Hyperbolic Dynamics: Milnikov's Method

1. Non-autonomous Periodic ODEs

A. *Autonomous Systems*

Let $z \in \mathbb{R}^n$ be the phase variable and t be the time. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping. We call the following ordinary equation

$$\frac{dz}{dt} = f(z) \tag{1}$$

an *autonomous system*.

Phase portrait

– Let $z(t, z_0)$ be the solution of equation (1) satisfying $z(0, z_0) = z_0$. Then for any $t_0 \in \mathbb{R}$, we have

$$z(t + t_0, z_0) = z(t, z(t_0, z_0))$$

This is because on both side we have a solution starting at $z(t_0, z_0)$ at $t = 0$. Hence they are the same by the uniqueness of the solutions for a given initial condition.

– For $z(t, z_0)$ given, let

$$O(z_0) = \{z(t, z_0) : \quad t \in \mathbb{R}\}$$

We call $O(z_0)$ an orbit of equation (1) in phase space $\mathbb{R}^n$ through z_0.

Claim 1.1: If $O(z_1) \cap O(z_2) \neq \emptyset$, then $O(z_1) = O(z_2)$.

Proof: For $p \in O(z_1) \cap O(z_2)$, let $p_1 = z(t_1, z_1) = z(t_2, z_2)$ then $\forall t \in \mathbb{R}$,

$z(t, z_1) = z(t - t_1, z(t_1, z_1)) = z(t - t_1, z(t_2, z_2)) = z(t - t_1 + t_2, z_2) \in O(z_2)$

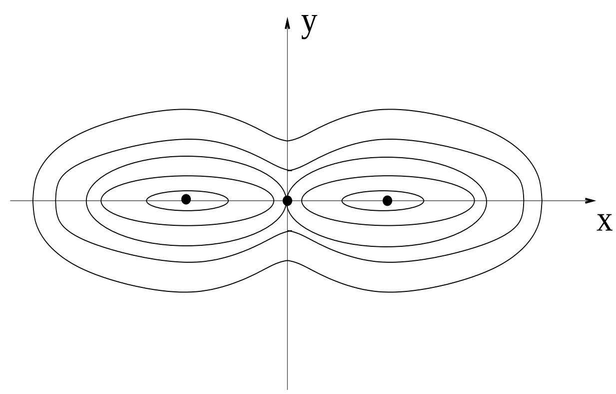
So $O(z_1) \subset O(z_2)$. Similarly we have $O(z_2) \subset O(z_1)$.

We obtain a **phase portrait** by drawing all orbits in phase space $\mathbb{R}^n$. By the claim in the above, phase portrait consists *non-intersecting* curves, each of which representing an orbit.

Ex: For $z = (x, y) \in \mathbb{R}^2$, let

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^3 \end{aligned}$$

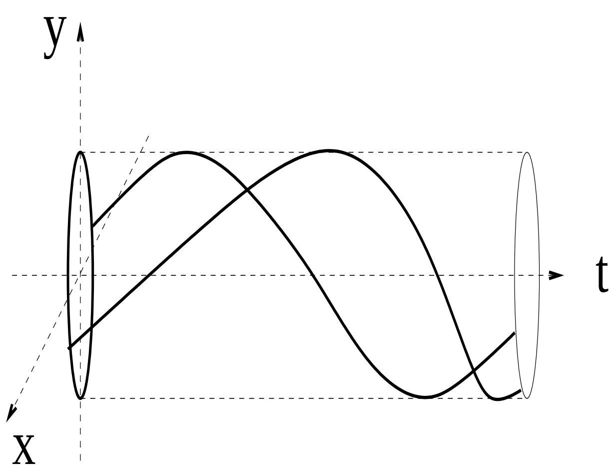
be an autonomous system defined on $\mathbb{R}^2$. The phase portrait is as follows



Space of solution curves

We can also view each solution $z = z(t, z_0)$ as a curve in space $(t, z) \in \mathbb{R} \times \mathbb{R}^n$. We have a *portrait of solutions curves*.

– For autonomous systems, phase portrait is obtained by projecting the portrait of solutions in $\mathbb{R}^n \times \mathbb{R}$ on to the phase space $\mathbb{R}^n$. There exists an entire family of solution curves corresponding to one orbit in the phase portrait.



– Note that these two portraits are drew in different spaces.

Time-T map

Let $z(t, z_0)$ be the solution of equation (1) satisfying $z(0, z_0) = z_0$. For $T > 0$ let $F_T : \mathbb{R}^n \to \mathbb{R}^n$ be define as

$$F_T(z_0) = z(T, z_0).$$

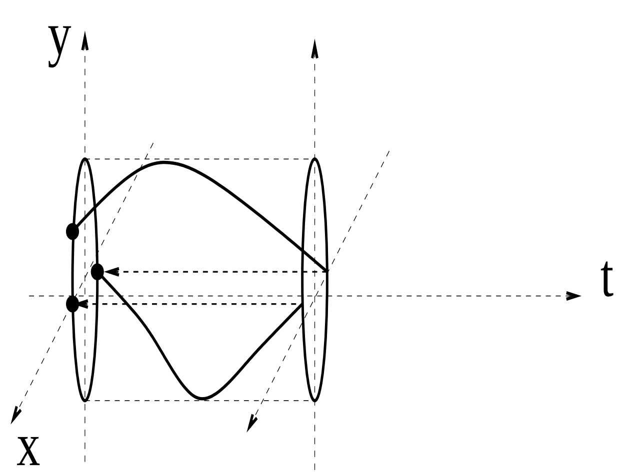
We call F_T the *time T map* induced by equation (1).

Claim 1.2: Let $z_0 \in \mathbb{R}^n$, and $\{z_i\}_{i=-\infty}^{\infty}$ be the orbit of z_0 under the iterations of F_T. Then $z_i = z(iT, z_0)$.

Proof: We prove by induction. Assume that $z_i = z(iT, z_0)$, we have

$$z((i+1)T, z_0) = z(T, z(iT, z_0)) = F(z_i).$$

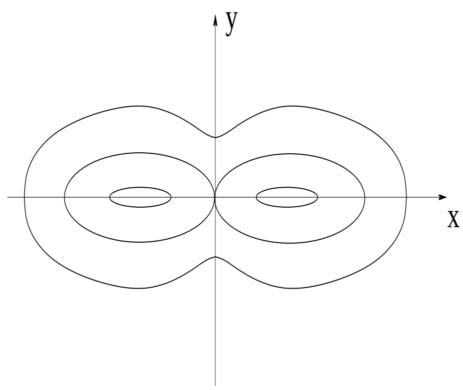
The relationship between an orbit under F and its corresponding solution curve is shown in the next picture.



Phase portrait and the time-T map

Phase portrait of an autonomous system can also be used to show the orbit structure of the time-T map.

Ex: Let us use the same example as in the above.



– We can read this picture as the phase portrait. We can also read this same picture as a description of the orbit structure of the time-T map F_T.

– Though we have the same picture, these two interpretations are entirely different.

B *Non-autonomous systems*

Let us now consider *non-autonomous* differential equations

$$\frac{dz}{dt} = f(z,t) \tag{2}$$

in which $f(z,t)$ is a dependent of t.

Very little of the discussions we made on autonomous systems in the above make sense!

– It is not true that $z(t+t_0, z_0) = z(t, z(t_0, z_0))$.

– Consequently, orbits do intersect in phase space. So it does not make sense to talk about *phase portrait* in phase space $\mathbb{R}^n$.

– The time-T map $F_T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(z_0) = z(T, z_0)$ is not useful: the orbit starting from $z_0 \in \mathbb{R}^n$ under F_T does not represent any given solution.

There is, however, one thing that survives: the portrait of solution. According to theorem on the uniqueness of solutions of a given initial value problem, we know that in $\mathbb{R}^n \times \mathbb{R}$, solutions curves are non-intersecting graphs from $\mathbb{R} \to \mathbb{R}^n$.

Non-autonomous systems of period T

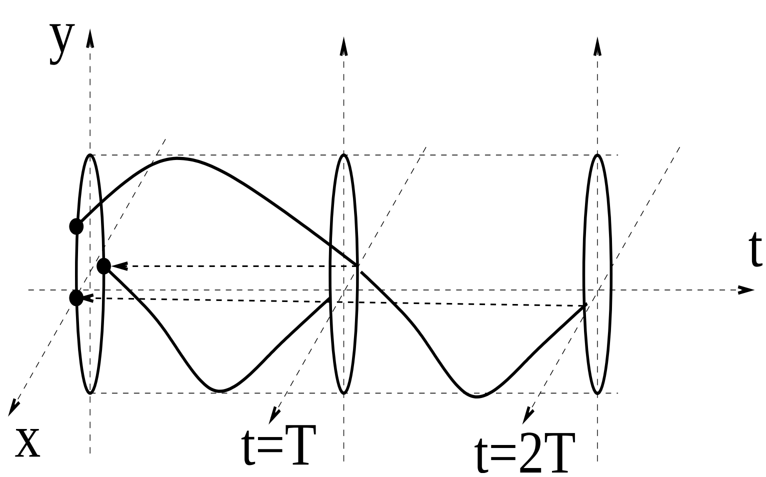
Let us assume that in equation (2), $f(z, t) = f(z, t + T)$ for some fixed $T > 0$ for all z and t. We call such system an non-autonomous periodic system.

For a non-autonomous periodic system of period T,

– we have $z(t+T, z_0) = z(t, z(T, z_0))$.

– Though it does not makes much sense to obtain a phase portrait in $\mathbb{R}^n$. It does make sense to study the time-T map F_T where T is the period of the function $f(z,t)$. i.e., let $F_T : \mathbb{R}^n \to \mathbb{R}^n$ be such that $F_T(z_0) = z(T, z_0)$, we have

$$F^i(z_0) = z(iT, z_0).$$



Note that again, a given orbit $\{z_i\} \in \mathbb{R}^n$ under F_T represent a solution curve in $\mathbb{R}^n \times \mathbb{R}$.

2 The Milnikov's Method

Let $z \in \mathbb{R}^2$ be the phase space, and

$$\frac{dz}{dt} = f(z) + \varepsilon g(z,t) \tag{3}$$

is such that $g(z, t+T) = g(z,t)$. This is non-autonomous periodic equation of period T. Let us also assume that $f(z), g(z,t)$ are both C^2, and ε is a small parameter. When $\varepsilon = 0$, this is an autonomous system which we will regard as our "unperturbed system". The time dependent term $\varepsilon g(z,t)$ is regarded as a term of perturbation.

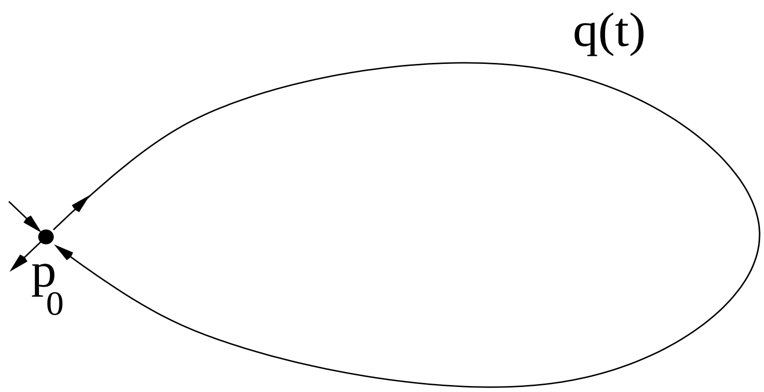
A. *Properties of the time-T map*

Unperturbed system and homoclinic loop
For the unperturbed system

$$\frac{dz}{dt} = f(z)$$

we assume that

(H) p_0 is a saddle point, and there exists a solution, which we denote as $q(t)$, such that $\lim_{t\to\pm\infty} q(t) = p_0$.



We call the orbit representing $q(t)$ in phase space $\mathbb{R}^2$ a *homoclinic loop*.

Note that, for any $t_0 \in \mathbb{R}$ given, $q(t-t_0)$ is also a solution of the unperturbed equation represented by the same homoclinic loop in phase portrait.

Time-T map

We now study the perturbed non-autonomous system. Let $z_\varepsilon(t, t_0, z_0)$ be the solution of the equation (3) of the parameter value ε satisfying $z(t_0, t_0, z_0) = z_0$. For the moment we fix t_0 and let

$$F_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$$

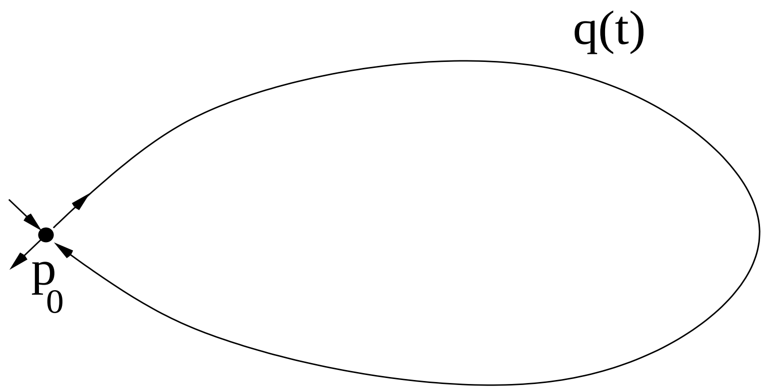
be the time-T map defined by

$$F_\varepsilon(z_0) = z(t_0 + T, t_0, z_0).$$

We have

(i) Orbits of F_ε represents solutions of the corresponding differential equation.

(ii) p_0 is a hyperbolic fixed point for F_0 ($\varepsilon = 0$). Furthermore, the curve represented by the orbit of $q(t)$ is invariant under the iterations of F_0. i.e., it follows from assumption (H) that we have the following picture for the orbit structure of F_0.



Remarks

(a) Though this picture is identical to the one in assumption (H), they represent completely different things. The loop in previous picture represents one **continuous** orbit of the unperturbed equation. Here it represents a collection of discrete orbits that is invariant under the iterations of the mapping F_0.

(b) F_ε is smooth dependent of ε and $z \in \mathbb{R}^2$. The smooth dependency of F_ε in ε is from the fact that the solutions of differential equations are smooth dependents of parameters.

(c) We also involved a parameter t_0 in the definition of F_ε. So F_ε is also a function of t_0.

Local orbit structure

We now study the orbit structure of F_ε around p_0.

Claim 2.1: There exists a unique hyperbolic fixed point p_ε for F_ε such that

(i) $|p_\varepsilon - p_0| = \mathcal{O}(\varepsilon)$, and

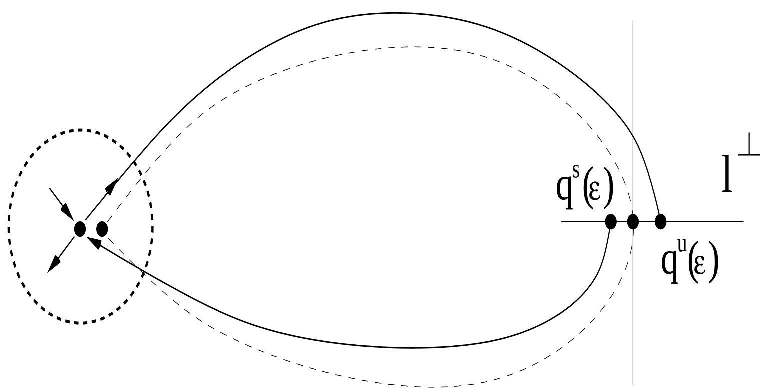
(ii) There exist neighborhood of U of p_ε of fixed size (independent of ε), such that $W^u_{loc}(p_\varepsilon)$ for F_ε and $W^u_{loc}(p_0)$ are $\mathcal{O}(\varepsilon)$ close in C^1-norms in U. So are $W^s_{loc}(p_\varepsilon)$ and $W^s_{loc}(p_0)$.

Proof: (i) To find p_ε we solve the equation $F_\varepsilon(z) = z$. According to the implicit function theorem, this equation has a unique solution in a neighborhood of p_0 for ε sufficiently small if $DF_0 - ID$ is non-singular at p_0. This is the case here because p_0 is a hyperbolic fixed point.

(ii) This is proved by a computation similar to what we did in the proof of the smoothness of the local stable and unstable manifold. This is a good exercise.

Global structure

We now move to a more global picture. Recall that $q(t)$ is an solution of the unperturbed system assumed in (H). It is also a parameterized representation of the homoclinic loop.



At the point $q(0)$, let us take a line segment $l^{\perp}$, that is perpendicular to the tangent direction of the homoclinic loop at $q(0)$. The images of $W^u_{loc}(p_\varepsilon)$ under the iterations of F_ε will intersect $l^{\perp}$. Let us denote the first intersection as $q^u(\varepsilon)$. Similarly, the images of $W^s_{loc}(p_\varepsilon)$ under T^{-1} will also intersect $l^{\perp}$. Let us denote the first intersection as $q^s(\varepsilon)$.

We further observe that

$$|q^s(\varepsilon) - q(0)|, |q^u_\varepsilon - q(0)| < K\varepsilon$$

For some K independent of ε. This is because (a) $W^u_{loc}(p_\varepsilon)$ and $W^u_{loc}(p_0)$ are of a fixed size independent of ε, (b) The number of iterations it takes to get the image of W^u_{loc} to intersect $l^\perp$ is also independent of ε.

We aim in computing

$$d(t_0) := |q^s(\varepsilon) - q^u(\varepsilon)|.$$

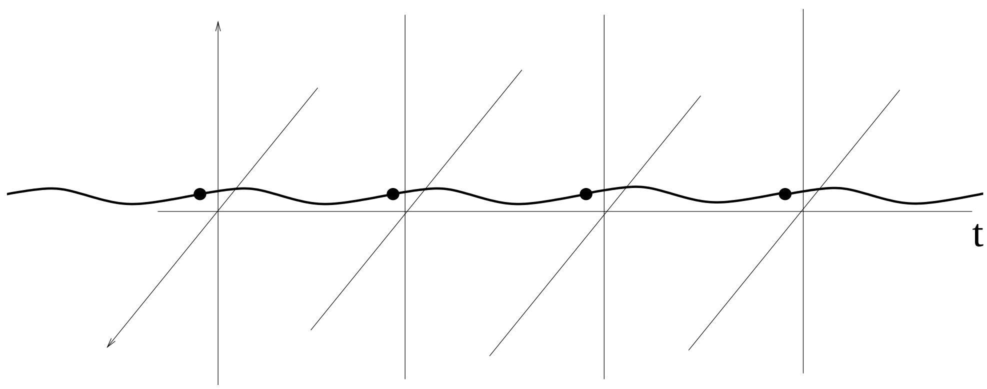
We write $d(t_0)$ here to indicate that these are all quantities depending on t_0.

B. *Solutions through $q^u(\varepsilon)$ and $q^s(\varepsilon)$*

– Corresponding to p_ε, the indicated hyperbolic fixed point of F_ε, there is a solution $z_\varepsilon(t, t_0, p_\varepsilon)$ satisfying

$$z_\varepsilon(t_0, t_0, p_\varepsilon) = z_\varepsilon(t_0 + T, t_0, p_\varepsilon) = p_\varepsilon.$$

We will denote this solution as $\gamma_\varepsilon(t, t_0)$. It is clear that $\gamma_0(t, t_0) = p_0$.



– Let $q^s_\varepsilon(t, t_0) = z_\varepsilon(t, t_0, q^s(\varepsilon))$ be the solution satisfying the initial condition $z_\varepsilon(t, t_0, q^s(\varepsilon)) = q^s(\varepsilon)$. We have

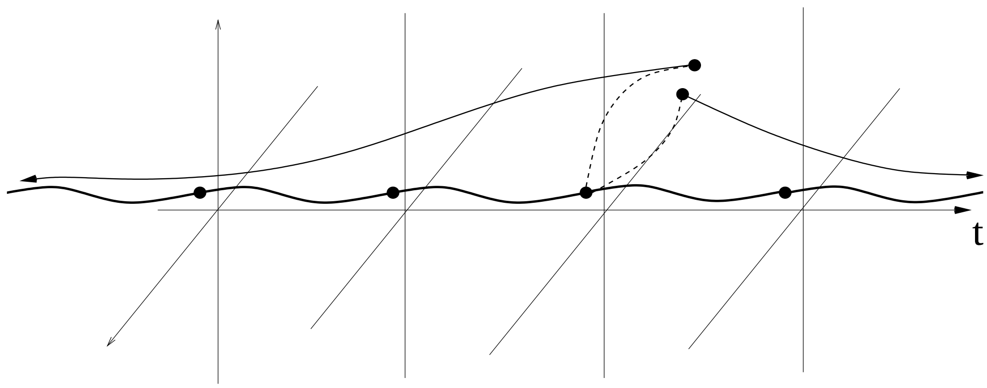
$$|q^s_\varepsilon(t, t_0) - \gamma_\varepsilon(t, t_0)| \to 0$$

exponentially fast as $t \to +\infty$.

Similarly, we have $q^u_\varepsilon(t, t_0)$ such that

$$|q^u_\varepsilon(t, t_0) - \gamma_\varepsilon(t, t_0)| \to 0$$

exponentially fast as $t \to -\infty$.



Note that at $\varepsilon = 0$, $q_0^s(t, t_0) = q_0^u(t, t_0) = q(t - t_0)$.

Claim 2.3: There exists $K > 0$ independent of ε such that for all ε sufficiently small, and all $t \in [t_0, \infty)$, we have

$$|q_\varepsilon^s(t, t_0) - q(t - t_0)| < K\varepsilon$$

Similarly we have for all $t \in (-\infty, t_0]$,

$$|q_\varepsilon^u(t, t_0) - q(t - t_0)| < K\varepsilon.$$

Proof: We have proved that the indicated inequalities hold for $t = t_0$. Also, there exists t_1 independent of ε, such that $q(t_1 - t_0) \in U$. By continuity, we have

$$|q_\varepsilon^s(t, t_0) - q(t - t_0)| < K\varepsilon$$

for $t \in [t_0, t_1]$ for a different K.

Inside of U we have

$$|q_\varepsilon^s(t,t_0) - \gamma_\varepsilon(t,t_0)| < K|q_\varepsilon^s(t_1,t_0) - \gamma_\varepsilon(t_1,t_0)|e^{-\lambda t}$$

for some $\lambda > 0$. Similarly at $\varepsilon = 0$,

$$|q(t-t_0) - \gamma_0(t,t_0)| < K|q(t_1-t_0) - \gamma_0(t_1,t_0)|e^{-\lambda t}.$$

We also have

$$|\gamma_\varepsilon(t,t_0) - \gamma_0(t,t_0)| < K\varepsilon$$

These three inequalities are combined to prove the first inequality of this claim. A proof for the second inequality is similar.

We can now write $q_\varepsilon^s(t,t_0)$ on $[t_0,\infty)$ as as

$$q_\varepsilon^s(t,t_0) = q(t-t_0) + \varepsilon h(t,t_0) + \mathcal{O}(\varepsilon^2).$$

Claimi 2.4: The function $h := h(t,t_0)$ in the above satisfies

$$\frac{d}{dt}h = Df(q(t-t_0))h + g(q(t-t_0),t).$$

Proof: Substitute the formula for $q_\varepsilon^s(t,t_0)$ into

$$\frac{dz}{dt} = f(z) + \varepsilon g(z,t)$$

we obtain

$$\varepsilon \frac{dh}{dt} + \mathcal{O}(\varepsilon^2) = f(q_\varepsilon^s(t_0, t)) - f(q(t - t_0)) + \varepsilon g(q_\varepsilon^s(t_0, t), t)$$

We now obtain the indicated equation by dividing ε both side first then let $\varepsilon \to 0$.

Similarly

Claim 2.5: On $(-\infty, t_0]$ we have

$$q_\varepsilon^u(t, t_0) = q(t - t_0) + \varepsilon h(t, t_0) + \mathcal{O}(\varepsilon^2).$$

where $h := h(t, t_0)$ satisfies the same equation in the above.

C. *Computing* $d(t_0) = |q_\varepsilon^u - q_\varepsilon^s|$

A notation: For two vectors $a = (a_1, a_2), b = (b_1, b_2)$ in $\mathbb{R}^2$, let $a \wedge b = a_1 b_2 - a_2 b_1$. Notice that $a^\perp = (-a_2, a_1)$ is perpendicular to a, and $a \wedge b = a^\perp \circ b$ by definition. $a \wedge b$ is the projection of vector b on $a^\perp$.

Proposition 2.1: Let us assume that $tr(Df(z)) = 0$ for all $z \in \mathbb{R}^2$. Then

$$d(t_0) = \frac{M(t_0)}{|f(q(0))|}\varepsilon + \mathcal{O}(\varepsilon^2)$$

where

$$M(t_0) := \int_{-\infty}^{\infty} f(q(t-t_0)) \wedge g(q(t-t_0), t)dt.$$

Proof: For $t \in [t_0, \infty)$, let

$$\Delta^s(t, t_0) := f(q(t-t_0)) \wedge h(t, t_0)$$

where $h(t, t_0)$ is such that

$$q_\varepsilon^s(t, t_0) = q(t-t_0) + \varepsilon h(t, t_0) + \mathcal{O}(\varepsilon^2)$$

as defined in the above. Similarly, for $t \in (-\infty, t_0]$, let

$$\Delta^u(t, t_0) := f(q(t-t_0)) \wedge h(t, t_0).$$

We have by definition that

$$d(t_0) = \varepsilon\frac{|\Delta^u(t_0, t_0) - \Delta^s(t_0, t_0)|}{|f(q(0))|} + \mathcal{O}(\varepsilon^2).$$

Note that we have

$$\begin{aligned}
\frac{d}{dt}\Delta^s(t,t_0) &= Df(q(t-t_0))f(q(t-t_0)) \wedge h(t,t_0) \\
&\quad + f(q(t-t_0)) \wedge \frac{d}{dt}h(t,t_0) \\
&= Df(q(t-t_0))f(q(t-t_0)) \wedge h(t,t_0) \\
&\quad + f(q(t-t_0)) \wedge (Df(q(t-t_0))h + g(q(t-t_0),t)) \\
&= tr(Df)\Delta^s(t,t_0) + f(q(t-t_0)) \wedge g(q(t-t_0),t) \\
&= f(q(t-t_0)) \wedge g(q(t-t_0),t)
\end{aligned}$$

The last equality is obtain by using the assumption that $tr(Df) = 0$.

Note that as $\Delta^s(\infty, t_0) = 0$ because as $t \to \infty$, $f(q(t-t_0)) \to f(p_0) = 0$. It follows then

$$\Delta^s(t_0,t_0) = -\int_{t_0}^{\infty} f(q(t-t_0)) \wedge g(q(t-t_0),t)dt$$

Similarly we have

$$\Delta^u(t_0,t_0) = \int_{-\infty}^{t_0} f(q(t-t_0)) \wedge g(q(t-t_0),t)dt$$

Our Proposition follows from combining these two.

C. *Applications*

Existence of homoclinic orbits

The following is a direct consequence of this Proposition.

Claim 3.1: The assumptions are the same as in Proposition 2.1. If $M(t_0)$ as a function of t_0 has a simple zero, i.e. there exists $\hat{t}$ such that $M(\hat{t}) = 0, M'(\hat{t}) \neq 0$. Then F_ε for $t_0 = \hat{t}$ has a transversal homoclinic orbit.

Proof: Let us fix $t_0 = \hat{t}$ in the above to get the time-T map F_ε. However, instead of using $q(0)$ to determine q_ε^s and q_ε^u, we use $q(\delta)$ for a δ inside a small interval around zero to compute $d(\hat{t})$. We denote this distance as $d(\hat{t}, \delta)$. It then follows from the same computation as in the above that

$$d(\hat{t}, \delta) = \varepsilon \frac{M(\hat{t}, \delta)}{|f(q(\delta))|} + \mathcal{O}(\varepsilon)$$

Our assumption implies that, for all ε sufficiently small, $d(\hat{t}, \delta)$ as a function in δ has a simple zero around $\delta = 0$.

Remember transversal homoclinic orbit implies the existence of a Horseshoe map.

Note that by a change of variable $t \to t - t_0$, we have

$$M(t_0) = \int_{-\infty}^{\infty} f(q(t)) \wedge g(q(t), t + t_0) dt.$$

This is usually more convenient in computing $M(t_0)$.

Hamiltonian system

Let $q, p \in \mathbb{R}^n$ and $H(q,p) : \mathbb{R}^{2n} \to \mathbb{R}$ be a given function. We call the following system of ordinary equations

$$\frac{dq}{dt} = \partial_p H(q,p), \quad \frac{dp}{dt} = -\partial_q H(q,p)$$

a *Hamiltonian System* and $H(q,p)$ the Hamiltonian of this system.

Claim 3.2: Hamiltonian systems satisfying $tr(Df) = 0$.

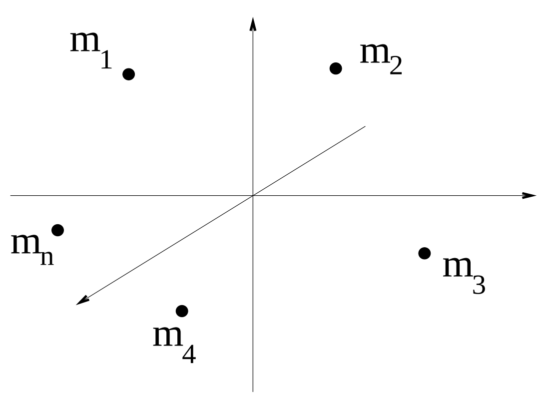
Proof: $tr(Df) = \partial_{pq}H - \partial_{qp}H = 0$.

Mathematical theories on Hamiltonian systems is one of the major subject in the history. This is because **all energy conservative mechanical systems** are Hamiltonian systems with

$$H = T + V$$

where T is the kinetic energy and V is the potential energy of the mechanical systems subjected to study.

Ex: The n-body problem.



This is a Hamiltonian system with

$$H = \frac{1}{2}\sum_{i=1}^{n} m_i^{-1}|p_i|^2 - \sum_{i<j}\frac{m_i m_j}{|r_{ij}|}$$

where $p_i = m_i \frac{d}{dt} r_i$.

Geometric implication of $tr(Df) = 0$

The condition $tr(Df) = 0$ has the following geometric interpretation:

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth 1-1 mapping. We denote the volume of an open subset $U \in \mathbb{R}^n$ as $m(U)$.

Definition: We say that F is volume preserving if, for any open subset $U \in \mathbb{R}^n$, we have $m(U) = m(F(U))$.

Let $z(t, z_0)$ be the solution of a given equation $\frac{dz}{dt} = f(z)$ satisfying $z_0 = z(0, z_0)$. Recall that, for t_0 given as a fixed number, the time-t_0 map $F_{t_0} : \mathbb{R}^n \to \mathbb{R}^n$ is define by

$$F_{t_0}(z_0) = z(t_0, z_0).$$

Claim 3.3: Let $t_0 \in \mathbb{R}$ be fixed, and $F_{t_0} : \mathbb{R}^n \to \mathbb{R}^n$ be the time-t_0 map. Then under the assumption that $tr(Df) = 0$, F_{t_0} is volume preserving.

Note that this implies that time-t mapping defined by any given Hamiltonian system is volume preserving.

Proof: Let $\xi \in \mathbb{R}^n$ be a given initial condition, and $z = z(t, \xi)$ be the solution satisfying $\xi = z(0, \xi)$. We are in need of writing vectors explicitly in their components. So $z = (z_1, \cdots, z_n)$, $\xi = (\xi_1, \cdots, \xi_n)$ and $f = (f_1, \cdots, f_n)$. The solution is now written in component as

$$z(t, \xi) = (z_1(t, \xi_1, \cdots, \xi_n),\ \cdots, z_n(t, \xi_1, \cdots, \xi_n))$$

Let $J := D_\xi z = (\partial_{x_j} z_i)$ be the Jacobi matrix of z with respect to ξ. For this claim to hold it suffices to prove that

$$\det(J) = 1.$$

Write $J = (J_{ij}(t))$ where $J_{ij} = \partial_{\xi_j} z_i$. By taking partial derivatives with respect to ξ on the equation give, we have

$$\frac{dJ_{ij}}{dt} = \sum_{k=1}^{n} (\partial_{z_k} f_i) J_{k,j}$$

Let X_{ij} be the *complement* of J_{ij} of the matrix J, i.e., $(-1)^{i+j} X_{ij}$ be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleted the j-th raw and the i-th column of the matrix J. Let M_i be the matrix we obtain by replacing J_{ij} by $\frac{d}{dt} J_{ij}$ for $j = 1, \cdots n$.

We have

$$\det M_i = \sum_{j=1}^{n} X_{ij} \frac{d}{dt} J_{ij},$$

and

$$\begin{aligned}
\frac{d}{dt} \det(J) &= \sum_{i=1}^{n} \det M_i = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} \frac{d}{dt} J_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} X_{ij} (\partial_{z_k} f_i) J_{k,j} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} (\partial_{z_k} f_i) \sum_{j=1}^{n} X_{ij} J_{kj}
\end{aligned}$$

Note that we have

$$\sum_{j=1}^{n} X_{ij} J_{kj} = \det J \delta_{ik}$$

where $\delta_{ik} = 1$ if $i = k$ but $\delta_{ik} = 0$ if $i \neq k$. Hence it follows that

$$\frac{d}{dt} \det J = (\sum_{i=1}^{n} (\partial_{z_i} f_i)) \det J = tr(Df) \det J.$$

So under the assumption that $tr(Df) = 0$, we have

$$\det J(t) = \det J(0) = 1.$$

The last equality holds because $J(0) = ID$.

Explicit computation of $M(t_0)$: an example

We finish by giving an explicit application of Milnikov's method.

For $z = (x, y) \in \mathbb{R}^2$, let

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^3 + \varepsilon(\gamma \cos \omega t - \delta y) \end{aligned}$$

where $\varepsilon, \gamma, \omega$ and δ are parameters.

For $\varepsilon = 0$, we have $f(x, y) = (y, x - x^3)$ and

$$Df = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}.$$

So $tr(Df) = 0$. It is easy to check that $p_0 = (0, 0)$ is a saddle point for the unperturbed equation. Let

$$\begin{aligned} x(t) &= \frac{2\sqrt{2}}{e^t + e^{-t}} \\ y(t) &= \frac{2\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}. \end{aligned}$$

Then

$$q(t) = (x(t), y(t))$$

is a homoclinic loop for the unperturbed system.

By definition we have

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} y(t)[\gamma \cos \omega(t + t_0) - \delta y(t)]dt \\ &= -2\sqrt{2}\gamma \int_{-\infty}^{\infty} \frac{(e^t - e^{-t})}{(e^t + e^{-t})^2} \cos \omega(t + t_0) dt \\ &\quad -8\delta \int_{-\infty}^{\infty} \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^4} dt. \end{aligned}$$

We evaluate these two integrals separately to obtain

$$M(t_0) = -\frac{4\delta}{3} + 2\sqrt{2}\gamma\pi\omega\frac{\sin\omega t_0}{e^{\frac{\pi\omega}{2}} + e^{-\frac{\pi\omega}{2}}}$$

Note that we have a simple zero for $M(t_0)$ if

$$\frac{\gamma}{\delta} > \frac{\sqrt{2}(e^{\frac{\pi\omega}{2}} + e^{-\frac{\pi\omega}{2}})}{3\pi\omega}.$$