1D Dynamics: Circle Diffeomorphisms

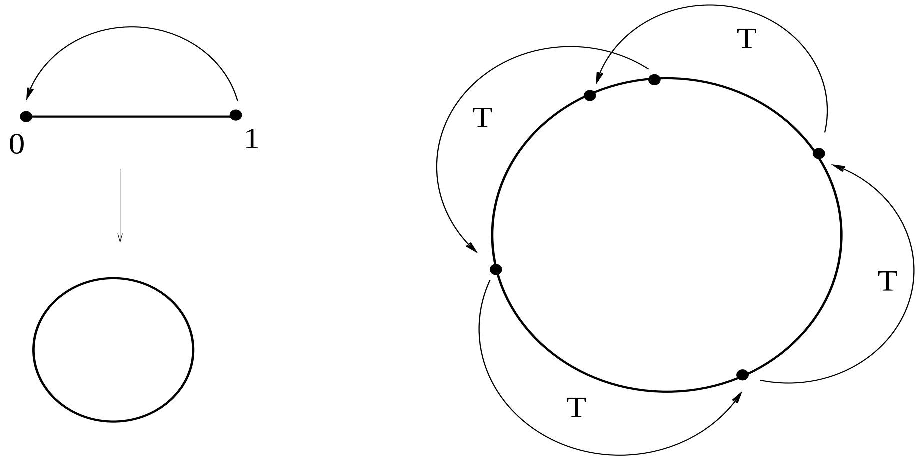
1. Rigid Rotations

Let $\alpha \in (0, \frac{1}{2})$ be irrational and $x \in S^1$ where

$$S^1 = \{e^{2\pi x i}, x \in [0,1)\}.$$

$T : S^1 \to S^1$ is defined by

$$T(x) = x + \alpha \; mod(1).$$



(i) Continuous fractions of α

- Let $x_0 = 0, x_j = T^j x_0 = \alpha j \ mod(1)$. x_j is on the left of 0 if $x_j \in (0, \frac{1}{2})$ and it is on the right if $x_j \in (\frac{1}{2}, 1)$.

- We define inductively a sequence $\{q_n\}$:

– $q_0 = 1$;

– Assume that q_n is defined, q_{n+1} is the first $j > q_n$ such that $d(x_{q_{n+1}}, 0) < d(x_{q_n}, 0)$. (Note that q_{n+1} always exists because $\{x_j\}$ is dense in S^1.)

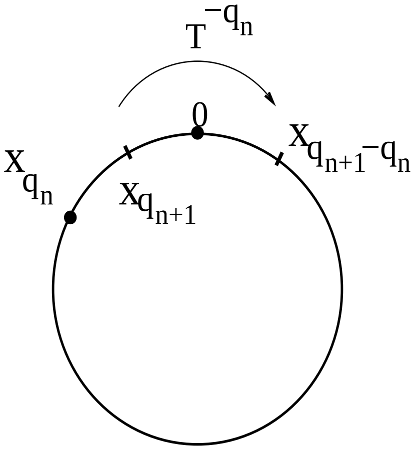
- Let $d_n = d(x_{q_n}, 0)$; and $a_n = \left[\frac{d_{n-1}}{d_n}\right]$ with $d_{-1} = 1$. $\{a_n\}$ is the continuous fraction for α:

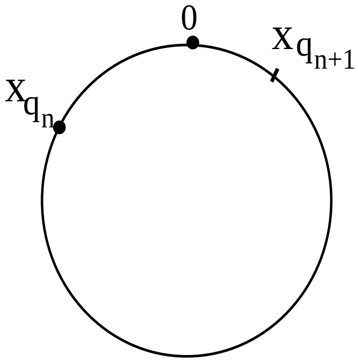
$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

(The proof of this fact is left as a homework.)

(ii) Some useful facts

Fact 1: x_{q_n}, $x_{q_{n-1}}$ are on different sides of 0.





Proof: If they are on the same side, then

$$d(x_{q_n - q_{n-1}}, 0) < d(x_{q_{n-1}}, 0),$$

contradicting to the definition of q_n.

Fact 2: For any $0 \le j < j' \le q_n$,

$$T^j((0, x_{q_{n-1}})) \cap T^{j'}((0, x_{q_{n-1}})) = \emptyset.$$

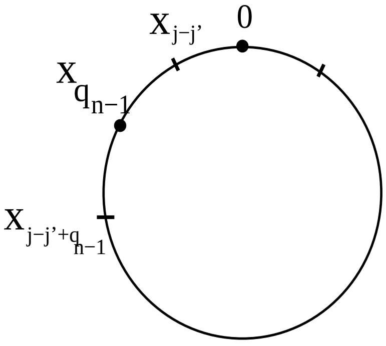
Proof: If not, we have

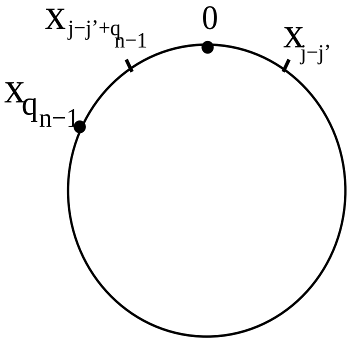
$$(0, x_{q_{n-1}}) \cap (x_{j'-j}, x_{j'-j+q_{n-1}}) \neq \emptyset.$$

This implies

$$d(x_{j'-j}, 0) < d(x_{q_{n-1}}, 0),$$

against the definition of q_n.





(iii) Tower structure

$x_0 = 0, x_1, \cdots, x_{q_n}$ divide S^1 into mutually disjoint intervals. We denote them as

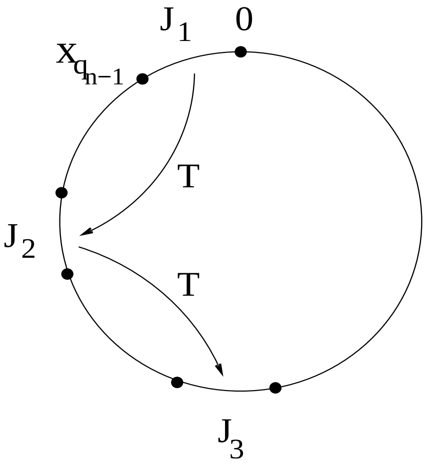
$$J_1^{(n)} \to J_2^{(n)} \to \cdots \to J_{q_n}^{(n)}$$

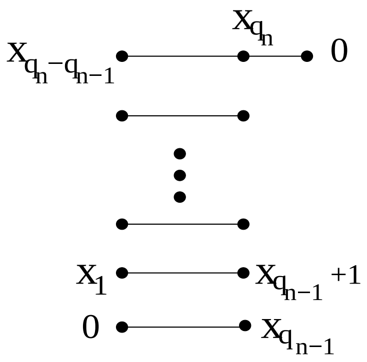
where

- $J_1^{(n)} = (0, x_{q_{n-1}})$;

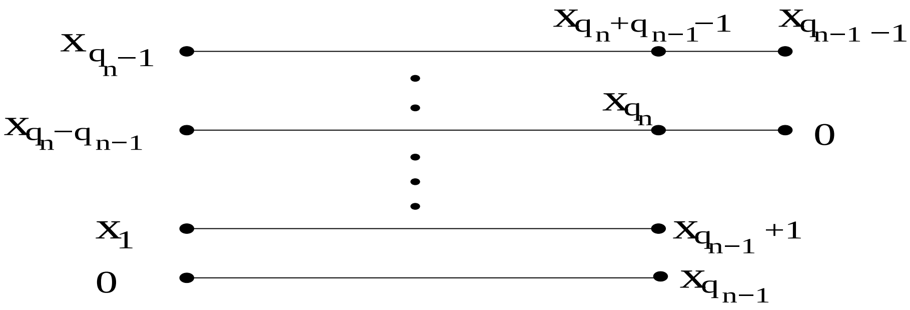
- $J_{i+1}^{(n)} = TJ_i^{(n)}$.

Note that by *Fact 2* in the above, $J_i \neq J_j$ for $i \neq j$. We put J_1 in the bottom, and J_2, J_3 on top one by one. Let us trace the end points of these intervals with care.





At $i = q_n - q_{n-1}$, one end of J_i is x_{q_n}. 0 must be on the other side, forcing the following tower.



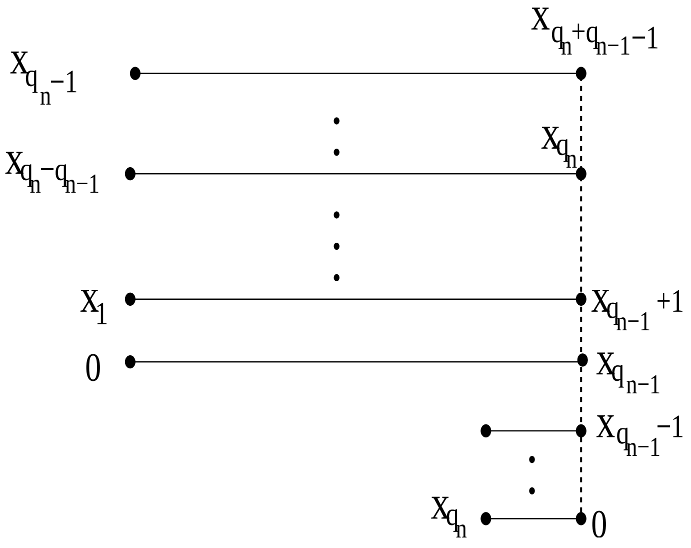
Characters of this tower:

- Size of the base: $|J_1^{(n)}| = d_{n-1}$;
- Height: q_n;
- Size of the balcony: d_n;
- height of the balcony q_{n-1};

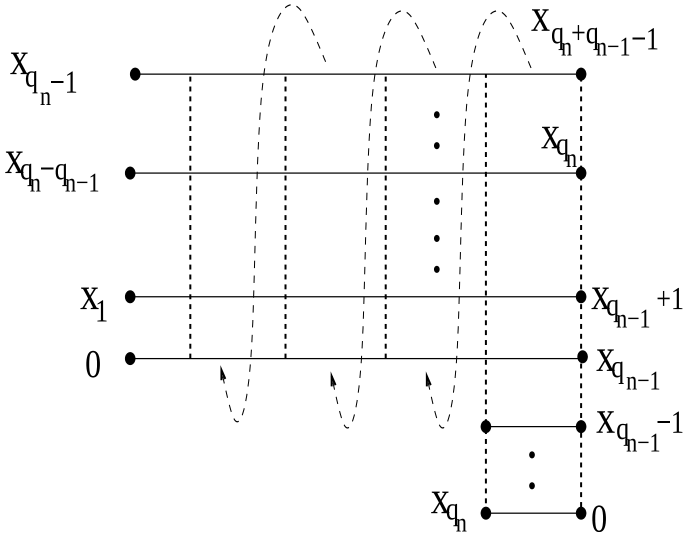
(iv): From n-th tower to $n+1$-th tower

We obtain $\{J^{(n+1)}\}$ from $\{J^{(n)}\}$ by performing the following:

- On the n-th tower, cut the balcony and put it under $J_1^{(n)}$, with the right ends aligned.



– Cut along the left end of balcony to reach the top, take the stack obtained and put then under what is left of $J_1^{(n)}$. Align on the right and cut along the left. Repeat until the size of the remains do not allow new cut.



– The new tower obtained is the $n+1$-th tower.

Number of cuts: a_n;

Length of balcony: d_{n+1}.

Height: $q_{n+1} = a_n q_n + q_{n-1}$.

The union of all these intervals: S^1.

2: Circle Homeomorphism

Let $T : S^1 \to S^1$ be a homeomorphism. (1-1; onto; both T and T^{-1} exists). Assume that T has **NO** periodic orbits.

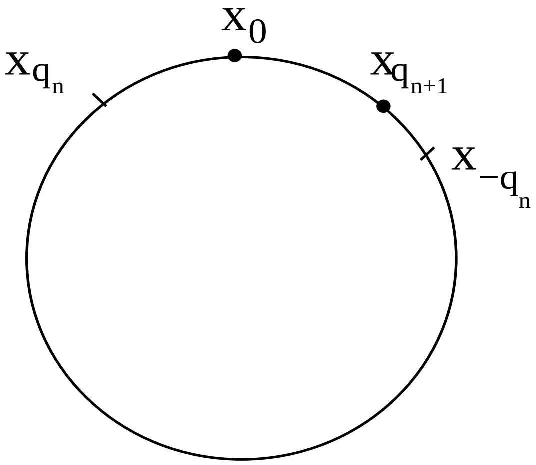
(i) The sequence $\{q_n\}$

Let $x_0 = 0$ and $x_j = T^j(x_0)$. Define $\{q_n\}$ inductively as follows:

– $q_0 = 1$,

– Assume that q_n is defined. q_{n+1} is the first integer $j > q_n$ such that $x_{q_{n+1}}$ is closer to 0 than x_{q_n}.

Remark: "Closer" here means $x_{q_{n+1}} \in (x_{q_n}, x_{-q_n})$.

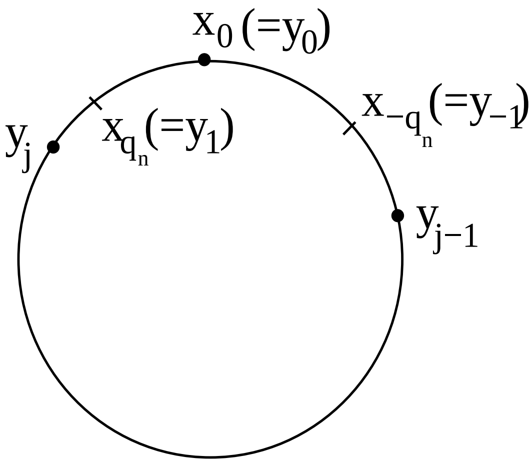


Claim: q_n is well-defined for all $n > 0$.

Proof: Let $F = T^{q_n}$, $y_0 = x_{q_n}$. Assume x_{q_n} is on the left of 0, Let $y_j = F^j y_0$:

If y_j never cross 0 from right, then $\{y_j\}$ is an increasing sequence and $\hat{y} = \lim y_j$ is a fixed point of F, which is a periodic orbit of T. Contradict to the assumption that T has no periodic orbit.

Let y_{i-1} and y_i be on different side of 0. Either y_i or y_{i-1} has to be in (x_{q_n}, x_{-q_n}) (Otherwise there is again a periodic orbit for T). So q_{n+1} exists.



(ii) Some basic facts

Fact 1: x_{q_n}, $x_{q_{n-1}}$ are on different sides of 0.

Proof: (The same as before) If not, then $x_{q_n - q_{n-1}}$ is closer to 0 than x_{q_n}.

Fact 2: For all $0 \leq j \leq j' \leq q_n$

$$T^j(0, x_{q_{n-1}}) \cap T^{j'}(0, x_{q_{n-1}}) = \emptyset.$$

Proof: the same as before.

(iii) Rotation number

Build the towers inductively following the same process using the images of the left end of the balcony to cut in forming the $n+1$-th tower from the n-th.

a_n: the number of cut needed from the n-th tower to the $n+1$-th tower.

Rotation number for T:

$$\rho(T) = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

(iv) T and $R_{\rho(T)} : x \to \rho(T) \mod (1)$

Theorem: T conjugates to $R_{\rho(T)}$ if and only if

$$\lim_{n\to\infty} (\max_i |J_i^{(n)}|) = 0.$$

Proof: Let $\alpha = \rho(T)$, $y_j = \alpha j \ mod(1)$, $x_j = T^j 0$. Both $\{x_j\}$ and $\{y_j\}$ are dense in S^1.

Define $h : S^1 \to S^1$ as follows:

– Let $h(x_j) = y_j$;

– For $x \in S^1$, $\exists k_n$ such that $x_{k_n} \to x$. Let $h(x) = \lim_{n\to\infty} y_{k_n}$.

We need to show that h is well-defined, one to one and continuous.

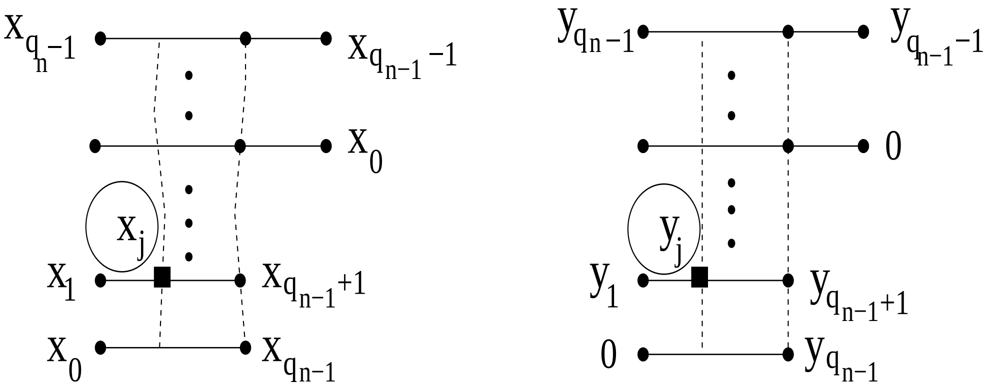
Observation 1: h preserves the order in S^1 on $\{x_j\}$. i.e. On S^1,

$$x_{j_1} < x_{j_2} < x_{j_3} \to y_{j_1} < y_{j_2} < y_{j_3}.$$

Proof: Inductively assume that h preserves the order of $\{x_j\}$ for all $j < q_n$. Then h preserves the order of $\{x_j\}$ for all $j < q_{n+1}$. This is because

(a) the end points of n-th tower of both T and R_α are identically labelled; and

(b) the cutting points to in forming the $n+1$-th tower for both mapping are also identically labelled.



Observation 2: $h(x)$ is uniquely defined for all $x \in S^1$.

Proof: If not, then $\exists x_{k_m} \to x$; $x_{k'_m} \to x$ such that $y_{k_m} \to y$ $y_{k'_m} \to y'$ and $y \neq y'$. Let n be large enough so that y and y' be on the different piece of the n-th tower.

This implies; $x_{k_m}, x_{k'_m}$ are on the same piece of the n-th tower for T but y_{k_m}, $y_{k'_m}$ are on different level. violating the order preserving property.

Observation 3: $h(x)$ is one to one.

Proof: Similar to Observation 2.

Observation 4: $h(x)$ maps intervals in n-th tower to n-th tower (Implies continuity of h and h^{-1}).

Proof: Again similar to observation 2.

Observation 5: We have $h \circ T = R_\alpha \circ h$.

Proof: Let $x = \lim x_{k_m}$, then $h(x) = \lim y_{k_m}$.

$R_\alpha h(x) = \lim y_{k_m} + \alpha = \lim(y_{k_m} + \alpha)$

$= \lim y_{k_m+1} = \lim h(x_{k_m+1}) = \lim h(T(x_{k_m})) = h(T(x))$.

Here we used continuities of both T and h.

3. Circle Diffeomorphisms

Theorem (Denjoy) If $T : S^1 \to S^1$ is a diffeomorphism of irrational rotation number, and $\log |DT|$ is of bounded variation. The T is conjugate to $R_{\rho(T)}$.

Recall: $f : S^1 \to \mathbb{R}$ is a function of bounded variation if there exists a constant $K > 0$, such that, for any division of S^1: $0 = t_1 < t_2 \cdots < t_n < t_{n+1} = 1$,

$$\sum |f(t_{i+1}) - f(t_i)| < K.$$

Proof: We show that there exists a $\lambda < 1$, such that the length of the base of the n-th tower for T is $\leq \lambda^n$.

Observation: There exists $K > 0$, such that

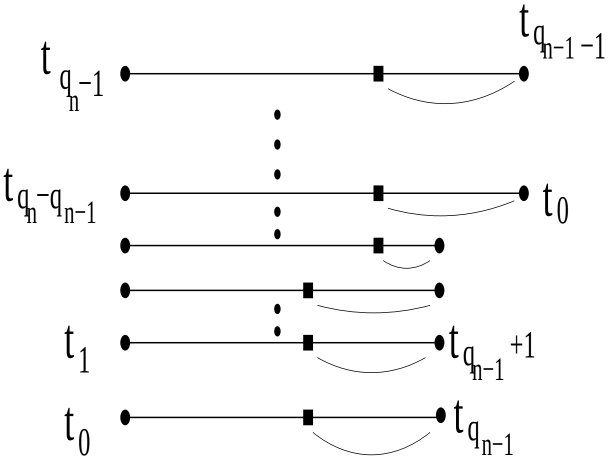
$$K^{-1}| < \log |DT^{q_n}(x)|| < K$$

for all $x \in S^1$.

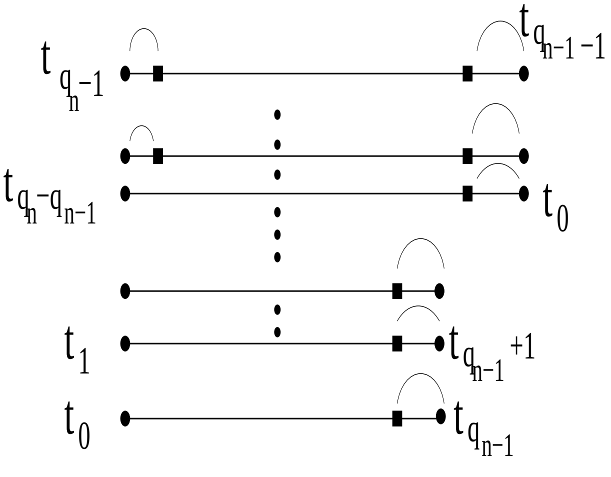
Proof: Let n be fixed. First we have $t \in S^1$ such $|DT^{q_n}(t)| = 1$ (Otherwise $|T^{q_n}(S^1)| \neq 1$ therefore can not be a homeomorphism).

We now using $\{t_i\}$ as end points to build the n-th tower, matching $\{x_i\}$ to $\{t_i\}$ so that $\{(t_i, x_{k(i)})\}$ are mutually disjoint sub-intervals of S^1:

Case 1:



Case 2:



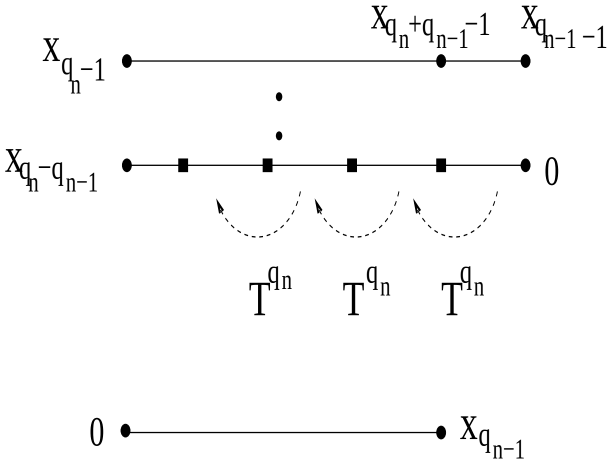
Then this observation follows from

$$\begin{aligned} |\log|DT^{q_n}(x)|| &= |\log \frac{DT^{q_n}(x)}{DT^{q_n}(t)}| \\ &\leq \sum |\log|DT(x_{k(i)})| - \log|DT(t_i)|| \\ &< K. \end{aligned}$$

It follows from this observation, and the way the n-th tower is cut to form the $n+1$-th tower that

$$|J^{(n+1)}|+(a_n-1)(\max|(DT^{q_n})|)|J^{(n+1)}| < |J^{(n)}|,$$

proving the claim if $a_n > 1$ for all n.



Remark: The possibility that $a_n = 1$ is not a problem. We can go from $J^{(n)}$ to $J^{(n+2)}$.

4. Rational Rotations

If $T : S^1 \to S^1$ is a homeomorphism allowing an periodic solution of period r. Assume that

$$t_0, t_1, \cdots t_r$$

is an orbit of the smallest period allowed. We define $\{q_n\}$ by using t_0. The process stop at time r, and we obtain $q_1 = 1 < q_2 < \cdots q_m = r$.

Rotation number for T:

$$\rho(T) = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_m}}}}.$$

Remark: In general, one can not conjugate T to $R_{\rho(T)}$.

Homework

1. Let $\{a_n\}$ be defined as on page 2. Prove that

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

2. Prove facts 1 and 2 claimed for circle homeomorphisms.

3. Prove that the definition of the rotation number for a given circle homeomorphism is independent of the orbit used in the construction of the towers.

4. Prove that if f and g are two circle homeomorphisms conjugating to each other, then $\rho(f) = \rho(g)$.

5. Prove Observations 3 and 4 for h that is constructed to conjugate T and $R_{rho(T)}$.

6. Prove that for any C^2 circle diffeomorphism T, $\log|DT|$ is in fact of bounded variation.