Conformal Mappings Around Fixed Points

System of Study: Let $\mathbb{C}$ be the complex plane and $U, V \subset \mathbb{C}$ be the neighborhoods of $z = 0$. Let $T : U \to V$ be a conformal mapping satisfying $T(0) = 0$.

Question: We write $T : U \to V$ as

$$T(z) = \lambda z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

and ask if there exists an analytic function $z = h(\xi)$,

$$h(\xi) = \xi + h_2 \xi^2 + \cdots + h_n \xi^n + \cdots$$

defined on a small neighborhood of $\xi = 0$, such that h conjugate T to $S : \xi \to \lambda \xi$.

Remark: For conformal mappings, questions on *analytic* conjugations are much easier to study than those on *topological* or C^r conjugations because looking for analytic conjugations allow us to work with power series.

1. A formal construction

For h to conjugate T and S, we need

$$h(S(\xi)) = T(h(\xi)).$$

To write it explicitly, we obtain

$$\lambda\xi + h_2\lambda^2\xi^2 + \cdots + h_n\lambda^n\xi^n + \cdots =$$

$$\lambda h(\xi) + a_2(h(\xi))^2 + \cdots + a_n(h(\xi))^n + \cdots .$$

From this it follows that

$$\lambda^n h_n = \lambda h_n + a_n + p_n(a_2, \cdots a_{n-1}; h_2, \cdots h_{n-1}) \qquad (1)$$

where $p_n, n = 1, 2, \cdots$ is a sequence of polynomials in variables $a_2, \cdots, a_{n-1}; h_2, \cdots, h_{n-1}$ of *positive coefficients*.

p_n are *universal* in the sense that it does not depend on the mappings of study.

Claim 1.1: (i) If λ is not a root of unity, then all h_n are uniquely determined by using (1).

(ii) If λ is a root of unity, then the desired analytic conjugate does not exist in general.

Remarks

(a) Assume there exists n such that $\lambda^n = 1$, then $S : \xi \to \lambda\xi$ satisfies $S^n(\xi) = \xi$. If T conjugates to h (Even in the case of a topological conjugate), we must have $T^n(z) = h \circ S^n \circ h^{-1}(z) = z$. So if $T^n \neq ID$, then T and its linear part are not conjugate.

Ex: $T(z) = \frac{z}{1-z}$ does not conjugate to $S(\xi) = \xi$.

(b) We can not conclude from the first item of this claim that if λ is not a root of unity, then an analytic conjugate exists. There is a *problem of convergence*. The formal series $h(\xi)$ constructed here might be divergent.

2. The issue of convergence

Hyperbolic case: $|\lambda| \neq 1$

Claim 2.1: If $|\lambda| \neq 1$, then the formal power series h constructed above converges in a sufficiently small neighborhood around $\xi = 0$.

Proof: We prove by the following observations.

– Let $f(z) = a_1 z + a_2 z^2 + \cdots$, $g(z) = b_1 z + b_2 z^2 + \cdots$. Assume that all $b_n, n \geq 1$ are real and positive. We say that $g(z)$ is a **majorant** of $f(z)$ (denoted as $f \triangleright g$) if $|a_n| \leq b_n$ for all n.

Assume that $f \triangleright g$. If g is convergent on a small disk U centered at $z = 0$, then f is convergent on U as well.

– Since $T(z)$ is analytic, there exists $a > 0$ such that $a_n < a^n$ for all n. Also, by assuming $|\lambda| \neq 1$, there exists $c > 0$ such that $|\lambda^n - \lambda| > c$ for all $n > 0$.

– $h_i, i = 1, \cdots$ are computed inductively by using (1). We now make the following changes in the computation to obtain another sequence H_i: (i) we replace all a_n by a^n, and (ii) we replace $\lambda^n - \lambda$ by c. Because all coefficients of p_n are positive, and $|a_n| < a^n$, $|\lambda^n - \lambda| > c$, it is easy to inductively prove that $|h_n| < H_n$ for all $n > 0$. So

$$H(\xi) := \xi + H_2 \xi^2 + \cdots + H_n \xi^n + \cdots$$

is a majorant of $h(z)$.

– It now suffices to show that $H(\xi)$ is convergent. By the way $H(\xi)$ is constructed, it satisfies the equation

$$c(H(\xi) - \xi) = \sum_{n=2}^{\infty} (aH(\xi))^n$$

from which we obtain

$$c(H(\xi) - \xi) = \frac{a^2 H^2(\xi)}{1 - aH(\xi)}.$$

Hence $H(\xi)$ is explicitly defined and is analytic at $\xi = 0$.
$\square$

Elliptic case: $|\lambda| = 1$

A. *An overview*

What is left for consideration: $|\lambda| = 1$, and λ is not a root of unity.

Let us write $\lambda = e^{2\pi\alpha}$. Then λ is a root of unity if and only if α is rational.

Question 1: Assume that α is irrational. Is it true that for all $T(z) = e^{2\pi\alpha}z + a_2z^2 + \cdots$, the power series h constructed above converge?

Answer: No. There is a dense set of irrational numbers and maps associated with them such that h constructed above diverge.

Question 2: Do we have a set of irrational numbers, such that for all $T(z) = e^{2\pi\alpha}z + a_2z^2 + \cdots$, the h constructed above converge?

Answer: Yes. In fact, the set of irrational numbers for which the answer for Question 2 is positive has full measure. i.e. for almost all α, the answer to question 2 is yes.

B. *Diophantine and Louvilles numbers*

Definition: We call $\alpha \in [0, 1]$ a Diophantine number if there exists $\varepsilon, \mu > 0$, such that for all $m, n \in \mathbb{Z}^+$,

$$|m\alpha - n| > \frac{\varepsilon}{n^\mu} \tag{2}$$

We call a number that is not Diophantine a Louville number.

Remark: From (2) we have

$|\lambda^n - 1| = |e^{2\pi n\alpha} - 1| = |e^{\pi n\alpha} - e^{-\pi n\alpha}|$

$= 2|\sin(\pi n\alpha)| = 2|\sin\pi(n\alpha - m)|$

where $m \in \mathbb{Z}^+$ is such that $|n\alpha - m| < \frac{1}{2}$. So if α is Diophantine, we have

$$|\lambda^n - 1| > K\frac{\varepsilon}{n^\mu}$$

for all $n > 0$.

Observe that the convergence of h is a problem because we can not bound $|\lambda^n - 1|$ uniformly away from 0 (We can in the case $|\lambda| \neq 1$, in which the majorant argument worked). This is the so called **small divisor** problem.

For Diophantine numbers, $|\lambda^n - 1|$ is allowed to approach to zero but in a controlled way. This is certainly good for h to converge.

Diophantine numbers are regarded as *far away* from rational numbers (very irrational), and Louville numbers are regarded as *close* to rational numbers (not very irrational).

C. *Conclusions on h*

To answer Question 2, we prove

Claim A: (i) If α is a Diophantine number, then $h(\xi)$ constructed in the above converge.

(ii) The measure of the set of all Diophantine numbers in $[0, 1]$ is one.

Our next Claim confirms our answer to Question 1.

Claim B: (i) There is a dense set L of irrational numbers in $[0, 1]$, such that associated with each $\alpha \in L$, there is a sequence $q_k \to \infty$, and $p_k \in \mathbb{Z}^+$, such that

$$|q_k \alpha - p_k| < \frac{1}{q_k!}.$$

(ii) For every $\alpha \in L$, there exists $T(z)$ such that h constructed in the above divergent.

D. *Proof of Claim A(ii)*

– Let $\Delta \subset [0, 1]$ be the set of all Diophantine numbers, and $\alpha \in \Delta$. Then for all pairs $\varepsilon, \mu >$

0, there exists n, m, such that $|n\alpha - m| < \frac{\varepsilon}{n^\mu}$. This implies that for all $\varepsilon > 0$, there exists n, m such that

$$|n\alpha - m| < \frac{\varepsilon}{n^2}. \tag{3}$$

– Let $B(\varepsilon, n, m)$ be the set of all α such that (3) hold for ε, n, m. For $\varepsilon > 0$ let

$$B(\varepsilon) = \cup_{n,m} B(\varepsilon, n, m).$$

We have

$$|\Delta| < |B(\varepsilon)| < \sum_{n,m} |B(\varepsilon, n, m)|.$$

– From (3) we have

$$|B(\varepsilon, n, m)| < \frac{2\varepsilon}{n^3}$$

We also have

$$-\varepsilon < m < n + \varepsilon$$

(For the upper bound use (3) with and $\alpha < 1$. For the lower bound, use (3) and $\alpha = 0$)

– It then follows that

$$|B(\varepsilon)| < \sum_{n=1}^{\infty} (n + 3\varepsilon) \frac{2\varepsilon}{n^3} < K\varepsilon$$

But $|\Delta|$ is independent of ε. So $|\Delta| = 0$. $\square$

E. *Proof of Claim B(i)*

– Let $\omega \in [0, 1]$ be an irrational number, and

$$\omega = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}. \tag{4}$$

$a_k \geq 1$ for all k is the continuous fractions for ω. For simplicity in notation, we write

$$\omega = [a_1, \cdots, a_k, \cdots].$$

– We set $a_n = 0$ for all $n > k$ in (4) to obtain a rational number, which we denote as $\frac{p_k}{q_k}$. q_k

and a_k are the same as in the tower pictures in Lecture 4 for ω. Further,

$$|q_k\omega - p_k| < \frac{1}{q_{k+1}} \qquad (5)$$

is the distance from $q_k\omega$ to zero in S^1.

(The proof of (5) is as follows: by definition of q_k, we have $d(n\omega, 0) > d(q_k\omega, 0)$ on S^1 for all $n < q_{k+1}$. This implies $d(n\omega, m\omega) > d(q_k\omega, 0)$ on S^1 for all $n, m < q_{k+1}, n \neq m$. Note that $\{n\omega\}, 0 \leq n < q_{k+1}$ cuts S^1 into q_{k+1} pieces. So the smallest, of size $d(q_k\omega, 0)$, can not be larger than $\frac{1}{q_{k+1}}$.)

– For $\varepsilon > 0$ given, let n_0 be the integer part for $2\varepsilon^{-1}$. Let $\omega \in [0, 1]$ is such that $\omega = [a_1, \cdots, a_k, \cdots]$. We define $b_k, k = 1, 2, \cdots$ as follows: (i) $b_k := a_k$ for $k \leq n_0$; (ii) for $k > n_0$, let $\frac{p_k}{q_k}$ be the rational number determined by $b_1, \cdots, b_k$. Then $b_{k+1} = q_k!$.

– Let $\alpha = [b_1, \cdots, b_k, \cdots]$, we have

(a) $|\alpha - \omega| < \varepsilon$.

(Let p_{n_0} and q_{n_0} are determined by $a_1, \cdots a_{n_0}$, we have from (5),

$$|\omega - \frac{p_{n_0}}{q_{n_0}}|, \quad |\alpha - \frac{p_{n_0}}{q_{n_0}}| < \frac{1}{n_0},$$

from which (a) follow.)

(b) There exists p_k, q_k such that $|q_k\alpha - p_k| < \frac{1}{q_k!}$ for all $k > n_0$.

(This is from (5) and the definition of $b_k, k > n_0$.) $\square$

F. Proof of Claim B(ii)

– Let α be as in Claim B(i), and $T(z) = e^{2\pi\alpha}z \pm z^2 \pm z^3 \pm \cdots$ where the sign in front of z^n will be determined inductively. They are so picked that

$$|(\lambda^n - \lambda)h_n| > 1.$$

(Recall that $(\lambda^n - \lambda)h_n = \pm 1 + p_n$. We take the positive sign if $Re(p_n) > 0$ and the negative sign otherwise.)

– It follows that $|h_{q_k}| > K^{-1}(q_k)!$. $h(\xi)$ diverges.

□

3. Proof of Claim A(i)

Let $T(z) = \lambda z + \widehat{f}(z)$, $h(\xi) = \xi + \widehat{h}(\xi)$ where $\widehat{f}$ and $\widehat{h}$ are terms of second order and higher. To find $z = h(\xi)$ to conjugate T and S is to find $\widehat{h}(\xi)$ satisfying

$$\widehat{h}(\lambda\xi) = \lambda\widehat{h}(\xi) + \widehat{f}(\xi + \widehat{h}(\xi)) \qquad (6)$$

In the above we adopted a way of finding h: first we solve h_n inductively, then prove that the power series obtained converge. In the case of $|\lambda| = 1$, h_n might grow so fast that h_n fails to converge (Claim B) because of the problem of small divisors. We now restricted to Diophantine numbers, meaning that we allow small divisors go to zero, but in a controlled fashion. We hope that such control will cause h to converge.

It is, however, still hard for us to see why h converge. h_n are determined *inductively* in a very complicated way, making it almost impossible to determine, for example, the effect of h_{10} on the size of $h_{1,000,000}$. *It is in fact not clear for us on how to control the growth of $|h_n|$ by the proposed control of small divisors*.

Pushing along this direction (to control the growth of $|h_n|$), Siegel proved Claim A(i) through hard analysis. Here we adopt a much better approach.

A new approach: Instead solving (6), we solve

$$\hat{h}(\lambda\xi) = \lambda\hat{h}(\xi) + \hat{f}(\xi) \tag{7}$$

Note that (7) is obtained by changing $f(\xi + \hat{h}(\xi))$ in (6) to $f(\xi)$.

(7) is much easier to solve. In fact we have explicitly

$$h_n = \frac{a_n}{(\lambda^n - \lambda)}.$$

By the assumption that λ is Diophantine, the convergency of $h(\xi) = \xi + \hat{h}(\xi)$ is not a problem.

However, this $h(\xi)$ is not what we wanted: it does not conjugate T to S.

Here is a key observation: Though $h(\xi)$ so obtained does not conjugate T to S, it does, as a coordinate change, transfer T to a mapping $S_1 : \xi \to S_1(\xi)$ and S_1 is much closer to S than T.

To obtain a real conjugate of T to S, we construct h_1 by using (7) for S_1 instead of T. Similarly we construct h_2 for S_2, and so on. At the end we will have $S_n \to S$ and a conjugate of T to S as

$$H = h_1 \circ h_2 \circ \cdots \circ \cdots .$$

We now present a formal proof.

Step 1 Let $T(z) = \lambda z + \widehat{f}(z)$ where

(A1) For all $n \in \mathbb{Z}^+$,

$$|\lambda^n - 1| > n^{-2}.$$

(A2) $\hat{f}(z)$ satisfies

$$|\hat{f}'(z)| < \delta, \quad on \quad |z| < r.$$

Remark: Under the assumption that $\lambda = e^{2\pi\alpha}$ and α is a Diophantine number, we have

$$|\lambda^n - 1| > \varepsilon n^{-\mu},$$

for all $n \in \mathbb{Z}^+$ for a fixed pair of ε and μ. In (A1) we take $\varepsilon = 1$, $\mu = 2$ to remove unnecessary distractions from the main lines of the proof.

Main Proposition: Let $\theta < \frac{1}{5}$ be such that

$$\delta < \theta^4.$$

Then we have

(a) The solution $\hat{h}(\xi)$ of (7) satisfies

$$|\hat{h}'(\xi)| < \frac{\delta}{\theta^3}, \quad on \quad |\xi| < r(1-\theta).$$

(b) Let $S = h \circ T \circ h^{-1}$. Write $S(\xi)$ as

$$S(\xi) = \lambda\xi + \hat{g}(\xi).$$

Then $S(\xi)$ is well defined on $|\xi| < r(1-5\theta)$ and

$$|\widehat{g}'(\xi)| < \frac{2r}{\theta^4}\delta^2.$$

Proof: The main technical tool we use is the Cauchy's formula

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f'(\eta)}{(\eta - z)}d\eta$$

and

$$f^{(n)}(z) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f'(\eta)}{(\eta - z)^n}d\eta$$

where $z \in D_\gamma$, the region bounded by a simple closed γ.

– For $\widehat{f} = \sum_{n=2}^{\infty} a_n z^n$, we use $\gamma = \{|z| = r\}$ in Cauchy formula to estimate $f^{(n)}(0)$, obtaining

$$n \cdot |a_n| < \frac{\delta}{r^{n-1}}$$

– From (7) we have

$$\widehat{h}(\xi) = \sum_{n=2}^{\infty} \frac{a_n}{\lambda(\lambda^{n-1}-1)} \xi^n.$$

So

$$|\widehat{h}'(\xi)| \le \delta \sum_{n=2}^{\infty} \frac{n^2}{r^{n-1}} |\xi|^{n-1}.$$

On $\{|\xi| < r(1-\theta)\}$, we have

$$|\widehat{h}'(\xi)| < \delta \sum_{n=2}^{\infty} n^2 (1-\theta)^{n-1} < \frac{\delta}{\theta^3}.$$

This proves (a).

Remark: It is important to note that, in order to obtain the indicated upper bound on $|\widehat{h}'(\xi)|$, the radius of the domain is shrank from r to $r(1-\theta)$. Here is a crucial technical point: if we introduce infinitely many coordinate transforms based (7), the domain will be shrank infinitely many times, causing potentially radius of convergence go to zero. The entire scheme will fail if this happens.

For (b) we note that $S = h^{-1} \circ T \circ h$. Since h is now defined on $|\xi| < r(1-\theta)$, it suffices for us to show that

(i) h maps $|\xi| < r(1-4\theta)$ to $|z| < r(1-3\theta)$;

(ii) T maps $|z| < r(1-3\theta)$ to $|z| < r(1-2\theta)$; and

(iii) h^{-1} maps $|z| < r(1-2\theta)$ to $|\xi| < r(1-\theta)$.

(i) holds because

$$|h(\xi)| < |\xi| + |\widehat{h}(\xi)| < r(1-4\theta) + \frac{\delta}{\theta^3} r < r(1-3\theta)$$

where the last inequality uses the assumption $\delta < \theta^4$.

(ii) is similar to (i).

(iii) is left as an exercise. (Hint: Let $\xi_{n+1} = z - \widehat{h}(\xi_n)$ to construct $\xi_n \to h^{-1}(\xi)$.)

We now move to the last part of (b). Let us write $S(\xi) = \lambda\xi + \widehat{g}(\xi)$. From

$$h \circ S = T \circ h,$$

we obtain

$$\widehat{g}(\xi) + \widehat{h}(\lambda\xi + \widehat{g}(\xi)) = \lambda\widehat{h}(\xi) + \widehat{f}(\xi + \widehat{h}(\xi))$$

By (7), this leads to

$$\widehat{g}(\xi) = \widehat{f}(\xi + \widehat{h}(\xi)) - \widehat{f}(\xi) + \widehat{h}(\lambda\xi) - \widehat{h}(\lambda\xi + \widehat{g}(\xi)).$$

Let $\gamma := \sup_{|\xi|<r(1-4\theta)} \widehat{g}(\xi)$. We have

$$\gamma \leq \sup_{|\xi|<r(1-4\theta)} |\widehat{h}'|\gamma + \delta \sup_{|\xi|<r(1-4\theta)} |\widehat{h}|,$$

from which it follows

$$\gamma < \frac{5}{4} \cdot \delta \cdot \frac{r\delta}{\theta^3} = \frac{2r}{\theta^3} \cdot \delta^2$$

To get an upper bound for $|\widehat{g}'(\xi)|$ we shrink the size of the domain again to $|\xi| < r(1-5\theta)$. Use Cauchy estimate and the above estimate on $|\widehat{h}(\xi)|$, we finally obtain

$$|\widehat{g}'(\xi)| < \frac{2r}{\theta^4} \cdot \delta^2$$

Remark: The last estimate implies that $|\widehat{g}'(\xi)|$ is in the magnitude of $|\widehat{f}'(z)|^2$. So $h(\xi)$ constructed by using (7) transfers $T(z)$ to something that is **much closer** to $S: \xi \to \lambda\xi$.

Step 2 Let $T(z) = \lambda z + \widehat{f}(z)$, $S(\xi) = \lambda\xi$. We construct h conjugating T and S following the procedure below:

(a) Let $\widehat{h}_1$ be the solution of (7), and $S_1 = h_1^{-1} \circ T \circ h_1$. Write S_1 as

$$S_1(z) = \lambda z + \widehat{g}_1(z).$$

(b) Inductively construct $\widehat{h}_n$ using $\widehat{g}_{n-1}$ as $\widehat{f}$ n (7). Let

$$S_n(z) = h_n^{-1} S_{n-1} h_n := \lambda z + \widehat{g}_n(z).$$

This last inequality defines $\widehat{g}_n(z)$.

(c) Let $H_n = h_1 \circ h_2 \circ \cdots \circ h_n$, We have

$$S_n = H_n^{-1} T H_n.$$

Let $n \to \infty$, we hope to have

$$S_n \to S, \quad H_n \to h \tag{8}$$

on a domain in complex plane of radius > 0.

(d) To prove the convergence indicated in (8) we use the Main Proposition above repeatedly.

– For the n-th iteration, let r_n be the radius of convergence for $\widehat{h}_n$, and $\delta_n = \sup|\widehat{g}_n|$. To use the Main Proposition we need to pick a value of θ, which we denote as θ_n.

– Note that, the shrinkage of the radius of the domain is determined by θ_n at each step. In fact, we have

$$r_{n+1} = r_n(1 - 5\theta_n).$$

So at the end, the radius of convergency for all h_n is

$$> r \prod_n (1 - 5\theta_n)$$

To make the end result $> \frac{1}{2}r$, let us set $\theta_n = \frac{1}{5 \cdot 2^n}$ in applying the Main Proposition.

– Two things to worry: (i) Is it true that $\delta_n \to 0$? (ii) Is it always true that $\delta_n < \theta_n^{-4}$, which must be checked before we apply the Main Proposition.

Claim: The two things listed above are true.

Proof: Let $p_n = 10^{4n+8}\delta_n$. From $\delta_n < \frac{\delta_{n-1}^2}{\theta_n^4}$ and $\theta_n^4 > 10^{-4n}$ we have $\delta_n < 10^{4n}\delta_{n-1}^2$. So

$$p_n = 10^{4n+8}\delta_n < 10^{8n+8}\delta_{n-1}^2 = (p_{n-1})^2.$$

This leads to

$$p_n < (p_1)^{2^n}.$$

Since we can make p_1 as small as we wish. (i) and (ii) follows from the last estimate. □

Remark: We remark that not only $\delta_n \to 0$. δ_n approaches to zero extremely fast.

– From the above we conclude $S_n \to S$. For the convergence of H_n we note that

$$|H_n - H_{n-1}| < \sup |\widehat{h}_n|.$$

The rest is left as an exercise (Need to prove H_n is a Cauchy sequence by using the last inequality).

This finishes our proof of Claim A(i). $\square$