1D Dynamics: Circle Diffeomorphisms

1. Rigid Rotations

Let \( \alpha \in (0, \frac{1}{2}) \) be irrational and \( x \in S^1 \) where

\[
S^1 = \{e^{2\pi i x}, x \in [0, 1)\}.
\]

\( T : S^1 \rightarrow S^1 \) is defined by

\[
T(x) = x + \alpha \mod(1).
\]
(i) Continuous fractions of $\alpha$

- Let $x_0 = 0, x_j = T^j x_0 = \alpha j \mod (1)$. $x_j$ is on the left of 0 if $x_j \in (0, \frac{1}{2})$ and it is on the right if $x_j \in (\frac{1}{2}, 1)$.

- We define inductively a sequence $\{q_n\}$:
  - $q_0 = 1$;
  - Assume that $q_n$ is defined, $q_{n+1}$ is the first $j > q_n$ such that $d(x_{q_n+1}, 0) < d(x_{q_n}, 0)$. (Note that $q_{n+1}$ always exists because $\{x_j\}$ is dense in $S^1$.)

- Let $d_n = d(x_{q_n}, 0)$; and $a_n = \left[\frac{d_{n-1}}{d_n}\right]$ with $d_{-1} = 1$. $\{a_n\}$ is the continuous fraction for $\alpha$:
  $$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}.$$

(The proof of this fact is left as a homework.)
(ii) Some useful facts

**Fact 1:** \( x_{q_n}, x_{q_n-1} \) are on different sides of 0.

![Diagram showing \( x_{q_n}, x_{q_n+1}, x_{q_n-1}, x_{q_n-2} \) and the transformation \( T^{-q_n} \).]

**Proof:** If they are on the same side, then

\[
d(x_{q_n-q_n-1}, 0) < d(x_{q_n-1}, 0),
\]
contradicting to the definition of \( q_n \).

**Fact 2:** For any \( 0 \leq j < j' \leq q_n \),

\[
T^j((0, x_{q_n-1})) \cap T^{j'}((0, x_{q_n-1})) = \emptyset.
\]

**Proof:** If not, we have

\[
(0, x_{q_n-1}) \cap (x_{j'-j}, x_{j'-j+q_n-1}) \neq \emptyset.
\]
This implies
\[ d(x_{j'-j}, 0) < d(x_{q_n}, 0), \]
against the definition of \( q_n \).

(iii) Tower structure

\( x_0 = 0, x_1, \ldots, x_{q_n} \) divide \( S^1 \) into mutually disjoint intervals. We denote them as

\[ J_1^{(n)} \rightarrow J_2^{(n)} \rightarrow \cdots \rightarrow J_{q_n}^{(n)} \]

where

\[ - \quad J_1^{(n)} = (0, x_{q_n-1}); \]

\[- \quad J_{i+1}^{(n)} = T J_i^{(n)}. \]
Note that by Fact 2 in the above, $J_i \neq J_j$ for $i \neq j$. We put $J_1$ in the bottom, and $J_2$, $J_3$ on top one by one. Let us trace the end points of these intervals with care.

At $i = q_n - q_{n-1}$, one end of $J_i$ is $x_{q_n}$. 0 must be on the other side, forcing the following tower.
Characters of this tower:
- Size of the base: $|J_1^{(n)}| = d_{n-1}$;
- Height: $q_n$;
- Size of the balcony: $d_n$;
- Height of the balcony $q_{n-1}$;

(iv): From $n$-th tower to $n+1$-th tower

We obtain $\{J^{(n+1)}\}$ from $\{J^{(n)}\}$ by performing the following:

- On the $n$-th tower, cut the balcony and put it under $J_1^{(n)}$, with the right ends aligned.
– Cut along the left end of balcony to reach the top, take the stack obtained and put then under what is left of \( J_1^{(n)} \). Align on the right and cut along the left. Repeat until the size of the remains do not allow new cut.

– The new tower obtained is the \( n+1 \)-th tower.

Number of cuts: \( a_n \);

Length of balcony: \( d_{n+1} \).

Height: \( q_{n+1} = a_nq_n + q_{n-1} \).

The union of all these intervals: \( S^1 \).
2: Circle Homeomorphism

Let $T : S^1 \to S^1$ be a homeomorphism. (1-1; onto; both $T$ and $T^{-1}$ exists). Assume that $T$ has NO periodic orbits.

(i) The sequence $\{q_n\}$

Let $x_0 = 0$ and $x_j = T^j(x_0)$. Define $\{q_n\}$ inductively as follows:

- $q_0 = 1$,

- Assume that $q_n$ is defined. $q_{n+1}$ is the first integer $j > q_n$ such that $x_{q_{n+1}}$ is closer to 0 than $x_{q_n}$.

Remark: "Closer" here means $x_{q_{n+1}} \in (x_{q_n}, x_{-q_n})$. 

[Diagram showing the sequence $\{q_n\}$ on a circle with points labeled $x_{q_n}$, $x_0$, $x_{q_{n+1}}$, and $x_{-q_n}$.]
**Claim:** $q_n$ is well-defined for all $n > 0$.

**Proof:** Let $F = T^{q_n}$, $y_0 = x_{q_n}$. Assume $x_{q_n}$ is on the left of 0, Let $y_j = F^j y_0$:

If $y_j$ never cross 0 from right, then $\{y_j\}$ is an increasing sequence and $\tilde{y} = \lim y_j$ is a fixed point of $F$, which is a periodic orbit of $T$. Contradict to the assumption that $T$ has no periodic orbit.

Let $y_{i-1}$ and $y_i$ be on different side of 0. Either $y_i$ or $y_{i-1}$ has to be in $(x_{q_n}, x_{-q_n})$ (Otherwise there is again a periodic orbit for $T$). So $q_{n+1}$ exists.

(ii) Some basic facts

**Fact 1:** $x_{q_n}$, $x_{q_n-1}$ are on different sides of 0.
Proof: (The same as before) If not, then \( x_{q_n - q_{n-1}} \) is closer to 0 than \( x_{q_n} \).

Fact 2: For all \( 0 \leq j \leq j' \leq q_n \)

\[ T^j(0, x_{q_{n-1}}) \cap T^{j'}(0, x_{q_{n-1}}) = \emptyset. \]

Proof: the same as before.

(iii) Rotation number

Build the towers inductively following the same process using the images of the left end of the balcony to cut in forming the \( n + 1 \)-th tower from the \( n \)-th.

\( a_n \): the number of cut needed from the \( n \)-th tower to the \( n + 1 \)-th tower.

Rotation number for \( T \):

\[ \rho(T') = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}. \]
(iv) \( T \) and \( R_{\rho(T)} : x \to \rho(T) \mod (1) \)

**Theorem:** \( T \) conjugates to \( R_{\rho(T)} \) if and only if

\[
\lim_{n \to \infty} \left( \max_i |J_i^{(n)}| \right) = 0.
\]

**Proof:** Let \( \alpha = \rho(T) \), \( y_j = \alpha j \mod(1), x_j = T^j0 \). Both \( \{x_j\} \) and \( \{y_j\} \) are dense in \( S^1 \).

Define \( h : S^1 \to S^1 \) as follows:

- Let \( h(x_j) = y_j \);

- For \( x \in S^1 \), \( \exists k_n \) such that \( x_{k_n} \to x \). Let \( h(x) = \lim_{n \to \infty} y_{k_n} \).

We need to show that \( h \) is well-defined, one to one and continuous.
**Observation 1:** $h$ preserves the order in $S^1$ on $\{x_j\}$. i.e. On $S^1$,

$$x_{j_1} < x_{j_2} < x_{j_3} \rightarrow y_{j_1} < y_{j_2} < y_{j_3}.$$

**Proof:** Inductively assume that $h$ preserves the order of $\{x_j\}$ for all $j < q_n$. Then $h$ preserves the order of $\{x_j\}$ for all $j < q_{n+1}$. This is because

(a) the end points of $n$-th tower of both $T$ and $R_\alpha$ are identically labelled; and

(b) the cutting points to in forming the $n+1$-th tower for both mapping are also identically labelled.
**Observation 2:** $h(x)$ is uniquely defined for all $x \in S^1$.

**Proof:** If not, then $x_{k_m} \rightarrow x$; $x_{k'_m} \rightarrow x$ such that $y_{k_m} \rightarrow y$ $y_{k'_m} \rightarrow y'$ and $y \neq y'$. Let $n$ be large enough so that $y$ and $y'$ be on the different piece of the $n$-th tower.

This implies; $x_{k_m}, x_{k'_m}$ are on the same piece of the $n$-th tower for $T$ but $y_{k_m}, y_{k'_m}$ are on different level. violating the order preserving property.

**Observation 3:** $h(x)$ is one to one.

**Proof:** Similar to Observation 2.

**Observation 4:** $h(x)$ maps intervals in $n$-th tower to $n$-th tower (Implies continuity of $h$ and $h^{-1}$).

**Proof:** Again similar to observation 2.

**Observation 5:** We have $h \circ T = R_\alpha \circ h$.

**Proof:** Let $x = \lim x_{k_m}$, then $h(x) = \lim y_{k_m}$.

$R_\alpha h(x) = \lim y_{k_m} + \alpha = \lim (y_{k_m} + \alpha)$

$= \lim y_{k_m+1} = \lim h(x_{k_m+1}) = \lim h(T(x_{k_m})) = h(T(x))$.

Here we used continuities of both $T$ and $h$. 
3. Circle Diffeomorphisms

**Theorem** (Denjoy) If $T : S^1 \to S^1$ is a diffeomorphism of irrational rotation number, and $\log |DT|$ is of bounded variation. The $T$ is conjugate to $R_{\rho(T)}$.

**Recall:** $f : S^1 \to \mathbb{R}$ is a function of bounded variation if there exists a constant $K > 0$, such that, for any division of $S^1$: $0 = t_1 < t_2 \cdots < t_n < t_{n+1} = 1$,

$$\sum |f(t_{i+1}) - f(t_i)| < K.$$

**Proof:** We show that there exists a $\lambda < 1$, such that the length of the base of the $n$-th tower for $T$ is $\leq \lambda^n$.

**Observation:** There exists $K > 0$, such that

$$K^{-1} < \log |DT^q_n(x)| < K$$

for all $x \in S^1$.

**Proof:** Let $n$ be fixed. First we have $t \in S^1$ such $|DT^q(t)| = 1$ (Otherwise $|T^q(S^1)| \neq 1$ therefore can not be a homeomorphism).
We now using \( \{t_i\} \) as end points to build the \( n \)-th tower, matching \( \{x_i\} \) to \( \{t_i\} \) so that \( \{(t_i, x_{k(i)})\} \) are mutually disjoint sub-intervals of \( S^1 \):

**Case 1:**

**Case 2:**
Then this observation follows from

$$| \log |DT^{q_n}(x)|| = | \log \frac{DT^{q_n}(x)}{DT^{q_n}(t)}|$$

$$\leq \sum | \log |DT(x_{k(i)})| - \log |DT(t_i)||$$

$$< K.$$

It follows from this observation, and the way the \(n\)-th tower is cut to form the \(n+1\)-th tower that

$$|J^{(n+1)}| + (a_{n-1})(\max |(DT^{q_n})|) |J^{(n+1)}| < |J^{(n)}|,$$

proving the claim if \(a_n > 1\) for all \(n\).

\[
\begin{array}{c}
X_{q_{n-1}} \quad X_{q_{n+q_{n-1}^{-1}}} \quad X_{q_{n-1}} \\
|\quad \quad \quad | \\
X_{q_{n-q_{n-1}}} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
\]

**Remark:** The possibility that \(a_n = 1\) is not a problem. We can go from \(J^{(n)}\) to \(J^{(n+2)}\).
4. Rational Rotations

If $T : S^1 \rightarrow S^1$ is a homeomorphism allowing an periodic solution of period $r$. Assume that

$$t_0, t_1, \cdots t_r$$

is an orbit of the smallest period allowed. We define $\{q_n\}$ by using $t_0$. The process stop at time $r$, and we obtain $q_1 = 1 < q_2 < \cdots q_m = r$.

Rotation number for $T$:

$$\rho(T) = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_m}}}}.$$

Remark: In general, one can not conjugate $T$ to $R_{\rho(T)}$. 
Homework

1. Let \( \{a_n\} \) be defined as on page 2. Prove that
\[
\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.
\]

2. Prove facts 1 and 2 claimed for circle homeomorphisms.

3. Prove that the definition of the rotation number for a given circle homeomorphism is independent of the orbit used in the construction of the towers.

4. Prove that if \( f \) and \( g \) are two circle homeomorphisms conjugating to each other, then \( \rho(f) = \rho(g) \).

5. Prove Observations 3 and 4 for \( h \) that is constructed to conjugate \( T \) and \( R_{\text{rho}(T)} \).

6. Prove that for any \( C^2 \) circle diffeomorphism \( T \), \( \log |DT^r| \) is in fact of bounded variation.