Hyperbolic Dynamics: Milnikov’s Method

1. Non-autonomous Periodic ODEs

A. Autonomous Systems

Let $z \in \mathbb{R}^n$ be the phase variable and $t$ be the time. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth mapping. We call the following ordinary equation

$$\frac{dz}{dt} = f(z)$$

(1)

an autonomous system.

Phase portrait

Let $z(t, z_0)$ be the solution of equation (1) satisfying $z(0, z_0) = z_0$. Then for any $t_0 \in \mathbb{R}$, we have

$$z(t + t_0, z_0) = z(t, z(t_0, z_0))$$
This is because on both side we have a solution starting at $z(t_0, z_0)$ at $t = 0$. Hence they are the same by the uniqueness of the solutions for a given initial condition.

- For $z(t, z_0)$ given, let

$$O(z_0) = \{z(t, z_0) : t \in \mathbb{R}\}$$

We call $O(z_0)$ an orbit of equation (1) in phase space $\mathbb{R}^n$ through $z_0$.

**Claim 1.1:** If $O(z_1) \cap O(z_2) \neq \emptyset$, then $O(z_1) = O(z_2)$.

**Proof:** For $p \in O(z_1) \cap O(z_2)$, let $p_1 = z(t_1, z_1) = z(t_2, z_2)$ then $\forall t \in \mathbb{R},$

$$z(t, z_1) = z(t - t_1, z(t_1, z_1)) = z(t - t_1, z(t_2, z_2)) = z(t - t_1 + t_2, z_2) \in O(z_2)$$

So $O(z_1) \subset O(z_2)$. Similarly we have $O(z_2) \subset O(z_1)$.

We obtain a **phase portrait** by drawing all orbits in phase space $\mathbb{R}^n$. By the claim in the
above, phase portrait consists *non-intersecting* curves, each of which representing an orbit.

**Ex:** For \( z = (x, y) \in \mathbb{R}^2 \), let

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= x - x^3
\end{align*}
\]

be an autonomous system defined on \( \mathbb{R}^2 \). The phase portrait is as follows

![Phase Portrait](image)

**Space of solution curves**

We can also view each solution \( z = z(t, z_0) \) as a curve in space \( (t, z) \in \mathbb{R} \times \mathbb{R}^n \). We have a *portrait of solutions curves*. 
– For autonomous systems, phase portrait is obtained by projecting the portrait of solutions in $\mathbb{R}^n \times \mathbb{R}$ on to the phase space $\mathbb{R}^n$. There exists an entire family of solution curves corresponding to one orbit in the phase portrait.

![Phase Portrait Diagram]

– Note that these two portraits are drew in different spaces.

**Time-T map**

Let $z(t, z_0)$ be the solution of equation (1) satisfying $z(0, z_0) = z_0$. For $T > 0$ let $F_T : \mathbb{R}^n \to \mathbb{R}^n$ be define as

$$F_T(z_0) = z(T, z_0).$$
We call $F_T$ the *time $T$ map* induced by equation (1).

**Claim 1.2:** Let $z_0 \in \mathbb{R}^n$, and $\{z_i\}_{i=-\infty}^{\infty}$ be the orbit of $z_0$ under the iterations of $F_T$. Then $z_i = z(iT, z_0)$.

**Proof:** We prove by induction. Assume that $z_i = z(iT, z_0)$, we have

$$z((i + 1)T, z_0) = z(T, z(iT, z_0)) = F(z_i).$$

The relationship between an orbit under $F$ and its corresponding solution curve is shown in the next picture.
Phase portrait and the time-\(T\) map

Phase portrait of an autonomous system can also be used to show the orbit structure of the time-\(T\) map.

\textbf{Ex: } Let us use the same example as in the above.

\begin{itemize}
  \item We can read this picture as the phase portrait. We can also read this same picture as a description of the orbit structure of the time-\(T\) map \(F_T\).
\end{itemize}
– Though we have the same picture, these two interpretations are entirely different.

B  Non-autonomous systems

Let us now consider non-autonomous differential equations

\[
\frac{dz}{dt} = f(z, t) \tag{2}
\]

in which \( f(z, t) \) is a dependent of \( t \).

Very little of the discussions we made on autonomous systems in the above make sense!

– It is not true that \( z(t + t_0, z_0) = z(t, z(t_0, z_0)) \).

– Consequently, orbits do intersect in phase space. So it does not make sense to talk about phase portrait in phase space \( \mathbb{R}^n \).
The time-$T$ map $F_T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(z_0) = z(T, z_0)$ is not useful: the orbit starting from $z_0 \in \mathbb{R}^n$ under $F_T$ does not represent any given solution.

There is, however, one thing that survives: the portrait of solution. According to theorem on the uniqueness of solutions of a given initial value problem, we know that in $\mathbb{R}^n \times \mathbb{R}$, solutions curves are non-intersecting graphs from $\mathbb{R} \to \mathbb{R}^n$.

**Non-autonomous systems of period $T$**

Let us assume that in equation (2), $f(z, t) = f(z, t + T)$ for some fixed $T > 0$ for all $z$ and $t$. We call such system an non-autonomous periodic system.

For a non-autonomous periodic system of period $T$,
we have $z(t + T, z_0) = z(t, z(T, z_0))$.

Though it does not makes much sense to obtain a phase portrait in $\mathbb{R}^n$. It does make sense to study the time-$T$ map $F_T$ where $T$ is the period of the function $f(z, t)$. i.e., let $F_T : \mathbb{R}^n \to \mathbb{R}^n$ be such that $F_T(z_0) = z(T, z_0)$, we have

$$F^i(z_0) = z(iT, z_0).$$

Note that again, a given orbit $\{z_i\} \in \mathbb{R}^n$ under $F_T$ represent a solution curve in $\mathbb{R}^n \times \mathbb{R}$. 
2 The Milnikov’s Method

Let $z \in \mathbb{R}^2$ be the phase space, and

$$\frac{dz}{dt} = f(z) + \varepsilon g(z, t) \quad (3)$$

is such that $g(z, t + T) = g(z, t)$. This is non-autonomous periodic equation of period $T$. Let us also assume that $f(z), g(z, t)$ are both $C^2$, and $\varepsilon$ is a small parameter. When $\varepsilon = 0$, this is an autonomous system which we will regard as our “unperturbed system”. The time dependent term $\varepsilon g(z, t)$ is regarded as a term of perturbation.

A. Properties of the time-$T$ map

Unperturbed system and homoclinic loop

For the unperturbed system

$$\frac{dz}{dt} = f(z)$$

we assume that
(H) $p_0$ is a saddle point, and there exists a solution, which we denote as $q(t)$, such that $\lim_{t \to \pm \infty} q(t) = p_0$.

We call the orbit representing $q(t)$ in phase space $\mathbb{R}^2$ a homoclinic loop.

Note that, for any $t_0 \in \mathbb{R}$ given, $q(t - t_0)$ is also a solution of the unperturbed equation represented by the same homoclinic loop in phase portrait.
**Time-T map**

We now study the perturbed non-autonomous system. Let \( z_\varepsilon(t, t_0, z_0) \) be the solution of the equation (3) of the parameter value \( \varepsilon \) satisfying \( z(t_0, t_0, z_0) = z_0 \). For the moment we fix \( t_0 \) and let

\[
F_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2
\]

be the time-T map defined by

\[
F_\varepsilon(z_0) = z(t_0 + T, t_0, z_0).
\]

We have

(i) Orbits of \( F_\varepsilon \) represents solutions of the corresponding differential equation.

(ii) \( p_0 \) is a hyperbolic fixed point for \( F_0 (\varepsilon = 0) \). Furthermore, the curve represented by the orbit of \( q(t) \) is invariant under the iterations
of $F_0$. i.e., it follows from assumption (H) that we have the following picture for the orbit structure of $F_0$.

![Diagram](image)

**Remarks**

(a) Though this picture is identical to the one in assumption (H), they represent completely different things. The loop in previous picture represents one **continuous** orbit of the unperturbed equation. Here it represents a collection of discrete orbits that is invariant under the iterations of the mapping $F_0$.

(b) $F_\varepsilon$ is smooth dependent of $\varepsilon$ and $z \in \mathbb{R}^2$. The smooth dependency of $F_\varepsilon$ in $\varepsilon$ is from the fact that the solutions of differential equations are smooth dependents of parameters.

(c) We also involved a parameter $t_0$ in the definition of $F_\varepsilon$. So $F_\varepsilon$ is also a function of $t_0$. 
Local orbit structure

We now study the orbit structure of $F_\varepsilon$ around $p_0$.

**Claim 2.1:** There exists a unique hyperbolic fixed point $p_\varepsilon$ for $F_\varepsilon$ such that

(i) $|p_\varepsilon - p_0| = O(\varepsilon)$, and

(ii) There exist neighborhood of $U$ of $p_\varepsilon$ of fixed size (independent of $\varepsilon$), such that $W_{loc}^u(p_\varepsilon)$ for $F_\varepsilon$ and $W_{loc}^u(p_0)$ are $O(\varepsilon)$ close in $C^1$-norms in $U$. So are $W_{loc}^s(p_\varepsilon)$ and $W_{loc}^s(p_0)$.

**Proof:** (i) To find $p_\varepsilon$ we solve the equation $F_\varepsilon(z) = z$. According to the implicit function theorem, this equation has a unique solution in a neighborhood of $p_0$ for $\varepsilon$ sufficiently small if $DF_0 - ID$ is non-singular at $p_0$. This is the case here because $p_0$ is a hyperbolic fixed point.

(ii) This is proved by a computation similar to what we did in the proof of the smoothness of the local stable and unstable manifold. This is a good exercise.
Global structure

We now move to a more global picture. Recall that \( q(t) \) is an solution of the unperturbed system assumed in (H). It is also a parameterized representation of the homoclinic loop.

At the point \( q(0) \), let us take a line segment \( l^\perp \), that is perpendicular to the tangent direction of the homoclinic loop at \( q(0) \). The images of \( W_{loc}^u(p_\epsilon) \) under the iterations of \( F_\epsilon \) will intersect \( l^\perp \). Let us denote the first intersection as \( q^u(\epsilon) \). Similarly, the images of \( W_{loc}^s(p_\epsilon) \) under \( T^{-1} \) will also intersect \( l^\perp \). Let us denote the first intersection as \( q^s(\epsilon) \).
We further observe that

$$|q^s(\varepsilon) - q(0)|, |q^u_\varepsilon - q(0)| < K\varepsilon$$

For some $K$ independent of $\varepsilon$. This is because (a) $W^u_{loc}(p_\varepsilon)$ and $W^u_{loc}(p_0)$ are of a fixed size independent of $\varepsilon$, (b) The number of iterations it takes to get the image of $W^u_{loc}$ to intersect $l^\perp$ is also independent of $\varepsilon$.

We aim in computing

$$d(t_0) := |q^s(\varepsilon) - q^u(\varepsilon)|.$$  

We write $d(t_0)$ here to indicate that these are all quantities depending on $t_0$.

**B. Solutions through $q^u(\varepsilon)$ and $q^s(\varepsilon)$**

- Corresponding to $p_\varepsilon$, the indicated hyperbolic fixed point of $F_\varepsilon$, there is a solution $z_\varepsilon(t, t_0, p_\varepsilon)$ satisfying

$$z_\varepsilon(t_0, t_0, p_\varepsilon) = z_\varepsilon(t_0 + T, t_0, p_\varepsilon) = p_\varepsilon.$$
We will denote this solution as $\gamma_\varepsilon(t, t_0)$. It is clear that $\gamma_0(t, t_0) = p_0$.

\[-\text{Let } q_\varepsilon^s(t, t_0) = z_\varepsilon(t, t_0, q^s(\varepsilon)) \text{ be the solution satisfying the initial condition } z_\varepsilon(t, t_0, q^s(\varepsilon)) = q^s(\varepsilon). \text{ We have}
\]
\[|q_\varepsilon^s(t, t_0) - \gamma_\varepsilon(t, t_0)| \to 0\]

exponentially fast as $t \to +\infty$.

Similarly, we have $q_\varepsilon^u(t, t_0)$ such that
\[|q_\varepsilon^u(t, t_0) - \gamma_\varepsilon(t, t_0)| \to 0\]

exponentially fast as $t \to -\infty$. 
Note that at $\varepsilon = 0$, $q^s_0(t, t_0) = q^u_0(t, t_0) = q(t - t_0)$.

**Claim 2.3:** There exists $K > 0$ independent of $\varepsilon$ such that for all $\varepsilon$ sufficiently small, and all $t \in [t_0, \infty)$, we have

$$|q^s_\varepsilon(t, t_0) - q(t - t_0)| < K\varepsilon$$

Similarly we have for all $t \in (-\infty, t_0]$,

$$|q^u_\varepsilon(t, t_0) - q(t - t_0)| < K\varepsilon.$$

**Proof:** We have proved that the indicated inequalities hold for $t = t_0$. Also, there exists $t_1$ independent of $\varepsilon$, such that $q(t_1 - t_0) \in U$. By continuity, we have

$$|q^s_\varepsilon(t, t_0) - q(t - t_0)| < K\varepsilon$$
for \( t \in [t_0, t_1] \) for a different \( K \).

Inside of \( U \) we have
\[
|q^s_\varepsilon(t, t_0) - \gamma_\varepsilon(t, t_0)| < K|q^s_\varepsilon(t_1, t_0) - \gamma_\varepsilon(t_1, t_0)|e^{-\lambda t}
\]
for some \( \lambda > 0 \). Similarly at \( \varepsilon = 0 \),
\[
|q(t - t_0) - \gamma_0(t, t_0)| < K|q(t_1 - t_0) - \gamma_0(t_1, t_0)|e^{-\lambda t}.
\]
We also have
\[
|\gamma_\varepsilon(t, t_0) - \gamma_0(t, t_0)| < K \varepsilon
\]
These three inequalities are combined to prove the first inequality of this claim. A proof for the second inequality is similar.

We can now write \( q^s_\varepsilon(t, t_0) \) on \( [t_0, \infty) \) as as
\[
q^s_\varepsilon(t, t_0) = q(t - t_0) + \varepsilon h(t, t_0) + \mathcal{O}(\varepsilon^2).
\]

**Claim 2.4:** The function \( h := h(t, t_0) \) in the above satisfies
\[
\frac{d}{dt} h = Df(q(t - t_0))h + g(q(t - t_0), t).
\]

**Proof:** Substitute the formula for \( q^s_\varepsilon(t, t_0) \) into
\[
\frac{dz}{dt} = f(z) + \varepsilon g(z, t)
\]
we obtain
\[ \varepsilon \frac{dh}{dt} + \mathcal{O}(\varepsilon^2) = f(q^s_\varepsilon(t_0, t)) - f(q(t - t_0)) \\
+ \varepsilon g(q^s_\varepsilon(t_0, t), t) \]

We now obtain the indicated equation by dividing \( \varepsilon \) both side first then let \( \varepsilon \to 0 \).

Similarly

**Claim 2.5:** On \((-\infty, t_0]\) we have

\[ q^u_\varepsilon(t, t_0) = q(t - t_0) + \varepsilon h(t, t_0) + \mathcal{O}(\varepsilon^2). \]

where \( h := h(t, t_0) \) satisfies the same equation in the above.

**C. Computing** \( d(t_0) = |q^u_\varepsilon - q^s_\varepsilon| \)

**A notation:** For two vectors \( a = (a_1, a_2), b = (b_1, b_2) \) in \( \mathbb{R}^2 \), let \( a \wedge b = a_1 b_2 - a_2 b_1 \). Notice that \( a^\perp = (-a_2, a_1) \) is perpendicular to \( a \), and \( a \wedge b = a^\perp \circ b \) by definition. \( a \wedge b \) is the projection of vector \( b \) on \( a^\perp \).
**Proposition 2.1:** Let us assume that $tr(Df(z)) = 0$ for all $z \in \mathbb{R}^2$. Then

$$d(t_0) = \frac{M(t_0)}{|f(q(0))|}\varepsilon + \mathcal{O}(\varepsilon^2)$$

where

$$M(t_0) := \int_{-\infty}^{\infty} f(q(t-t_0)) \wedge g(q(t-t_0), t) dt.$$ 

**Proof:** For $t \in [t_0, \infty)$, let

$$\Delta^s(t, t_0) := f(q(t-t_0)) \wedge h(t, t_0)$$

where $h(t, t_0)$ is such that

$$q_\varepsilon^s(t, t_0) = q(t-t_0) + \varepsilon h(t, t_0) + \mathcal{O}(\varepsilon^2)$$

as defined in the above. Similarly, for $t \in (-\infty, t_0]$, let

$$\Delta^u(t, t_0) := f(q(t-t_0)) \wedge h(t, t_0).$$

We have by definition that

$$d(t_0) = \varepsilon \frac{|\Delta^u(t_0, t_0) - \Delta^s(t_0, t_0)|}{|f(q(0))|} + \mathcal{O}(\varepsilon^2).$$

Note that we have
\[ \frac{d}{dt} \Delta^s(t, t_0) = Df(q(t - t_0))f(q(t - t_0)) \land h(t, t_0) \]
\[ + f(q(t - t_0)) \land \frac{d}{dt} h(t, t_0) \]
\[ = Df(q(t - t_0))f(q(t - t_0)) \land h(t, t_0) \]
\[ + f(q(t - t_0)) \land (Df(q(t - t_0))h + g(q(t - t_0), t)) \]
\[ = \text{tr}(Df) \Delta^s(t, t_0) + f(q(t - t_0)) \land g(q(t - t_0), t) \]
\[ = f(q(t - t_0)) \land g(q(t - t_0), t) \]

The last equality is obtained by using the assumption that \( \text{tr}(Df) = 0 \).

Note that as \( \Delta^s(\infty, t_0) = 0 \) because as \( t \to \infty \), \( f(q(t - t_0)) \to f(p_0) = 0 \). It follows then

\[ \Delta^s(t_0, t_0) = - \int_{t_0}^{\infty} f(q(t - t_0)) \land g(q(t - t_0), t) dt \]

Similarly we have

\[ \Delta^u(t_0, t_0) = \int_{-\infty}^{t_0} f(q(t - t_0)) \land g(q(t - t_0), t) dt \]

Our Proposition follows from combining these two.
C. Applications

Existence of homoclinic orbits

The following is a direct consequence of this Proposition.

Claim 3.1: The assumptions are the same as in Proposition 2.1. If $M(t_0)$ as a function of $t_0$ has a simple zero, i.e. there exists $\hat{t}$ such that $M(\hat{t}) = 0, M'(\hat{t}) \neq 0$. Then $F_\varepsilon$ for $t_0 = \hat{t}$ has a transversal homoclinic orbit.

Proof: Let us fix $t_0 = \hat{t}$ in the above to get the time-$T$ map $F_\varepsilon$. However, instead of using $q(0)$ to determine $q^s_\varepsilon$ and $q^u_\varepsilon$, we use $q(\delta)$ for a $\delta$ inside a small interval around zero to compute $d(\hat{t})$. We denote this distance as $d(\hat{t}, \delta)$. It then follows from the same computation as in the above that

$$d(\hat{t}, \delta) = \varepsilon \frac{M(\hat{t}, \delta)}{|f(q(\delta))|} + O(\varepsilon)$$

Our assumption implies that, for all $\varepsilon$ sufficiently small, $d(\hat{t}, \delta)$ as a function in $\delta$ has a simple zero around $\delta = 0$. 
Remember transversal homoclinic orbit implies the existence of a Horseshoe map.

Note that by a change of variable $t \rightarrow t - t_0$, we have

$$M(t_0) = \int_{-\infty}^{\infty} f(q(t)) \wedge g(q(t), t + t_0) dt.$$  

This is usually more convenient in computing $M(t_0)$.

**Hamiltonian system**

Let $q, p \in \mathbb{R}^n$ and $H(q, p) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a given function. We call the following system of ordinary equations

$$\frac{dq}{dt} = \partial_p H(q, p), \quad \frac{dp}{dt} = -\partial_q H(q, p)$$

a *Hamiltonian System* and $H(q, p)$ the Hamiltonian of this system.

**Claim 3.2:** Hamiltonian systems satisfying $\text{tr}(Df) = 0$.  

Proof: \( tr(Df) = \partial_{pq} H - \partial_{qp} H = 0. \)

Mathematical theories on Hamiltonian systems is one of the major subject in the history. This is because all energy conservative mechanical systems are Hamiltonian systems with

\[ H = T + V \]

where \( T \) is the kinetic energy and \( V \) is the potential energy of the mechanical systems subjected to study.

Ex: The \( n \)-body problem.

![Diagram of n-body problem](image)
This is a Hamiltonian system with

\[ H = \frac{1}{2} \sum_{i=1}^{n} m_i^{-1} |p_i|^2 - \sum_{i<j} m_i m_j |r_{ij}| \]

where \( p_i = m_i \frac{d}{dt} r_i \).

**Geometric implication of** \( tr(Df) = 0 \)

The condition \( tr(Df) = 0 \) has the following geometric interpretation:

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth 1-1 mapping. We denote the volume of an open subset \( U \in \mathbb{R}^n \) as \( m(U) \).

**Definition:** We say that \( F \) is volume preserving if, for any open subset \( U \in \mathbb{R}^n \), we have \( m(U) = m(F(U)) \).

Let \( z(t, z_0) \) be the solution of a given equation \( \frac{dz}{dt} = f(z) \) satisfying \( z_0 = z(0, z_0) \). Recall that,
for \( t_0 \) given as a fixed number, the time-\( t_0 \) map \( F_{t_0} : \mathbb{R}^n \to \mathbb{R}^n \) is define by

\[
F_{t_0}(z_0) = z(t_0, z_0).
\]

**Claim 3.3:** Let \( t_0 \in \mathbb{R} \) be fixed, and \( F_{t_0} : \mathbb{R}^n \to \mathbb{R}^n \) be the time-\( t_0 \) map. Then under the assumption that \( \text{tr}(Df) = 0 \), \( F_{t_0} \) is volume preserving.

Note that this implies that time-t mapping defined by any given Hamiltonian system is volume preserving.

**Proof:** Let \( \xi \in \mathbb{R}^n \) be a given initial condition, and \( z = z(t, \xi) \) be the solution satisfying \( \xi = z(0, \xi) \). We are in need of writing vectors explicitly in their components. So \( z = (z_1, \ldots, z_n) \), \( \xi = (\xi_1, \ldots, \xi_n) \) and \( f = (f_1, \ldots, f_n) \). The solution is now written in component as

\[
z(t, \xi) = (z_1(t, \xi_1, \ldots, \xi_n), \ldots, z_n(t, \xi_1, \ldots, \xi_n))
\]

Let \( J := Dz = (\partial_{x_i}z_i) \) be the Jacobi matrix of \( z \) with respect to \( \xi \). For this claim to hold it suffices to prove that

\[
det(J) = 1.
\]
Write $J = (J_{ij}(t))$ where $J_{ij} = \partial_{\xi_j} z_i$. By taking partial derivatives with respect to $\xi$ on the equation give, we have

$$\frac{dJ_{ij}}{dt} = \sum_{k=1}^{n} (\partial_{z_k} f_i) J_{k,j}$$

Let $X_{ij}$ be the complement of $J_{ij}$ of the matrix $J$, i.e., $(-1)^{i+j} X_{ij}$ be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleted the $j$-th raw and the $i$-th column of the matrix $J$. Let $M_i$ be the matrix we obtain by replacing $J_{ij}$ by $\frac{d}{dt} J_{ij}$ for $j = 1, \ldots, n$.

We have

$$\det M_i = \sum_{j=1}^{n} X_{ij} \frac{d}{dt} J_{ij},$$

and

$$\frac{d}{dt} \det(J) = \sum_{i=1}^{n} \det M_i = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} \frac{d}{dt} J_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} X_{ij} (\partial_{z_k} f_i) J_{k,j}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} (\partial_{z_k} f_i) \sum_{j=1}^{n} X_{ij} J_{k,j}$$

Note that we have

$$\sum_{j=1}^{n} X_{ij} J_{k,j} = \det J \delta_{ik}$$
where $\delta_{ik} = 1$ if $i = k$ but $\delta_{ik} = 0$ if $i \neq k$. Hence it follows that

$$\frac{d}{dt} \det J = \left( \sum_{i=1}^{n} (\partial z_i f_i) \right) \det J = tr(Df) \det J.$$  

So under the assumption that $tr(Df) = 0$, we have

$$\det J(t) = \det J(0) = 1.$$  

The last equality holds because $J(0) = ID$.

**Explicit computation of $M(t_0)$: an example**

We finish by giving an explicit application of Milnikov’s method.

For $z = (x, y) \in \mathbb{R}^2$, let

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x - x^3 + \varepsilon(\gamma \cos \omega t - \delta y)$$

where $\varepsilon, \gamma, \omega$ and $\delta$ are parameters.
For $\varepsilon = 0$, we have $f(x, y) = (y, x - x^3)$ and

$$Df = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}.$$ 

So $tr(Df) = 0$. It is easy to check that $p_0 = (0, 0)$ is a saddle point for the unperturbed equation. Let

$$x(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}$$
$$y(t) = \frac{2\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}.$$ 

Then

$$q(t) = (x(t), y(t))$$

is a homoclinic loop for the unperturbed system.

By definition we have

$$M(t_0) = \int_{-\infty}^{\infty} y(t)[\gamma \cos \omega(t + t_0) - \delta y(t)]dt$$

$$= -2\sqrt{2}\gamma \int_{-\infty}^{\infty} \frac{(e^t - e^{-t})}{(e^t + e^{-t})^2} \cos \omega(t + t_0)dt$$

$$-8\delta \int_{-\infty}^{\infty} \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^4}dt.$$
We evaluate these two integrals separately to obtain

\[ M(t_0) = -\frac{4\delta}{3} + 2\sqrt{2}\gamma \pi \omega \frac{\sin \omega t_0}{e^{\frac{\pi \omega}{2}} + e^{-\frac{\pi \omega}{2}}} \]

Note that we have a simple zero for \( M(t_0) \) if

\[ \frac{\gamma}{\delta} > \frac{\sqrt{2}(e^{\frac{\pi \omega}{2}} + e^{-\frac{\pi \omega}{2}})}{3\pi \omega} \]