1D Dynamics: Periodic Orbits

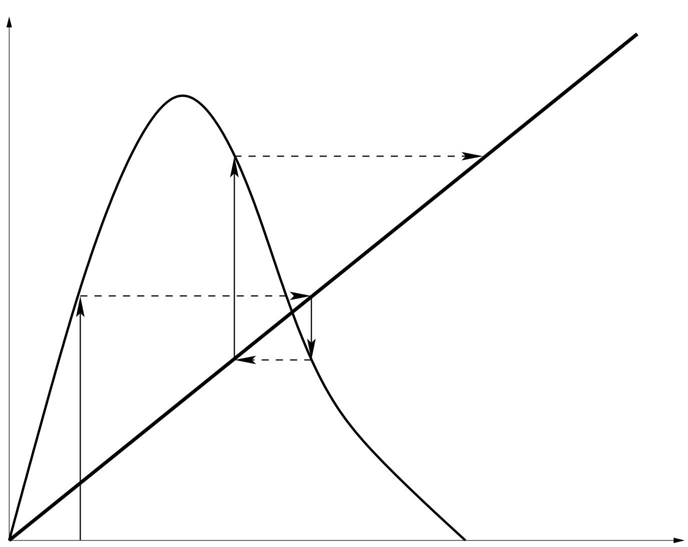
Maps of study: $T(x) : \mathbb{R} \to \mathbb{R}$.

$T(x)$ is as smooth as we need along the way.

1. Graph of $T(x)$ and orbits

– Fixed points: Intersection of the graph $y = T(x)$ and $y = x$.

– A given orbit: Trace the graph.



2. Stability of a fixed point

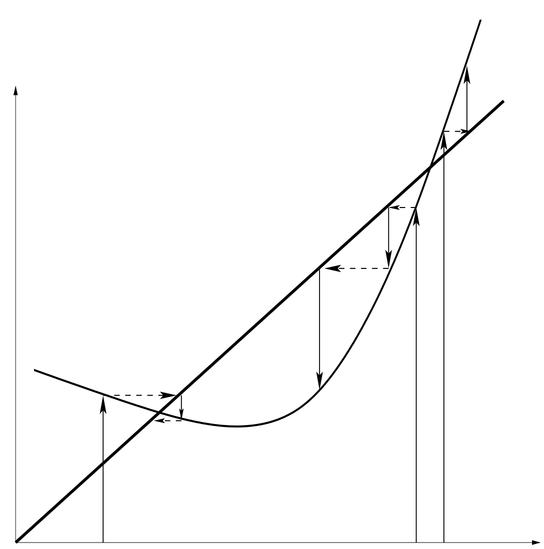
Claim: Let x_0 be a fixed point. x_0 is asymptotically stable if $|T'(x_0)| < 1$. It is unstable if $|T'(x_0)| > 1$.

Proof: If $|T'(x_0)| < 1$, then by continuity there exist $I(x_0)$ (an interval contains x_0), such that $|T'(x)| < \lambda < 1$ for all $x \in I(x_0)$. By mean value theorem then,

$$|T(x) - x_0| = |T(x) - T(x_0)| < \lambda |x - x_0|$$

for all $x \in I(x_0)$. This implies $|T^n(x) - x_0| < \lambda^n |x - x_0| \to 0$. The other half is similar.

A demonstration using graph



3. $|T'(x_0)| = 1$: Degenerate case

Ex: $T(x) = x + x^3$: unstable at $x = 0$.

Proof: $T'(x) = 1 + 3x^2 > 1$ around $x = 0$. So $|T(x) - 0| > x$ for all $x \neq 0$.

Ex: $T(x) = x - x^3$: asymptotically stable at $x = 0$.

Proof: $T'(x) = 1 - 3x^2 < 1$ around $x = 0$. So $|T(x) - 0| < |x - 0|$. Starting from, say, $x \neq 0$, $\{x_n\}$ is a increasing sequence. So $x_n \to \hat{x}$. $\hat{x}$ must be a fixed point. So $\hat{x} = 0$ and $x_n \to 0$.

Ex: $T(x) = x^3 \sin \frac{1}{x} + x$: Stable but not asymptotically stable at $x = 0$.

Proof: Fixed points of this $T(x)$ is defined by

$$x^3 \sin \frac{1}{x} = 0.$$

This is a case in which $T(x)$ has infinitely many fixed points accumulating at $x = 0$.

4. Existence of periodic orbits

Claim: Let $T : \mathbb{R} \to \mathbb{R}$. If T has a periodic orbit of period three, then it has periodic orbit of all periods.

Proof: Assume $a < b < c$ is such that $f(a) = b$, $f(b) = c$ and $f(c) = a$. Let $I_0 = [a, b]$, $I_1 = [b, c]$, we have $T(I_0) \supset I_1$ and $f(I_1) \supset I_0 \cup I_1$.

Basic observation: For any interval A such that $T^i(A) \supset I_1$, there are two sub-intervals $A^0, A^1 \subset A$, such that $T^{i+1}(A^0) = I_0$, $T^{i+1}(A^1) = I_1$.

Let n be fix, we will be able to find an sub-interval A in I_1 such that

(a) $T^i(A) \subset I_1$ for all $i < n - 1$;

(b) $T^{n-1}(A) = I_0$.

Since $T^n(A) = T(I_0) = I_1 \supset A$. T^n has a fixed point, which is a periodic orbit of T of period n. The period of this orbit can not be less than n by design.

5. Sarkovskii's Theorem

Sarkaovskii order

$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots$

$\rhd 2^m \cdot 3 \rhd 2^m \cdot 5 \rhd \cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots \rhd 2 \rhd 1.$

Remark: We can always write an integer n in the form $n = p2^m$ where $p \geq 1$ is odd and $m \geq 0$ be positive.

Theorem: Assume that $T : \mathbb{R} \to \mathbb{R}$ is continuous. If T has a periodic orbit of period n, then for all n' such that $n \rhd n'$, T has a periodic orbit of period n'.

– The previous claim (period three implies all period) is a special case of this claim.

– If a 1D map has only finitely many periodic solutions, their period has to be multiples of 2.

– This claim is true only for interval maps (not even true for maps from S^1 to S^1).

Case 1: Let us first consider the case that T has a periodic orbit of *odd* period.

A crucial observation:

– Let n be the smallest odd period T allows (assume n exists). List this periodic orbit as

$$x_1 < x_2 < \cdots < x_n$$

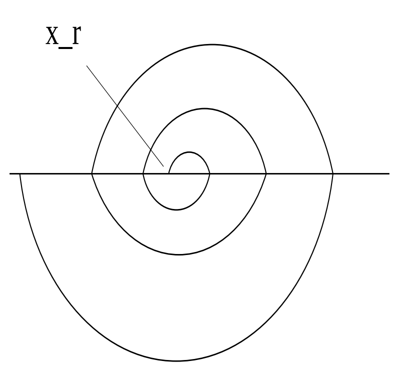
on $\mathbb{R}$.

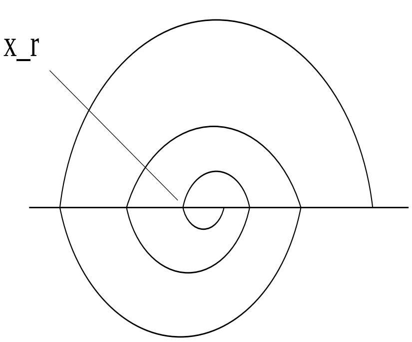
– Let r be the largest integer such that $T(x_r) > x_r$.

Then

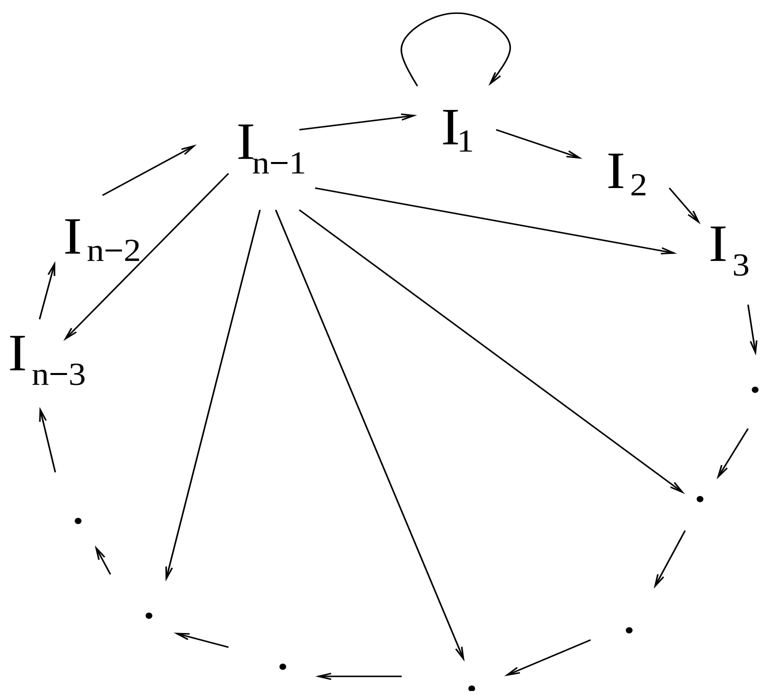
(a) $r = \frac{1}{2}(n-1)$; and

(b) T on $\{x_i\}$ is as one of the following two pictures:





Proof for Case 1:



All period $> n$*:* $I_1 \to I_2 \to \cdots I_{n-1} \to I_1 \to \cdots I_1$.

Other even period $< n$*:* Starting from I_{n-1} go to other vertices then come back.

Proof of the observation:

Let n be the smallest odd period T admits, and $x_1 < x_2 \cdots < x_n$ be a periodic orbit of period n. Let $I_i = [x_i, x_{i+1}]$.

Let $I_r := [x_r, x_{r+1}]$ where r is the largest such that $T(x_r) \geq x_{r+1}$.

Claim 1: There exists $i \neq r$ such that $f(I_i) \supset I_r$.

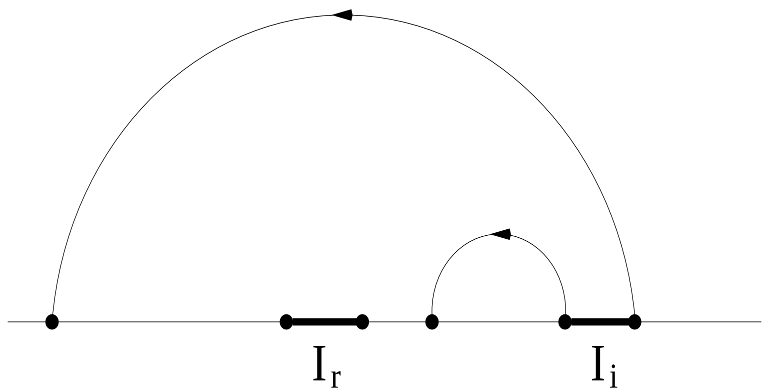
proof: Denote $L = \{x_1, \cdots, x_r\}$, $R = \{x_{r+1}, \cdots, x_n\}$. We observe that

– T does not map L into L or R into R. Otherwise $\{x_1, \cdots, x_n\}$ is not one single periodic orbit.

– T do not map L completely to R and R completely to L because R and L contains different number of point. (n is odd)

– Consequently, there exists $i \neq r$, such that $T(x_i)$ and $T(x_{i+1})$ are on different side of I_r.

Let $J_1 = I_r$ and $J_k = T^{k-1}(I_r)$. J_k, as a continuous image of an interval, is also an interval. We also have $J_k \subset J_{k+1}$ because $T(I_r) \supset I_r$.



Claim 2: Let k_0 be the smallest such that $J_{k_0} \supset [x_1, x_n]$. Then for $k < k_0$, we have

$$J_k \cap \{x_i\}_{i=1}^n \neq J_{k+1} \cap \{x_i\}_{i=1}^n.$$

Proof: By assumption $J_k \cap \{x_i\}_{i=1}^n$ is a proper subset of $\{x_i\}_{i=1}^n$. If this claim fails, then

$$T(J_k \cap \{x_i\}_{i=1}^n) = J_k \cap \{x_i\}_{i=1}^n,$$

claiming that a proper subset of $\{x_1, \cdots, x_n\}$ is invariant. A contradiction.

We caution that J_k is not necessarily a union of several I_i intervals.

Let I_i be as the interval asserted by Claim 1, and $k(i)$ be the smallest such that $I_i \subset J_{k(i)}$. Then

Claim 3: We have

$$k(i) = k_0 = n - 1.$$

Proof: $k_0 \leq n-1$ follows from Claim 2. $k(i) \leq k_0$ is by definition. So all we need is to show that $k(i) \geq n-1$.

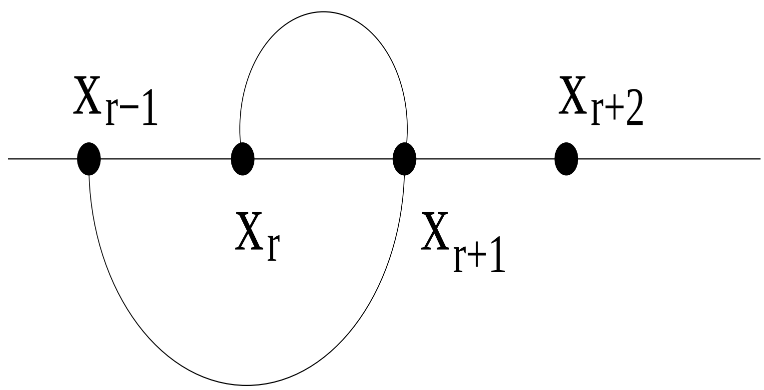
If $k(i) < n-1$, then

$$I_r \to J_{k(i)} \to I_r \quad (\to I_r)$$

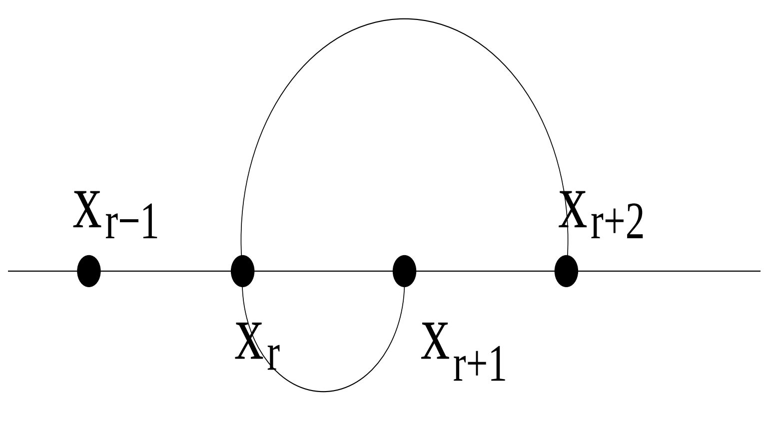
give a periodic orbit of odd period $< n$. Note that the last $\to I_r$ is optional. It is to make sure we end up with **odd** period.

Claim 3 implies that, every time we applying T to $J_k, k < n$, J_{k+1} picks up one and only one point from $\{x_i\}_{i=1}^n$. So we have two possibilities for $T(x_r)$ and $T(x_{r+1})$:

(i) $T(x_r) = x_{r+1}$, $T(x_{r+1}) = x_{r-1}$.



(ii) $T(x_r) = T(x_{r+2})$, $T(x_{r+1}) = x_r$.



From this point on, the two pictures are forced.

Case 2: For $n = p2^m$, let $f = T^{2^m}$. Then f allow p, therefore allow all $n' \lhd p$. Get back to T: T allows all $2^m n'$ with $n' \lhd p$. This finishes the part in Sarkovskii order except all powers of 2 less than 2^m.

To allow these numbers we need to:

(i) Prove that if T allow an even period, then it allows two.

(ii) Use it then on $f = T^{2^l}$ for all $1 \leq l < m - 1$ to allow 2^{l+1}.

To prove (i) we assume that $\{x_1, \cdots, x_n\}$ is a periodic orbit where n is the smallest even period allowed. We go back t see in our previous proof where the condition n is odd is used. It is, as one could observe, only used in proving Claim 1. So here we face the following two possibilities:

(a) Claim 1 still hold regardless of n is even now.

Then by repeat the same proof we conclude that n is odd. A contradiction.

(b) Claim 1 is false because R and L indeed have the same number of points, and T maps R to L, L to R. In this case we obtain $T([x_1, x_r]) \supset [x_{r+1}, x_n]$ and $T([x_{r+1}, x_n]) \supset [x_1, x_r]$, creating a periodic orbit of period two.

Homework

1. Discuss the stability of the fixed points of $T(x) = \mu x(1-x)$ for $2 < \mu < 5$.

2. Let $T(x) = x^3 - \lambda x$ for $\lambda > 0$.

(a) Find all periodic points and discuss their stabilities for $0 < \lambda < 1$.

(b) Prove that, if $|x|$ is sufficiently large, then $|f^n(x)| \to \infty$.

(c) Prove that if λ is sufficiently large, then the set of points which do not tend to infinity is a Cantor set.

3. Let

$$T(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \le x \le 1 \end{cases}$$

be the tent map on unit interval. Prove

(a) T has exactly 2^n periodic orbits of period n.

(b) The set of all perioic points of $T(x)$ are dense in $[0, 1]$.

4. Suppose $A_0, A_1, \cdots, A_n$ are closed intervals and $T(A_i) \supset A_{i+1}$ for $i = 0, \cdots, n-1$. Prove that there is a point $x \in A_0$ such that $T^i(x) \in A_i$ for all i.

5. Construct a map that has periodic orbit of period 2^j for $j < \ell$ but not period 2^ℓ.