ and $L$ contains different number of point. ($n$ is odd)

- Consequently, there exists $i \neq r$, such that $T(x_i)$ and $T(x_{i+1})$ are on different side of $I_r$.
Let $J_1 = I_r$ and $J_k = T^{k-1}(I_r)$. $J_k$, as a continuous image of an interval, is also an interval. We also have $J_k \subseteq J_{k+1}$ because $T(I_r) \supseteq I_r$.

Claim 2: Let $k_0$ be the smallest such that $J_{k_0} \supseteq [x_1, x_n]$. Then for $k < k_0$, we have

$$J_k \cap \{x_i\}_{i=1}^n \neq J_{k+1} \cap \{x_i\}_{i=1}^n.$$ 

Proof: By assumption $J_k \cap \{x_i\}_{i=1}^n$ is a proper subset of $\{x_i\}_{i=1}^n$. If this claim fails, then

$$T(J_k \cap \{x_i\}_{i=1}^n) = J_k \cap \{x_i\}_{i=1}^n,$$

claiming that a proper subset of $\{x_1, \ldots, x_n\}$ is invariant. A contradiction.
We caution that $J_k$ is not necessarily a union of several $I_i$ intervals.

Let $I_i$ be as the interval asserted by Claim 1, and $k(i)$ be the smallest such that $I_i \subset J_{k(i)}$. Then

**Claim 3:** We have

$$k(i) = k_0 = n - 1.$$  

**Proof:** $k_0 \leq n - 1$ follows from Claim 2. $k(i) \leq k_0$ is by definition. So all we need is to show that $k(i) \geq n - 1$.

If $k(i) < n - 1$, then

$$I_r \rightarrow J_{k(i)} \rightarrow I_r \ (\rightarrow I_r)$$

give a periodic orbit of odd period $< n$. Note that the last $\rightarrow I_r$ is optional. It is to make sure we end up with **odd** period.

Claim 3 implies that, every time we applying $T$ to $J_k, k < n, J_{k+1}$ picks up one and only one point from $\{x_i\}_{i=1}^{n}$. So we have two possibilities for $T(x_r)$ and $T(x_{r+1})$: 
(i) \( T(x_r) = x_{r+1}, \ T(x_{r+1}) = x_{r-1}. \)

(ii) \( T(x_r) = T(x_{r+2}), \ T(x_{r+1}) = x_r. \)

From this point on, the two pictures are forced.
Case 2: For $n = p2^m$, let $f = T^{2^m}$. Then $f$ allow $p$, therefore allow all $n' < p$. Get back to $T$: $T$ allows all $2^m n'$ with $n' < p$. This finishes the part in Sarkovskii order except all powers of 2 less than $2^m$.

To allow these numbers we need to:

(i) Prove that if $T$ allow an even period, then it allows two.

(ii) Use it then on $f = T^{2^l}$ for all $1 \leq l < m − 1$ to allow $2^{l+1}$.

To prove (i) we assume that $\{x_1, \ldots, x_n\}$ is a periodic orbit where $n$ is the smallest even period allowed. We go back to see in our previous proof where the condition $n$ is odd is used. It is, as one could observe, only used in proving Claim 1. So here we face the following two possibilities:
(a) Claim 1 still hold regardless of \( n \) is even now.

Then by repeat the same proof we conclude that \( n \) is odd. A contradiction.

(b) Claim 1 is false because \( R \) and \( L \) indeed have the same number of points, and \( T \) maps \( R \) to \( L \), \( L \) to \( R \). In this case we obtain \( T([x_1, x_r]) \supset [x_{r+1}, x_n] \) and \( T([x_{r+1}, x_n]) \supset [x_1, x_r] \), creating a periodic orbit of period two.
Homework

1. Discuss the stability of the fixed points of $T(x) = \mu x(1 - x)$ for $2 < \mu < 5$.

2. Let $T(x) = x^3 - \lambda x$ for $\lambda > 0$.
   (a) Find all periodic points and discuss their stabilities for $0 < \lambda < 1$.
   (b) Prove that, if $|x|$ is sufficiently large, then $|f^n(x)| \to \infty$.
   (c) Prove that if $\lambda$ is sufficiently large, then the set of points which do not tend to infinity is a Cantor set.

3. Let
   \[ T(x) = \begin{cases} 
   2x & 0 \leq x \leq \frac{1}{2} \\
   2 - 2x & \frac{1}{2} \leq x \leq 1 
   \end{cases} \]
   be the tent map on unit interval. Prove
   (a) $T$ has exactly $2^n$ periodic orbits of period $n$.
   (b) The set of all periodic points of $T(x)$ are dense in $[0, 1]$.

4. Suppose $A_0, A_1, \ldots, A_n$ are closed intervals and $T(A_i) \supset A_{i+1}$ for $i = 0, \ldots, n - 1$. Prove that there is a point $x \in A_0$ such that $T^i(x) \in A_i$ for all $i$.

5. Construct a map that has periodic orbit of period $2^j$ for $j < \ell$ but not period $2^\ell$. 