1D Dynamics: Symbolic Coding

1. Full shift of two symbols

– Let

$$X = \{\mathrm{s} = (s_0 s_1 s_2 \cdots s_i \cdots);\ s_i = 0 \text{ or } 1\}$$

be the collection of all sequences of 0 and 1.

– For $\mathrm{s} = (s_0 s_1 \cdots s_i \cdots)$, $\mathbf{t} = (t_0 t_1 \cdots t_i \cdots)$, let

$$d(\mathrm{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Claim: (i) (X, d) is a metric space.

(ii) $d(\mathrm{s}, \mathbf{t}) \leq \frac{1}{2^n}$ iff $s_i = t_i$ for all $i < n$.

Proof: (i) $d(\mathbf{s},\mathbf{t}) \geq 0$ and $d(\mathbf{s},\mathbf{t}) = d(\mathbf{t},\mathbf{s})$ are obvious. For the triangle inequality, observe

$$\begin{aligned} d(\mathbf{s},\mathbf{t}) &= \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{|s_i - v_i|}{2^i} + \sum_{i=0}^{\infty} \frac{|v_i - t_i|}{2^i} \\ &= d(\mathbf{s},\mathbf{v}) + d(\mathbf{v},\mathbf{t}). \end{aligned}$$

(ii) is straight forward from definition.

– Let $\sigma : X \to X$ be define by

$$\sigma(s_0 s_1 \cdots s_i \cdots) = (s_1 s_2 \cdots s_{i+1} \cdots).$$

Claim: $\sigma : X \to X$ is continuous.

Proof: For $\varepsilon > 0$ given let n be such that $\frac{1}{2^n} < \varepsilon$. Let $\delta = \frac{1}{2^{n+1}}$. For $d(\mathbf{s},\mathbf{t}) < \delta$, $d(\sigma(\mathbf{s}),\sigma(\mathbf{t})) < \varepsilon$.

– (X, σ) as a dynamical system:

Claim (a) $\sigma : X \to X$ have exactly 2^n periodic points of period n.

(b) Union of all periodic orbits are dense in X.

(c) There exists a transitive orbit.

Proof: (a) The number of periodic point in X is the same as the number of strings of $0, 1$ of length n, which is 2^n.

(b) $\forall \mathbf{s} = (s_0 s_1 \cdots s_n \cdots)$, let $\mathbf{s}_n = (s_0 s_1 \cdots s_n; s_0 s_1 \cdots s_n; \cdots)$. We have

$$d(\mathbf{s}_n, \mathbf{s}) < \frac{1}{2^{n-1}}.$$

So $\mathbf{s}_n \to \mathbf{s}$ as $n \to \infty$.

(c) List all possible strings of length 1, then of length 2, then of length 3, and so on. We obtain a orbit that is clearly dense in X.

2. Dynamical Equivalence

Let $f : X \to X$, $g : Y \to Y$ be two given dynamical system.

Question: When can we regard (X, f) and (Y, g) as the same?

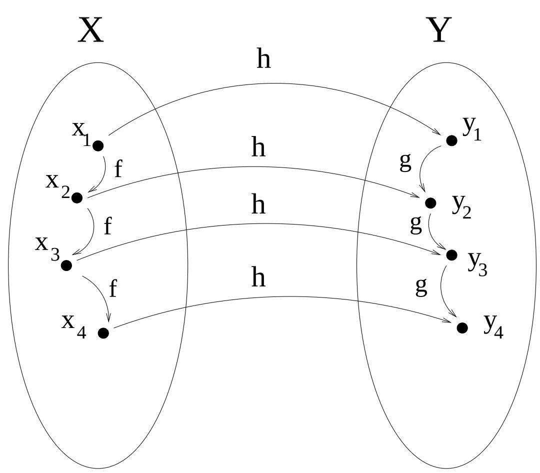
Definition: Two systems (X, f) and (Y, g) are topologically conjugate if there exists $h : X \to Y$, such that

(a) h is a homeomorphism (Both h and h^{-1} are well-defined and continuous); and

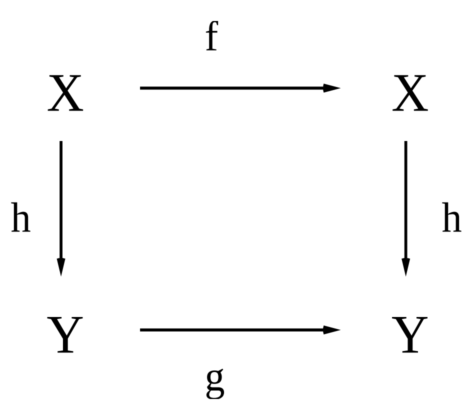
(b) $g \circ h = h \circ f$.

Remarks: Item (a) requires that, as topological spaces, X and Y are equivalent.

Item (b) requires that h maps orbit to orbit.



Item (b) is also represented by the following commuting diagram:



Claim: If (X, f) and (Y, g) are conjugated by h. Then

(i) They have the same number of periodic orbits of any given period.

(ii) The stability of the corresponding periodic orbits are the same.

(iii) If one has a transitive orbit so does the other.

The proof of this Claim is left as a homework.

Ex1: $f(x) = 2x$, $g(x) = \frac{1}{2}x$ are not conjugate.

Ex2: $f(\theta) = \theta + \pi$ and $f(\theta) = \theta + 1$ are not conjugate.

Ex3: How about $f(\theta) = \theta + r_1$ and $f(\theta) = \theta + r_2$, $r_1 \neq r_2$?

Not conjugate if $r_1 \neq r_2 mod(2\pi)$. Need more sophisticated tool to show it.

Ex 4: $f(x) = 2x$ and $g(x) = 3x$ are conjugate.

Proof: Construct $h : \mathbb{R} \to \mathbb{R}$ By conjugating *fundamental domains*.

– For $f(x) = 2x$, let $U = [-2, -1) \cup (1, 2]$. All orbit of $f(x)$ passing U one and only one time.

– For $g(x) = 3x$, $V = [-3, -1) \cup (1, 3]$, is the corresponding fundamental domain.

– To construct $h : \mathbb{R} \to \mathbb{R}$, we

- Construct $k : (1, 2] \to (1, 3]$: $k(x) = 2x - 1$
- For $x \in (2^n, 2^{n+1}]$, let $h(x) = g^n \circ k \circ f^{-n}(x)$.
- $h(x)$ on $x < 0$ are defined similarly using $[-2, -1)$ and $[-3, -1)$.
- Check that $h : \mathbb{R} \to \mathbb{R}$ so defined conjugates $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$.

3. Dynamics of $T(x) = 7x(1 - x)$

Let $\Lambda = \{x \in [0, 1]; T^i(x) \in [0, 1] \text{ for all } i > 0\}$.

Claim: (Λ, T) and (X, σ) conjugate where X is the space of sequences of two symbols and σ is the shift operator.

Proof: Let $I_0 = [0, \frac{1}{2}], I_1 = [\frac{1}{2}, 1]$.

– Address: let $a(x) = 0$ if $x \in I_0$, and $a(x) = 1$ if $x \in I_1$.

– Coding of orbit. For $x \in \Lambda$, let

$$\mathrm{s}(x) = a(x)a(T(x))a(T^2(x)) \cdots .$$

– $\mathrm{s} : \Lambda \to X$ satisfies $\mathrm{s} \circ T = \sigma \circ \mathrm{s}$:

$$\mathrm{s}(Tx) = a(Tx) \cdot a(T^n x) \cdots = \sigma(\mathrm{s}(x)).$$

– $\mathrm{s} : \Lambda \to X$ is one to one; onto; continuous.

Key observation: for any given sequence $a_0 a_1 \cdots a_n$, $a_i = 0$ or 1, there exists a unique interval $I_{a_0 a_1 \cdots a_n}$, such that

- $I_{a_0 a_1 \cdots a_n a_{n+1}} \subset I_{a_0 a_1 \cdots a_n}$;
- for all $x \in I_{a_0 a_1 \cdots a_n}$, $T^i(x) \in I_{a_i}, i \leq n$;
- $T^n(I_{a_0 a_1 \cdots a_n}) = [0, 1]$, $|(T^n)'(x)| > 2^n$.
- $T^n([0, 1] \setminus \cup_{a_0 a_1 \cdots a_n} I_{a_0 a_1 \cdots a_n}) \not\subset (0, 1)$.

Proof of the key observation: Inductively assume the above for $I_{a_0 a_1 \cdots a_n}$.

$\exists$ a sub-interval $I_{a_0 a_1 \cdots a_n, 0} \subset I_{a_0 a_1 \cdots a_n}$ such that

$$f(f^n(I_{a_0 a_1 \cdots a_n, 0})) = [0, \frac{1}{2}]$$

and $I_{a_0a_1\cdots a_n,1} \subset I_{a_0a_1\cdots a_n}$ such that

$$f(f^n(I_{a_0a_1\cdots a_n,1})) = [\frac{1}{2}, 1].$$

The properties of $\mathbf{s} : \Lambda \to X$ claimed (1-1, onto, continuous) follows easily from this observation. The details are left as a homework.

– Conclusion: $s : \Lambda \to X$ conjugates (Λ, T) to (X, σ).

Corollaries: (i) (Λ, T) has 2^n periodic orbits of period n; (ii) the collection of all periodic orbits of T is dense in Λ; and (iii) T has a transitive orbit.

4. Kneading theory

Let $I = [a, b]$, and $f : I \to I$, continuous, is such that

(i) $f(a) = f(b) = a$;

(ii) $\exists$ a unique $c \in I$, such that $f(x)$ is monotonically increasing on $[a, c)$, monotonically decreasing on $(c, b]$.

We call $f : I \to I$ a unimodal map.

– Let $I_0 = [a, c)$, $I_1 = (c, b]$ and define $a(x) = 0$ for $x \in I_0$ and $a(x) = 1$ for $x \in I_1$. Let $a(c) = C$.

– Let $\mathbf{s}(x) = a(x)a(f(x)) \cdots a(f^i(x)) \cdots$.

– Let $K(f) = \mathbf{s}(f(c))$: the kneading sequence for f.

Question: which sequence is allowed and which is not?

A partial answer: The kneading theory.

The kneading order

(1) $0 < C < 1$.

(2) For $\mathbf{s} = s_0 s_1 \cdots s_n \cdots$, $\mathbf{t} = t_0 t_1 \cdots t_n \cdots$, let n be the first integer such that $t_n \neq s_n$. Let

$\tau(\mathbf{s}, \mathbf{t}) = \#$ of 1's among $\{s_0 s_1 \cdots s_{n-1}\}$.

(3) The kneading order: $\mathbf{s} \lhd \mathbf{t}$ if

(a) $\tau(\mathbf{s}, \mathbf{t}) =$ even, and $s_n < t_n$; or

(b) $\tau(\mathbf{s}, \mathbf{t}) =$ odd, and $s_n > t_n$.

Ex: $(0101 \cdots) \lhd (010C \cdots) \lhd (0100 \cdots)$

Ex: $(110 \cdots) \lhd (11C \cdots) \lhd (111 \cdots)$.

Note: To compare $\mathbf{s}$ and $\mathbf{t}$ is to compare $s_0 s_1 \cdots s_n$ and $t_0 t_1 \cdots t_n$ where n is the smallest integer such that $t_n = s_n$.

Theorem: Assume that $K(f)$, the kneading sequence for f, is not periodic. If $\mathbf{t}$ is such that $\sigma^n \mathbf{t} \lhd K(f)$ for all $n \geq 1$, then there is an $x \in [a, b]$ such that $\mathbf{s}(x) = \mathbf{t}$.

Observation 1: $x \le y$ iff $\mathrm{s}(x) \lhd \mathrm{s}(y)$.

Proof: If x and y are in $I_0 = (a, c)$, then $f(x) \le f(y)$ iff $x \le y$.

If x and y are in $I_1 = (c, b)$ then $f(x) \le f(y)$ iff $x > y$.

So as long as the addresses of x and y are kept the same, then the number of switches in order before time n is the same as the number of 1's before time n in $\mathrm{s}(x)$.

This observation is now the same as the definition of the kneading order.

Observation 2: Let $A = \{x \in [a, b], \mathrm{s}(x) \lhd \mathbf{t}\}$, $B = \{x \in [a, b], \mathrm{s}(x) \rhd \mathbf{t}\}$. Then A and B are both non-empty.

Proof: This is because $\mathrm{s}(a) = 00 \cdots \lhd \mathbf{t}$ so $a \in A$. Similarly, $\mathrm{s}(b) = 10 \cdots \rhd \mathbf{t}$ so $b \in B$.

Observation 3: Both A and B are open in $[a, b]$.

First note that the theorem follows from observation 3 since $[a, b] \setminus A \cup B$ is a non-empty and closed subset.

Proof: We argue that A is open (The reason for B open is similar).

– Let $\mathbf{t} = t_0 t_1 \cdots t_i \cdots$, then $t_i \neq C$ for all i.

(If $t_i = C$ then $\sigma^i(\mathbf{t}) = K(f)$, against the assumption that $\sigma^i \mathbf{t} \lhd K(f)$).

– Let $z \in [a, b]$ be such that $\mathbf{s}(z) = \mathbf{s} \lhd \mathbf{t}$. Let n be the first index such that $s_n \neq t_n$. Either $t_n = 0$ or $t_n = 1$. Let us consider only the case $t_n = 1$, leaving the other case $t_n = 0$ as a homework problem.

It suffices for us to prove

Lemma: Let $z \in [a, b]$ be such that $\mathbf{s}(z) \lhd \mathbf{t}$ as in the above. Then there exists a sufficiently small open neighborhood $V(z)$ around z such that $\forall x \in V(z)$, $\mathbf{s}(x) \lhd \mathbf{t}$.

Proof of this lemma: We have two possible cases : (i) $s_n = 0$, or (ii) $s_n = C$.

(i) If $s_n = 0$, then there is a small open set $V(z)$ around z such that for any $x \in V(z)$, $s_i(x) := a(T^i(x)) = s_i$ for all $i \leq n$. (This is because $s_i = t_i \neq C$ for all $i \leq n$)

We have

$$\mathbf{s}(x) = s_1 s_2 \cdots s_n \cdots \lhd t_1 t_2 \cdots t_n \cdots .$$

(ii) If $s_n = C$, then $T^n z = c$. Let $V(z) = (z - \varepsilon, z + \varepsilon)$. Then $T^n V(z) = (T^n(z - \varepsilon), T^n(z + \varepsilon))$ is a small interval around c. Note that we have $T^n(z-\varepsilon) < T^n(z+\varepsilon)$ because

$$s_1 \cdots s_{n-1} C \lhd t_1 \cdots t_{n-1} 1,$$

implicating that there are even number of 1's in $s_1 \cdots s_{n-1}$.

Now for all $x \in (z-\varepsilon, z]$, $\mathbf{s}(x) \lhd \mathbf{t}$ is again from

$$s_1 \cdots s_{n-1} s_n(x) \lhd t_1 \cdots t_{n-1} 1.$$

For $x \in (z, z+\varepsilon)$, first we have

$$s_1(x) \cdots s_n(x) = t_1 \cdots t_n \tag{1}$$

and $t_1 \cdots t_n$ now have odd number of 1.

Let α be the first integer such that

$$t_{n+1} \cdots t_{n+\alpha} \lhd s_{n+1} \cdots s_{n+\alpha} \tag{2}$$

(The existence of the integer α is from the fact that $K(f) = s_{n+1} s_{n+2} \cdots$ because $s_n = C$. We must have $\sigma^n(\mathbf{t}) \lhd K(f)$.)

Putting (1) and (2) together we conclude

$$s_1 \cdots s_{n-1} 1 s_{n+1} \cdots s_{n+\alpha} \lhd t_1 \cdots t_n t_{n+1} \cdots t_{n+\alpha}.$$

Note that the left is the first $n+\alpha$ symbols for $\mathbf{s}(x)$.

Homework

1. Prove that, if (X, f) and (Y, g) conjugate, then

(a) They have the same number of periodic orbits of any given period.

(b) The stability of the corresponding periodic orbits are the same.

(c) If one has a transitive orbit so the other.

2. Prove that $h : \mathbb{R} \to \mathbb{R}$ constructed in Ex4 conjugates $f = 2x$ and $g = 3x$.

3. Prove that $f(x) = \frac{1}{2}x$ is structurally table in following sense: $f(x)$ conjugate to $g(x) = f(x) + k(x)$ where $k(x)$ is any given function from $\mathbb{R} \to \mathbb{R}$ such that $\|k(x)\|_{C_1} < \frac{1}{16}$.

4. Finish the writing that $\mathbf{s} : \Lambda \to X$ is one-to-one, onto and continuous.

5. Find all coding sequences allowed by $T(x) = 2x(1 - x)$. How about $T(x) = 4x(1 - x)$?

6. Prove that the set A in observation 3 above is open in the case $t_n = 0$.