DISSIPATIVE HOMOCLINIC LOOPS AND RANK ONE CHAOS

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Abstract. In this paper we prove that when subjected to periodic forcing of the form of \( p_{\mu,\rho}(t) = \mu(\rho h(x, y) + \sin \omega t) \), second order systems with a dissipative homoclinic saddle admit strange attractors with SRB measures for a positive measure set of forcing parameters \((\mu, \rho, \omega)\). Our proof applies a recent theory of rank one maps developed by Wang and Young ([WY1]-[WY3]) based on the analysis of Benedicks and Carleson on strongly dissipative Hénon maps [BC2].

The study of ordinary differential equations with homoclinic loop has inspired many significant findings in the modern theory of chaos and dynamical systems. The most important among these findings is the discovery of homoclinic tangles by H. Poincaré in his study of the Newtonian \( N \)-body problem [P]. Periodically-forced second order systems, such as the periodically perturbed nonlinear pendulum, Duffing’s equation and van der Pol’s equation, among others, have been studied extensively in the past ([A], [D], [V], [Lev], [Le], [GH]). When a homoclinic solution in a given second order system is periodically perturbed, transversal intersections of stable and unstable manifolds occur within a certain range of forcing parameters, generating homoclinic tangles and chaotic dynamics ([P], [S], [M]).

The setting of this paper is similar. Autonomous second order systems with a dissipative homoclinic saddle are subjected to periodic forcing of the form of \( p_{\mu,\rho}(t) = \mu(\rho h(x, y) + \sin \omega t) \) where \( \mu, \rho, \omega \) are forcing parameters. First we show that, if the system we study satisfies certain non-degeneracy conditions (See (H2) in Section 1), then there is an interval \([\rho_1, \rho_2]\) for \( \rho \), where the stable and unstable manifolds of the perturbed saddle do not intersect. Let \( \rho \in [\rho_1, \rho_2] \) be fixed, then the dynamical properties of the periodically perturbed systems in a neighborhood of the homoclinic solution of the unperturbed equation are determined by the magnitude of the forcing frequency \( \omega \). When the forcing frequency \( \omega \) is small, we obtain, for all \( \mu \) sufficiently small, an attracting tori in the extended phase space. In particular, we obtain an attracting tori consisting of quasi-periodic solutions for a set of \( \mu \) with positive Lebesgue density at \( \mu = 0 \). As \( \omega \) increases, the attracting tori are dis-integrated into isolated periodic sinks and saddles. Increasing the forcing frequency \( \omega \) further, the stable and unstable manifolds of these periodic saddles will fold, and intersect to create horseshoe and strange attractors. We prove, in particular, that these are strange attractors with SRB measures. SRB measures are constructed initially for uniformly hyperbolic systems by Sinai, Ruelle and Bowen ([Si], [R], [Bo]). They are observable objects.

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representing chaos in nonuniformly hyperbolic systems. We refer to [Y] for a review on the theory of SRB measures.

Our proof uses a theory of rank one maps developed in recent years by Wang and Young [WY1]-[WY3] based on the analysis of Benedicks and Carleson on strongly dissipative Hénon maps [BC2]. With respect to the analysis of concrete nonuniformly hyperbolic systems, two of the most influential theories in the last thirty years are Jakobson’s theory on the quadratic family [J] and the theory of Benedicks and Carleson on strongly dissipative Hénon maps [BC2]. Benedick and Young later constructed SRB measures for the good maps obtained through Benedicks and Carleson’s analysis [BY]. Mora and Viana applied the theory of Benedicks and Carleson to systems with dissipative saddles unfolding quadratic homoclinic tangencies [MV].

Recently, Wang and Young have published a sequence of papers ([WY1]-[WY3]) in an effort to develop a comprehensive dynamical theory for a class of nonuniformly hyperbolic maps based on the studies of Benedicks and Carleson. One major motivation for the development of this so-called theory of rank one maps is to introduce a nonuniformly hyperbolic setting that is more flexible in applications. Various systems of ordinary differential equations have been constructed in association with the applications of the theory of rank one maps ([WY4], [WY5], [L], [GWY], [WO]), among which the most significant are the ones associated with periodically kicked Hopf limit cycles ([WY5]). Admissible families of rank one maps are created through periodically kicked limit cycles with weak contraction and strong shearing ([WY4], [WY5]). Similar examples are also constructed for certain parabolic partial differential equations ([LWY]). These studies have illustrated that the theory of rank one maps can be used to rigorously prove the existence of strange attractors with SRB measures for a class of physically meaningful systems of differential equations.

In this paper we apply the theory of rank one maps to the classical scenario of periodically perturbed second order systems with homoclinic loops. The method of analysis of this paper is as follows. We write the forced equations as an autonomous system in \((x, y, \theta)\)-space where \(x\) and \(y\) are the original phase variables and \(\theta\) is an angular variable representing time. In a neighborhood of the unperturbed homoclinic loop, we construct a 2D Poincaré section in the space of \((x, y, \theta)\) by fixing the value of one of the original phase variable and we explicitly compute the return maps induced by the three-dimensional autonomous flow. It has turned out that, not only these maps are rank one maps, but also they fall naturally into the category of the specific rank one maps studied by Wang and Young in [WY4]. The dynamical scenario outlined in the above follows directly from the ones detailed in [WY4]. In the writings of this paper, we will focus exclusively on the scenario of rank one chaos. We refer the reader to Theorems 1 and 2 in [WY4] for other cases. We also remark that [WY1] contains not only the existence of SRB measures but also a comprehensive dynamical profile for the good maps, including geometric structures of the attractors and statistical properties. We have opted to limit our statement to SRB measures, but all aspects of this larger dynamical picture in fact apply.

This paper is not only about the generic existence of rank one attractors in periodically forced second order equations. Explicit, verifiable conditions are formulated
for the existence of rank one attractors in concrete systems of differential equations. Based on the theorems of this paper, the first named author has been able to prove the existence of rank one chaos in a Duffing’s equation in the form of
\[
\frac{d^2q}{dt^2} + (a - bq^2) \frac{dq}{dt} - q + q^3 = \mu \sin \omega t
\]
and in a system of periodically forced pendulum in the form of
\[
\frac{d^2\theta}{dt^2} + \sin \theta - (\alpha - \delta v) = \mu \sin \omega t.
\]
These results will be presented in separate papers.

Our method is motivated by a paper of V.S. Afraimovich and L.P. Shil’nikov published almost thirty years ago [AS]. Afraimovich and Shil’nikov observed that, for periodically-forced systems with dissipative homoclinic loops, the dissipation around the fixed point could potentially put the flow-induced return maps into the category (in our terminology) of rank one maps. In this paper we basically start from where they stopped, turning an insightful observation into a theorem one can use to analyze concrete systems of periodically-forced second order equations. The authors are deeply in debt to Afraimovich for bringing to our attention his previous work with Shil’nikov [AS]. See also [AH]. We would also like to thank Kening Lu and Lai-Sang Young for motivating conversations related to this work, and particularly Lai-Sang Young for connecting us to Afraimovich and his work with Shil’nikov.

1. Statement of Results

Let \((x, y) \in \mathbb{R}^2\) be the phase variables and \(t\) be the time. We start with an autonomous system

\[
\frac{dx}{dt} = -\alpha x + f(x, y), \quad \frac{dy}{dt} = \beta y + g(x, y)
\]

where \(f(x, y), g(x, y)\) are real analytic at \((x, y) = (0, 0)\) and \(f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0\). We assume that \(\alpha, \beta\) satisfy a certain Diophantine non-resonance condition and \((x, y) = (0, 0)\) is a dissipative saddle point. Namely,

**(H1)** (i) there exist \(d_1, d_2 > 0\) such that for all \(n, m \in \mathbb{Z}^+\),

\[
|n\alpha - m\beta| > d_1(|n| + |m|)^{-d_2};
\]

(ii) \(0 < \beta < \alpha\).

We also assume that the positive \(x\)-side of the local stable manifold and the positive \(y\)-side of the local unstable manifold of \((0, 0)\) are included as part of a homoclinic solution, which we denote as \(x = a(t), y = b(t)\). Let

\[
\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2 : t \in \mathbb{R}\}.
\]

We further assume that \(f(x, y), g(x, y)\) are \(C^4\) in a sufficiently small neighborhood of \(\ell\).
To the right side of equation (1.1) we add a time-periodic term to form a non-autonomous system

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha x + f(x, y) - \mu(\rho h(x, y) + \sin \omega t) \\
\frac{dy}{dt} &= \beta y + g(x, y) + \mu(\rho h(x, y) + \sin \omega t)
\end{align*}
\]

(1.2)

where \(\mu, \rho\) and \(\omega\) are parameters. We assume that \(h(x, y)\) is analytic at \((x, y) = (0, 0)\) and it is \(C^4\) in a small neighborhood of the homoclinic loop \(\ell\). The parameter \(\mu\) satisfies \(0 < \mu << 1\) and controls the magnitude of the forcing term. The pre-factor \(\rho\) and the forcing frequency \(\omega\) are much larger parameters, the ranges of which we will make explicit momentarily. Observe that the same forcing function is added to the equation for \(y\) but subtracted from the equation for \(x\). This is to facilitate the future applications of our theorem to concrete second order systems.

To study (1.2), we introduce an angular variable \(\theta \in S^1\) to write it as

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha x + f(x, y) - \mu(\rho h(x, y) + \sin \theta) \\
\frac{dy}{dt} &= \beta y + g(x, y) + \mu(\rho h(x, y) + \sin \theta) \\
\frac{d\theta}{dt} &= \omega.
\end{align*}
\]

(1.3)

We denote

\[ (u(t), v(t)) = \left| \frac{d}{dt} \ell(t) \right|^{-1} \frac{d}{dt} \ell(t) \]

where \(\ell(t) = (a(t), b(t))\) is the homoclinic loop of equation (1.1). The vector \((u(t), v(t))\) is a unit vector tangent to \(\ell\) at \(\ell(t)\). Define

\[
E(t) = v^2(t)(-\alpha + \partial_x f(a(t), b(t))) + u^2(t)(\beta + \partial_y g(a(t), b(t))) - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))).
\]

(1.4)

\(E(t)\) measures the rate of expansion of the solutions of equation (1.1) in the direction normal to \(\ell\) at \(\ell(t)\) (see Sect. 2.2).

Define

\[
\begin{align*}
A &= \int_{-\infty}^{\infty} (u(s) + v(s)) h(a(s), b(s)) e^{-\int_0^s E(r)dr} ds \\
C &= \int_{-\infty}^{\infty} (u(s) + v(s)) \cos(\omega s) e^{-\int_0^s E(r)dr} ds \\
S &= \int_{-\infty}^{\infty} (u(s) + v(s)) \sin(\omega s) e^{-\int_0^s E(r)dr} ds.
\end{align*}
\]

(1.5)

\(A, C\) and \(S\) are all well-defined (See Lemma 2.3), and as we will see, they are the characteristics that describe the relative positions of the stable and unstable manifolds of the perturbed saddle. \(\rho A \mu\) measures the average distance between the stable and
the unstable manifolds and \((C^2 + S^2)^{\frac{1}{2}} \mu\) measures the magnitude of relative oscillation of the one to the other. See Fig. 1.

\[\rho A \mu \]

\[\sqrt{(C^2 + S^2)^{1/2} \mu} \]

Fig. 1 The constants \(A, C\) and \(S\).

We assume that

\((H2)\)  
(i) \(A \neq 0\) and (ii) \(C^2 + S^2 \neq 0\).

For a given equation (1.2) satisfying (H1) and (H2), we let

\[\rho_1 = -\frac{202 \sqrt{C^2 + S^2}}{99 A}, \quad \rho_2 = -\frac{396 \sqrt{C^2 + S^2}}{101 A}.\]

We also let

\[I = \{z \in \mathbb{R}, \ |z| < K \mu\} \]

for some \(K > 1\) sufficiently large independent of \(\mu\) and

\[\Sigma = \{\ell(0) + (v(0), -u(0))z \in \mathbb{R}^2 : z \in I\} \times S^1.\]

Fig. 2 Poincaré section \(\Sigma\)

We have
Theorem 1. Assume (H1) and (H2)(i) for (1.2). Then there exists $\omega_0 > 0$ such that the following hold. If $\omega > \omega_0$ is such that (H2)(ii) is satisfied, then for every $\rho \in [\rho_1, \rho_2]$, we have

(i) for $\mu$ sufficiently small, equation (1.3) induces a well-defined family of return maps $F_\mu : \Sigma \to \Sigma$;

(ii) there exists a parameter set $\Delta_{\omega, \rho}$ for $\mu$ with positive Lebesgue density at $\mu = 0$, such that for every $\mu \in \Delta_{\omega, \rho}$, $F_\mu$ admits an ergodic SRB measure $m$ on $\Sigma$; and Lebesgue almost every point in $\Sigma$ is generic with respect to $m$.

We recall that an $F$-invariant Borel probability measure $m$ on $\Sigma$ is an SRB measure if (i) $F$ has a positive Lyapunov exponent $m$-a.e.; (ii) the conditional measures of $m$ on unstable manifolds are absolutely continuous with respect to the Riemannian measures on these unstable leaves. Also recall that $z \in \Sigma$ is generic with respect to an SRB measure $m$ if for every continuous function $\phi$ on $\Sigma$ we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(z)) \to \int \phi dm.$$ 

To say that Lebesgue almost every point in $\Sigma$ is generic with respect $m$ is to say that $m$ is a statistical law of time evolutions obeyed by almost all orbits of $F$ in $\Sigma$.

Our proof of Theorem 1 uses the recent theory of rank one maps developed by Wang and Young in [WY1] and [WY2]. We prove that, assuming (H1) and (H2), $F_\mu$, for $\mu$ sufficiently small, is a well defined admissible family of rank one maps on $\Sigma$. Consequently, for $\mu \in \Delta_{\omega, \rho}$, $F_\mu$ has all of the dynamical, geometrical and statistical properties proved in [WY1]. In particular, $F_\mu$ admits strange attractors with SRB measures $m$, and Lebesgue almost every point in $\Sigma$ is generic with respect to $m$.

Remark: For a given equation (1.1) satisfying (H1), (H2)(i) obviously holds for an open and dense set of $h(x, y)$ among the functions that are analytic at $(x, y) = (0, 0)$ and are $C^4$ around the homoclinic loop $\ell$. (H2)(ii) requires that, as a function of $s$, the Fourier spectrum of the function

$$R(s) = (u(s) + v(s))e^{-\int_0^s E(\tau)d\tau}$$

is not identically zero on the frequency range higher than $\omega_0$. As we will see, $R(s)$ decays exponentially as a function of $s$, and it follows that the Fourier transform $\hat{R}(\xi)$ is analytic in a strip contain the real $\xi$-axis. Consequently, $\hat{R}(\xi) = 0$ for at most a discrete set of values of $\xi$ because $R(s)$ is not identically zero.

2. Canonical forms of equation

In this section we introduce sequences of coordinate changes to transform equation (1.3) into certain standard forms. The part of the phase space $(x, y)$ we consider is divided into a small neighborhood $U_\varepsilon$ of $(0, 0)$ and a small neighborhood $D$ of the entire length of $\ell$ out of $U_{\frac{1}{2}}$. See Fig. 3. Also let $\sigma^\pm \in U_\varepsilon \cap D$ be two line segments in Fig. 3, both perpendicular to $\ell$. In extended phase space $(x, y, \theta)$ we denote

$$U_\varepsilon = U_\varepsilon \times S^1, \quad D = D \times S^1.$$
and let

$$\Sigma^\pm = \sigma^\pm \times S^1.$$  

Fig. 3 $U_\varepsilon$, $D$ and $\sigma^\pm$.

In Sect. 2.1 we derive a standard form for equation (1.3) on $U_\varepsilon$ and in Sect. 2.2 we derive a standard form for equation (1.3) on $D$. Let $N : \Sigma^+ \to \Sigma^-$ be the map induced by equation (1.3) on $U_\varepsilon$ and $M : \Sigma^- \to \Sigma^+$ be the maps induced by equation (1.3) on $D$. See Fig. 4. In Section 3, we will use the standard form obtained in Sect. 2.2 to compute $M$ and the standard form obtained in Sect. 2.1 to compute $N$. We then compose $N$ and $M$ to obtain an explicit formula for the return map $N \circ M : \Sigma^- \to \Sigma^-$. This composed map conjugates to the induced map $F : \Sigma \to \Sigma$ of Theorem 1.

Fig. 4 $N$ and $M$.

In the rest of this paper, $\alpha, \beta, \rho \in [\rho_1, \rho_2]$ and $\omega > \omega_0$ are all regarded as fixed constants. The size of the neighborhood on which all the coordinate transformations in Sect. 2.1 are valid is represented by a small number $\varepsilon > 0$. $\varepsilon$ is also regarded as a fixed constant. We regard $\mu$ as the only parameter of equation (1.3).
Two small scales: \( \mu << \varepsilon << 1 \) represent two small scales of different magnitude. \( \varepsilon \) represents the size of a small neighborhood of \((x, y) = (0, 0)\) on which the local analysis of Sect. 2.1 is valid. Define

\[
U_\varepsilon = \{(x, y) : x^2 + y^2 < 4\varepsilon^2\} \quad \text{and} \quad U_\varepsilon = U_\varepsilon \times S^1.
\]

Let \( L^+, -L^- \) be the respective times at which the homoclinic solution \( \ell(t) \) enters \( U_\varepsilon \) in the positive and the negative directions. \( L^+ \) and \( L^- \) are related, both determined completely by \( \varepsilon \) and \( \ell(t) \). The parameters \( \mu << \varepsilon \) controls the magnitude of the time-periodic perturbation.

**Notation:** Quantities that are independent of phase variables, time and \( \mu \) are regarded as constants and \( K \) is used to denote a generic constant, the precise value of which is allowed to change from line to line. On occasion, a specific constant is used in different places. We use subscripts to denote such constants as \( K_0, K_1, \ldots \). We will also make distinctions between constants depend on \( \varepsilon \) and those that do not by making such dependencies explicit. A constant that depends on \( \varepsilon \) is written as \( K(\varepsilon) \). A constant written as \( K \) is independent of \( \varepsilon \).

### 2.1. Standard form around the fixed point.

For the purpose of computing the map \( \mathcal{N} \), it would be the best if we could introduce coordinate changes that linearizes equation (1.3) on \( U_\varepsilon \). This is however not a route we could take. The time periodic perturbation \( \mu(\rho h(x, y) + \sin \omega t) \) makes the saddle point \((x, y) = (0, 0)\) of the unperturbed equation (1.1) a periodic solution, and the Floquet exponents of this periodic solution are dependents of forcing parameters \( \mu, \rho \) and \( \omega \). In order to achieve complete linearization, we would have to impose non-resonance condition on these characteristics, consequently impose sophisticated and largely uncontrollable restrictions on the forcing parameters \( \mu, \rho \) and \( \omega \).

The route we take in this subsection is as follows. First we use (H1)(i) to completely linearized the unforced equation (1.1) on \( U_\varepsilon \). We then move the center of the coordinate system to the periodic solution obtained from the perturbed saddle. The third step is to change the coordinates to make the local stable and unstable manifolds of this periodic solution flat. Finally we re-scale by the factor \( \mu^{-1} \). Computations are tedious in occasions but entirely routine.

#### A. Linearize the unperturbed equation: \((x, y) \rightarrow (\xi, \eta)\)

Let \((\xi, \eta)\) be such that

\[
\xi = x + q_1(x, y), \quad \eta = y + q_2(x, y)
\]

where \(q_1(x, y), q_2(x, y)\) are analytic terms of order at least two in \(x\) and \(y\). Formula (2.1) defines a near-identity coordinate transformation \((x, y) \rightarrow (\xi, \eta)\), the inverse of which we write as

\[
x = \xi + Q_1(\xi, \eta), \quad y = \eta + Q_2(\xi, \eta).
\]

We have
Proposition 2.1. Assume that $\alpha$ and $\beta$ satisfy the Diophantine non-resonance condition (H1)(i). Then there exists a neighborhood $U$ of $(0,0)$, the size of which is completely determined by equation (1.1) and $d_1, d_2$ in (H1)(i), such that on $U$ there exists an analytic coordinate transformation (2.1) that transforms equation (1.1) into the linear system

$$
\frac{d\xi}{dt} = -\alpha \xi, \quad \frac{d\eta}{dt} = \beta \eta.
$$

Proof: See [HPS] for a proof.

We now use the coordinate transformation of Proposition 2.1 to transform equation (1.3). Observe that by definition, $q_1(x,y), q_2(x,y)$ satisfy

$$
\begin{align*}
(1 + \partial_xq_1(x,y))(-\alpha x + f(x,y)) + \partial_yq_1(x,y)(\beta y + g(x,y)) &= -\alpha \xi \\
(1 + \partial_yq_2(x,y))(\beta y + g(x,y)) + \partial_xq_2(x,y)(-\alpha x + f(x,y)) &= \beta \eta.
\end{align*}
$$

We derive the form of (1.3) in terms of $\xi, \eta$. We have

$$
\frac{d\xi}{dt} = (1 + \partial_xq_1(x,y))(-\alpha x + f(x,y) - \mu(\rho h(x,y) + \sin \theta)) + \partial_yq_1(x,y)(\beta y + g(x,y) + \mu(\rho h(x,y) + \sin \theta))
$$

and

$$
\frac{d\eta}{dt} = (1 + \partial_yq_2(x,y))(\beta y + g(x,y) + \mu(\rho h(x,y) + \sin \theta)) + \partial_xq_2(x,y)(-\alpha x + f(x,y) - \mu(\rho h(x,y) + \sin \theta))
$$

where the first line in (2.3) is used for the second equality. Similarly, we have

$$
\frac{d\xi}{dt} = (1 + \partial_xq_1(x,y))(-\alpha x + f(x,y) - \mu(\rho h(x,y) + \sin \theta)) + \partial_yq_1(x,y)(\beta y + g(x,y) + \mu(\rho h(x,y) + \sin \theta))
$$

and

$$
\frac{d\eta}{dt} = (1 + \partial_yq_2(x,y))(\beta y + g(x,y) + \mu(\rho h(x,y) + \sin \theta)) + \partial_xq_2(x,y)(-\alpha x + f(x,y) - \mu(\rho h(x,y) + \sin \theta)).
$$

Writing the functions of $x, y$ as functions of $\xi, \eta$ using (2.2), the form of (1.3) in terms of $\xi$ and $\eta$ is given by

$$
\begin{align*}
\frac{d\xi}{dt} &= -\alpha \xi - \mu(1 + h_1(\xi, \eta))(\rho H(\xi, \eta) + \sin \theta) \\
\frac{d\eta}{dt} &= \beta \eta + \mu(1 + h_2(\xi, \eta))(\rho H(\xi, \eta) + \sin \theta) \\
\frac{d\theta}{dt} &= \omega
\end{align*}
$$

where $h_1(\xi, \eta) = \partial_xq_1(x,y) - \partial_yq_1(x,y), h_2(\xi, \eta) = \partial_yq_2(x,y) - \partial_xq_2(x,y)$ are such that $h_1(0,0) = h_2(0,0) = 0$ and $H(\xi, \eta) = h(x,y)$.

B. Move the periodic solution to the center: $(\xi, \eta) \to (X,Y)$

With the forcing added, the hyperbolic fixed point $(x,y) = (0,0)$ of equation (1.1) is perturbed to become a hyperbolic periodic solution of period $2\pi \omega^{-1}$ of (1.2). We denote this periodic solution in $(\xi, \eta, \theta)$-coordinates as $\xi = \mu \phi(\theta; \mu), \eta = \mu \psi(\theta; \mu)$.

Proposition 2.2. For equation (2.4), there exists a unique solution of the form of

$$
\xi = \mu \phi(\theta; \mu), \quad \eta = \mu \psi(\theta; \mu), \quad \theta = \omega t
$$
satisfying
\[ \phi(\theta; \mu) = \phi(\theta + 2\pi; \mu), \quad \psi(\theta; \mu) = \psi(\theta + 2\pi; \mu). \]

The $C^3$-norm of the functions $\phi(\theta; \mu)$ and $\psi(\theta; \mu)$, regarded as functions of $\theta$ and $\mu$, are bounded by a constant $K$.

**Proof:** Write $\phi = \phi(\theta; \mu), \psi = \psi(\theta; \mu)$. The functions $\phi, \psi$ should satisfy

\[ \frac{d\phi}{d\theta} = -\alpha \phi - (1 + h_1(\mu \phi, \mu \psi))(\rho H(\mu \phi, \mu \psi) + \sin \theta) \]
\[ \frac{d\psi}{d\theta} = \beta \psi + (1 + h_2(\mu \phi, \mu \psi))(\rho H(\mu \phi, \mu \psi) + \sin \theta). \]

From (2.5) it follows that

\[ \phi(\theta; \mu) = e^{-\alpha \omega^{-1}(\theta - \theta_0)} \phi(\theta_0; \mu) - \omega^{-1} \int_{\theta_0}^{\theta} e^{\alpha \omega^{-1}(s - \theta)} [1 + h_1(\mu \phi(s; \mu), \mu \psi(s; \mu))] \]
\[ \rho H(\mu \phi(s; \mu), \mu \psi(s; \mu)) + \sin s] ds \]
\[ \psi(\theta; \mu) = e^{\beta \omega^{-1}(\theta - \theta_0)} \psi(\theta_0; \mu) + \omega^{-1} \int_{\theta_0}^{\theta} e^{-\beta \omega^{-1}(s - \theta)} [1 + h_2(\mu \phi(s; \mu), \mu \psi(s; \mu))] \]
\[ \rho H(\mu \phi(s; \mu), \mu \psi(s; \mu)) + \sin s] ds. \]

To solve for $\phi$ and $\psi$ we let $\theta = \theta_0 + 2\pi$ and set $\phi(\theta_0 + 2\pi; \mu) = \phi(\theta_0; \mu), \psi(\theta_0 + 2\pi; \mu) = \psi(\theta_0; \mu)$ to obtain

\[ \phi(\theta; \mu) = \frac{-\omega^{-1}}{1 - e^{-2\omega^{-1}2\pi}} \int_{0}^{2\pi} e^{\alpha \omega^{-1}(s - 2\pi)} [1 + h_1(\mu \phi(s + \theta; \mu), \mu \psi(s + \theta; \mu))] \]
\[ \rho H(\mu \phi(s + \theta; \mu), \mu \psi(s + \theta; \mu)) + \sin (s + \theta)] ds \]
\[ \psi(\theta; \mu) = \frac{\omega^{-1}}{1 - e^{2\beta \omega^{-1}2\pi}} \int_{0}^{2\pi} e^{-\beta \omega^{-1}(s - 2\pi)} [1 + h_2(\mu \phi(s + \theta; \mu), \mu \psi(s + \theta; \mu))] \]
\[ \rho H(\mu \phi(s + \theta; \mu), \mu \psi(s + \theta; \mu)) + \sin (s + \theta)] ds. \]

The existence and uniqueness of $\phi(\theta; \mu)$ and $\psi(\theta; \mu)$ follows directly from an application of the contacting mapping theorem to (2.6). The asserted bound on partial derivatives with respect to $\theta$ and $\mu$ follows from differentiating (2.6) with respect to $\theta$ and $\mu$. \hfill \Box

We now introduce new variables $(X, Y)$ by letting

\[ X = \xi - \mu \phi(\theta; \mu), \quad Y = \eta - \mu \psi(\theta; \mu). \]

We have
\[ \frac{dX}{dt} = -\alpha X - \alpha \mu \phi - \mu \omega \frac{d\phi}{d\theta} \]
\[ - \mu(1 + h_1(X + \mu \phi, Y + \mu \psi))(\rho H(X + \mu \phi, Y + \mu \psi) + \sin \theta), \]
\[ \frac{dY}{dt} = \beta Y + \beta \mu \psi - \mu \omega \frac{d\psi}{d\theta} \]
\[ + \mu(1 + h_2(X + \mu \phi, Y + \mu \psi))(\rho H(X + \mu \phi, Y + \mu \psi) + \sin \theta). \]
Using (2.5), the form of (1.3) in terms of \( X, Y \) and \( \theta \) is given by

\[
\begin{align*}
\frac{dX}{dt} &= -\alpha X + \mu F(X, Y, \theta; \mu) \\
\frac{dY}{dt} &= \beta Y + \mu G(X, Y, \theta; \mu) \\
\frac{d\theta}{dt} &= \omega
\end{align*}
\]

(2.8)

where

\[
F(X, Y, \theta; \mu) = -[h_1(X + \mu \phi, Y + \mu \psi) - h_1(\mu \phi, \mu \psi)](\rho H(X + \mu \phi, Y + \mu \psi) \\
+ \sin \theta) - \rho(1 + h_1(\mu \phi, \mu \psi))(H(X + \mu \phi, Y + \mu \psi) - H(\mu \phi, \mu \psi))
\]

\[
G(X, Y, \theta; \mu) = [h_2(X + \mu \phi, Y + \mu \psi) - h_2(\mu \phi, \mu \psi)](\rho H(X + \mu \phi, Y + \mu \psi) \\
+ \sin \theta) + \rho(1 + h_2(\mu \phi, \mu \psi))(H(X + \mu \phi, Y + \mu \psi) - H(\mu \phi, \mu \psi))
\]

are such that \( F(0, 0, \theta; \mu) = G(0, 0, \theta; \mu) = 0 \). Observe that in the new coordinates \((X, Y, \theta)\), the solution \( \xi = \mu \phi(\theta; \mu), \eta = \mu \psi(\theta; \mu) \) is represented by \( X = Y = 0 \). We remark that on

\[
\{(X, Y, \theta; \mu) : |(X, Y)| < \varepsilon, \theta \in S^1, 0 \leq \mu < \mu_0\},
\]

(i) \( F(X, Y, \theta; \mu), G(X, Y, \theta; \mu) \) are analytic functions bounded by \( K\varepsilon \);

(ii) it follows from Proposition 2.2 that the \( C^3 \)-norms of both \( F \) and \( G \) as functions of \((X, Y, \theta)\) and \( \mu \) are bounded by a constant \( K \).

**C. Flat the local stable and unstable manifolds: \((X, Y) \rightarrow (X, Y)\)**

The periodic solution \((X, Y, \theta) = (0, 0, \omega t)\) of equation (2.8) has a local unstable manifold, which we write as

\[ X = \mu W^u(Y, \theta; \mu), \]

and a local stable manifold, which we write as

\[ Y = \mu W^s(X, \theta; \mu). \]

We have

**Proposition 2.3.** There exists \( \varepsilon > 0 \), and \( \mu_0 = \mu_0(\varepsilon) > 0 \) such that \( W^u(Y, \theta; \mu), W^s(X, \theta; \mu) \)

are analytically defined on

\[ (-\varepsilon, \varepsilon) \times S^1 \times [0, \mu_0) \]

and satisfy

\[ W^u(0, \theta; \mu) = 0, \quad W^s(0, \theta; \mu) = 0. \]

The \( C^3 \)-norm of \( W^u(Y, \theta; \mu) \) and \( W^s(X, \theta; \mu) \), regarded as a function of all three of their arguments, are bounded by a constant \( K \).

**Proof:** We regard \( X, Y, \theta \) and \( \mu \) in equation (2.8) as complex variables. The existence and smoothness of local stable and unstable manifolds follows from the standard argument based on the contraction mapping theorem. See [HPS] for instance. \( \square \)
By definition, \( W^u(Y, \theta; \mu) \) satisfies
\[
- \alpha W^u(Y, \theta; \mu) + F(\mu W^u(Y, \theta; \mu), Y, \theta; \mu) = \omega \partial_y W^u(Y, \theta; \mu)
+ \partial_x W^u(Y, \theta; \mu)(\beta Y + \mu G(\mu W^u(Y, \theta; \mu), Y, \theta; \mu)).
\]
(2.9)

Similarly \( W^{s}(X, \theta; \mu) \) satisfies
\[
\beta W^{s}(X, \theta; \mu) + G(X, \mu W^{s}(X, \theta; \mu), \theta; \mu) = \omega \partial_x W^{s}(X, \theta; \mu)
+ \partial_x W^{s}(X, \theta; \mu)(-\alpha X + \mu F(X, \mu W^{s}(X, \theta; \mu), \theta; \mu)).
\]
(2.10)

Define the new variables \((X, Y)\) by
\[
X = X - \mu W^u(Y, \theta; \mu), \quad Y = Y - \mu W^{s}(X, \theta; \mu).
\]
(2.11)

By using (2.8), (2.9) and (2.10), the form of (1.3) in terms of \((X, Y, \theta)\) is given by
\[
\frac{dX}{dt} = (-\alpha + \mu F(X, Y, \theta; \mu))X
\]
(2.12)
\[
\frac{dY}{dt} = (\beta + \mu G(X, Y, \theta; \mu))Y
\]
\[
\frac{d\theta}{dt} = \omega.
\]

where \(F, G\) are analytic functions of \(X, Y, \theta\) and \(\mu\) defined on \(U_{\varepsilon} \times S^1 \times [0, \mu_0)\). The \(C^3\) norms of \(F\) and \(G\) are bounded by a constant \(K\).

Tracing back to the variables \((\xi, \eta)\), we have
\[
X = \xi - \mu(\phi(\theta; \mu) + W^u(\eta - \mu \psi(\theta; \mu), \theta; \mu))
\]
\[
Y = \eta - \mu(\psi(\theta; \mu) + W^{s}(\xi - \mu \phi(\theta; \mu), \theta; \mu)).
\]
(2.13)

D. Re-scale by \(\mu^{-1}: (X, Y) \rightarrow (\bar{X}, \bar{Y})\)

The last coordinate change is a rescaling of \(X, Y\) by the factor \(\mu^{-1}\). Let
\[
X = \mu^{-1}X, \quad Y = \mu^{-1}Y.
\]
(2.14)

We write equation (2.12) in \(\bar{X}, \bar{Y}\) as
\[
\frac{d\bar{X}}{dt} = (-\alpha + \mu F(\bar{X}, \bar{Y}, \theta; \mu))\bar{X}
\]
(2.15)
\[
\frac{d\bar{Y}}{dt} = (\beta + \mu G(\bar{X}, \bar{Y}, \theta; \mu))\bar{Y}
\]
\[
\frac{d\theta}{dt} = \omega
\]

where
\[
F(\bar{X}, \bar{Y}, \theta; \mu) = F(\mu X, \mu Y, \theta; \mu) \quad G(\bar{X}, \bar{Y}, \theta; \mu) = G(\mu X, \mu Y, \theta; \mu)
\]
are analytic functions of \(X, Y, \theta\) and \(\mu\) defined on
\[
D = \{(X, Y, \theta; \mu) : \mu \in (0, \mu_0), \quad (X, Y, \theta) \in U_{\varepsilon}\}
\]
where
\[
U_{\varepsilon} = \{(X, Y, \theta) : |(X, Y)| < 2\varepsilon \mu^{-1}, \theta \in S^1\}.
\]
Remark: We observe that all constants represented by \( K \) in Sect. 2.1 are independent of \( \varepsilon \) and \( \mu \).

2.2. A standard form around the homoclinic loop. In this subsection we derive a standard form for equation (1.3) around the homoclinic loop of equation (1.1) outside of \( \mathcal{U}_{1\varepsilon} \). Some elementary estimates are also included.

A. Derivation of equations

Let us regard \( t \) in \( \ell(t) = (a(t), b(t)) \) not as time, but as a parameter that parameterizes the curve \( \ell \) in \((x, y)\)-space. We replace \( t \) by \( s \) and write this homoclinic loop as \( \ell(s) = (a(s), b(s)) \). We have

\[
\frac{da(s)}{ds} = -\alpha a(s) + f(a(s), b(s)), \quad \frac{db(s)}{ds} = \beta b(s) + g(a(s), b(s)).
\]

By definition,

\[
u(s) = \frac{-\alpha a(s) + f(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}},
\]

\[
v(s) = \frac{\beta b(s) + g(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}.
\]

Let

\[e(s) = (v(s), -u(s)).\]

We now introduce new variables \((s, z)\) such that

\[(x, y) = \ell(s) + z e(s)\]

This is to say that

\[
x = x(s, z) = a(s) + v(s)z, \quad y = y(s, z) = b(s) - u(s)z.
\]

We derive the form of (1.3) in terms of the new variables \((s, z)\) defined through (2.18). Differentiating (2.18), we obtain

\[
\frac{dx}{dt} = (\alpha a(s) + f(a(s), b(s)) + v'(s)z) \frac{ds}{dt} + v(s) \frac{dz}{dt}
\]

\[
\frac{dy}{dt} = (\beta b(s) + g(a(s), b(s)) - u'(s)z) \frac{ds}{dt} - u(s) \frac{dz}{dt}
\]

where \(u'(s) = \frac{du(s)}{ds}, v'(s) = \frac{dv(s)}{ds}\). Let us denote

\[F(s, z) = -\alpha (a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)),\]

\[G(s, z) = \beta (b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)),\]

\[H(s, z) = h(a(s) + zv(s), b(s) - zu(s)).\]
Using (1.3) and (2.19), we have

\[
\frac{ds}{dt} = \frac{v(s)G(s, z) + u(s)F(s, z) + \mu(v(s) - u(s))(\rho \mathbb{H}(s, z) + \sin \theta)}{\sqrt{F(s, 0)^2 + G(s, 0)^2 + z(u(s)v'(s) - v(s)u'(s))}}
\]

\[
\frac{dz}{dt} = v(s)F(s, z) - u(s)G(s, z) - \mu(u(s) + v(s))(\rho \mathbb{H}(s, z) + \sin \theta).
\]

We rewrite these equations as

\[
\frac{ds}{dt} = 1 + zw_1(s, z; \mu) + \frac{\mu(v(s) - u(s))(\rho \mathbb{H}(s, 0) + \sin \theta)}{\sqrt{F(s, 0)^2 + G(s, 0)^2}}
\]

\[
\frac{dz}{dt} = E(s)z + z^2w_2(s, z) - \mu(u(s) + v(s))(\rho \mathbb{H}(s, z) + \sin \theta)
\]

(2.20)

\[
\frac{d\theta}{dt} = \omega
\]

where

\[
E(s) = v^2(s)(-\alpha + \partial_x f(a(s), b(s))) + u^2(s)(\beta + \partial_y g(a(s), b(s)))
\]

\[
- u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s))),
\]

\[
\mathbb{H}(s, 0) = h(a(s), b(s)).
\]

Equation (2.20) is defined on

\[
\{ s \in [-2L^-, 2L^+], \; \mu \in [0, \mu_0], \; \theta \in S^1, \; |z| < K_0(\varepsilon)\mu \},
\]

where \( K_0(\varepsilon) \) is independent of \( \mu \). The \( C^3 \)-norms of the functions \( w_1(s, z; \mu) \), \( w_2(s, z) \) are bounded by a constant \( K(\varepsilon) \).

Finally, we re-scale the variable \( z \) by letting

\[
(2.21) \quad Z = \mu^{-1}z.
\]

We arrive at the equations

\[
\frac{ds}{dt} = 1 + \mu \tilde{w}_1(s, Z, \theta; \mu)
\]

\[
(2.22) \quad \frac{dZ}{dt} = E(s)Z + \mu \tilde{w}_2(s, Z, \theta; \mu) - (u(s) + v(s))(\rho \mathbb{H}(s, 0) + \sin \theta)
\]

\[
\frac{d\theta}{dt} = \omega
\]

defined on

\[
D = \{(s, Z, \theta; \mu) : s \in [-2L^-, 2L^+], \; \theta \in S^1, \; \mu \in [0, \mu_0], \; |Z| \leq K_0(\varepsilon)\}.
\]

We assume that \( \mu_0 \) is sufficiently small so that

\[
\mu << \min_{s \in [-2L^-, 2L^+]} (F(s, 0)^2 + G(s, 0)^2).
\]

The \( C^3 \)-norms of the functions \( \tilde{w}_1, \tilde{w}_2 \) are bounded by a constant \( K(\varepsilon) \) on \( D \).

Equation (2.22) is the one we need. The function \( E(t) \) appears in the integrals \( A, C, S \) in (H2).
Remark: Observe that all of the generic constants that have appeared thus far in this subsection have the form $K(\varepsilon)$.

B. Technical estimates

**Notation:** We adopt the following convention in comparing the magnitude of two functions $f(t)$ and $g(t)$. We write $f(t) \prec g(t)$ if there exists $K > 0$ independent of $t$ such that $|f(t)| < K|g(t)|$ as $t \to \infty$ (or $-\infty$). We write $f(t) \sim g(t)$ if in addition we have $|f(t)| > K^{-1}|g(t)|$. We also write $f(t) \approx g(t)$ if

$$\frac{f(t)}{g(t)} \to 1$$

as $t \to \infty$ (or $-\infty$).

Recall that $\ell(t) = (a(t), b(t))$ is the homoclinic solution for the hyperbolic fixed point $(0, 0)$ of equation (1.1). The vector $(u(t), v(t))$ is the unit tangent vector of $\ell$ at $\ell(t)$.

**Lemma 2.1.** As $t \to +\infty$,

$$a(t) \sim e^{-\alpha t}, \quad b(t) \prec e^{-2\alpha t}, \quad u(t) \approx -1, \quad v(t) \prec e^{-\alpha t};$$

$$a(-t) \prec e^{-2\beta t}, \quad b(-t) \sim e^{-\beta t}, \quad u(-t) \prec e^{-\beta t}, \quad v(-t) \approx 1.$$

**Proof:** We are simply restating the fact that $\ell(t) \to (0, 0)$ with an exponential rate $-\alpha$ in the positive time direction along the $x$-axis and with an exponential rate $\beta$ in the negative time direction along the $y$-axis. \hfill $\square$

Let $E(t)$ be as in (1.4).

**Lemma 2.2.** As $L^\pm \to +\infty$,

(i) $\int_{-L^-}^{0} (E(s) + \alpha) ds \prec 1, \quad \int_{0}^{L^+} (E(s) - \beta) ds \prec 1.$

(ii) $\int_{-L^-}^{0} E(s) ds \approx -\alpha L^-, \quad \int_{0}^{L^+} E(s) ds \approx \beta L^+.$

**Proof:** Statement (i) claims that the integrals are convergent as $L^\pm \to \infty$. For the first integral, we observe that by adding $\alpha$ to $E(t)$, we obtain $E(t) + \alpha$ as a collection of terms, each of which decays exponentially as $t \to -\infty$ by Lemma 2.1. Similarly, taking $\beta$ away from $E(t)$, we obtain $E(t) - \beta$ as a collection of terms, each of which decays exponentially as $t \to \infty$.

For (ii) we write

$$\int_{-L^-}^{0} E(s) ds = -\alpha L^- + \int_{-L^-}^{0} (E(s) + \alpha) ds$$

$$\int_{0}^{L^+} E(s) ds = \beta L^+ + \int_{0}^{L^+} (E(s) - \beta) ds.$$

Statement (ii) now follows from (i). \hfill $\square$

We also have
Lemma 2.3. All of the integrals defined in (1.5) are absolutely convergent.

Proof: Let us write

$$A = \int_{-\infty}^{-L_0} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau) \, d\tau} \, ds$$

$$+ \int_{-L_0}^{L_0} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau) \, d\tau} \, ds$$

$$+ \int_{L_0}^{\infty} (u(s) + v(s))h(a(s), b(s))e^{-\int_0^s E(\tau) \, d\tau} \, ds$$

We write the first integral as

$$\int_{-\infty}^{-L_0} (u(s) + v(s))h(a(s), b(s))e^{\alpha s} e^{-\int_0^s (E(\tau) + \alpha) \, d\tau} \, ds$$

and make $L_0$ sufficiently large so that $|E(\tau) + \alpha| < \frac{1}{2} \alpha$ for all $\tau \in (-\infty, -L_0)$. This integral is convergent since the integrand is $< Ke^{\frac{\alpha s}{2}}$ for all $s \in (-\infty, -L_0)$. For the convergence of the third integral we rewrite it as

$$\int_{L_0}^{\infty} (u(s) + v(s))h(a(s), b(s))e^{-\beta s} e^{-\int_0^s (E(\tau) - \beta) \, d\tau} \, ds$$

and observe that $|E(\tau) - \beta| < \frac{\beta}{2}$ for $\tau \in [L_0, \infty)$ provided that $L_0$ is sufficiently large. Proofs for $C$ and $S$ are similar. \square

2.3. Poincaré sections and conversion of coordinates. In this subsection we introduce the Poincaré sections $\Sigma^\pm$. In Section 3 we will compute the induced map $\mathcal{M} : \Sigma^- \rightarrow \Sigma^+$ using equation (2.22) in Sect. 2.2 and the induced map $\mathcal{N} : \Sigma^+ \rightarrow \Sigma^-$ by using equation (2.15) in Sect. 2.1D. To properly compose $\mathcal{N}$ and $\mathcal{M}$ so obtained, we need to be able to convert variables from $(\mathcal{X}, \mathcal{Y}, \theta)$ to $(s, Z, \theta)$ on $\Sigma^\pm$ and vice versa. This issue of coordinate conversion is also treated in this subsection.

A. The Poincaré Sections $\Sigma^\pm$

Recall that $\{\ell(s) : s \in (-\infty, \infty)\}$ is the homoclinic loop of equation (1.1). For a given $\varepsilon > 0$ sufficiently small, let $L^+$ and $-L^-$ be such that

$$\xi(-L^-) = a(-L^-) + q_1(a(-L^-), b(-L^-)) = 0,$$

$$\eta(-L^-) = b(-L^-) + q_2(a(-L^-), b(-L^-)) = \varepsilon,$$

$$\xi(L^+) = a(L^+) + q_1(a(L^+), b(L^+)) = \varepsilon,$$

$$\eta(L^+) = b(L^+) + q_2(a(L^+), b(L^+)) = 0.$$  \hspace{1cm} (2.23)

Let

$$K_0 = \max_{\theta \in S^1, \mu \in [0, \mu_0]} \{|\phi(\theta; \mu)|, |\psi(\theta; \mu)|\}$$
where \( \phi(\theta; \mu), \psi(\theta; \mu) \) are as in Sect. 2.1B. We define two sections in \( U_\varepsilon \), denoted \( \Sigma^- \) and \( \Sigma^+ \), as follows.

\[
\Sigma^- = \{(x, y, \theta) : s = -L^- , \ |z| \leq (K_0 + 1)\mu, \ \theta \in S^1 \}; \\
\Sigma^+ = \{(x, y, \theta) : s = L^+, \ \frac{1}{10}(-\rho A)(K_0 + 1)\epsilon^{K\ln 2} \mu \leq z \leq 10(-\rho A)(K_0 + 1)e^{2\Lambda} \mu, \ \theta \in S^1 \}
\]

where \( s, z \) are as in (2.18). We construct the flow-induced map \( F_\mu \) in two steps.

(i) Starting from \( \Sigma^- \), the solutions of equation (1.3) move out of \( U_\varepsilon \), following the homoclinic loop of equation (1.1) to eventually hit \( \Sigma^+ \). This defines a flow-induced map from \( \Sigma^- \) to \( \Sigma^+ \), which we denote as \( \mathcal{M} : \Sigma^- \to \Sigma^+ \). We will prove that \( \mathcal{M}(\Sigma^-) \subset \Sigma^+ \).

(ii) Starting from \( \Sigma^+ \), the solutions of equation (1.3) stay inside of \( U_\varepsilon \), carrying \( \Sigma^+ \) into \( \Sigma^- \). This map we denote as \( \mathcal{N} \).

We have \( F_\mu = \mathcal{N} \circ \mathcal{M} \). Observe that the variables \( (s, Z, \theta) \) of Sect. 2.2 are suitable for computing \( \mathcal{M} \) and \( (X, Y, \theta) \) are suitable for computing \( \mathcal{N} \). To properly compose \( \mathcal{N} \) and \( \mathcal{M} \), we also need to know how to convert from \( (s, Z, \theta) \) to \( (X, Y, \theta) \) on \( \Sigma^\pm \) and vice versa.

**The new parameter \( p \):** As stated earlier, we regard \( \mu \) as the only parameter of system (1.3). We make a coordinate change on this parameter by letting \( p = \ln \mu \) and regard \( p \), not \( \mu \), as our bottom-line parameter. In other words, we regard \( \mu \) as a shorthand for \( e^p \) and all functions written in \( \mu \) are regarded as functions in \( p \). Observe that \( \mu \in (0, \mu_0) \) corresponds to \( p \in (-\infty, \ln \mu_0) \). This is a very important conceptual point because by regarding a function \( F(\mu) \) of \( \mu \) as a function of \( p \), we have

\[
\partial_p F(\mu) = \mu \partial_\mu F(\mu).
\]

Therefore, regarding \( F(\mu) \) as a function of \( p \) gives us a \( C^3 \)-norm that is completely different from the one obtained by regarding \( F(\mu) \) as a function of \( \mu \).

**Notation:** In order to apply the theory of rank one maps ([WY1] and [WY2]), we need to control the \( C^3 \)-norms of \( F_\mu \). In particular, we must estimate the \( C^3 \)-norms of certain quantities with respect to various sets of variables on relevant domains. The derivation of the flow-induced maps \( F_\mu \) involves a composition of maps and multiple coordinate changes. To facilitate the presentation, from this point on we adopt specific conventions for indicating controls on magnitude. For a given constant, we write \( \mathcal{O}(1) \), \( \mathcal{O}(\varepsilon) \) or \( \mathcal{O}(\mu) \) to indicate that the magnitude of the constant is bounded by \( K, K\varepsilon \) or \( K(\varepsilon)\mu \), respectively. For a function of a set \( V \) of variables on a specific domain, we write \( \mathcal{O}_V(1), \mathcal{O}_V(\varepsilon) \) or \( \mathcal{O}_V(\mu) \) to indicate that the \( C^3 \)-norm of the function on the specified domain is bounded by \( K, K\varepsilon \) or \( K(\varepsilon)\mu \), respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in the notation. For example \( \mathcal{O}_{X_0, Y_0, \theta, \mu}(\varepsilon) \) represents a function of \( X_0, Y_0, \theta, \mu \), the \( C^3 \)-norm of which is bounded above by \( K\varepsilon \) on a domain explicitly given in the surrounding text. Similarly, \( \mathcal{O}_{Z, \theta, \mu}(\mu) \) represent a function of \( Z, \theta \) and \( p \), the \( C^3 \)-norm of which is bounded above by \( K(\varepsilon)\mu \).
B. Conversion on $\Sigma^-$

The section $\Sigma^-$ is defined by $s = -L^-$. A point $q \in \Sigma^-$ is uniquely determined by a pair $(Z, \theta)$. First we compute the coordinates $X$ and $Y$ for a point given in $(Z, \theta)$-coordinates on $\Sigma^-$. Recall that $p = \ln \mu$.

**Proposition 2.4.** For $\mu \in (0, \mu_0)$ and $(Z, \theta) \in \Sigma^-$, we have

$$X = (1 + \mathcal{O}_{\theta, \mu}(\varepsilon) + \mu\mathcal{O}_{Z, \theta, \mu}(1))Z - \mathcal{O}_{\theta, \mu}(1)$$

$$Y = \mu^{-1}\varepsilon + \mathcal{O}_{Z, \theta, \mu}(1).$$

**Proof:** By definition $s = -L^-$ on $\Sigma^-$. Let $q \in \Sigma^-$ be represented by $(z, \theta)$. We have from (2.23) that

$$a(-L^-) = Q_1(0, \varepsilon) = \mathcal{O}(\varepsilon^2), \quad b(-L^-) = \varepsilon + Q_2(0, \varepsilon) = \varepsilon + \mathcal{O}(\varepsilon^2).$$

We also have

$$u(-L^-) = \mathcal{O}(\varepsilon), \quad v(-L^-) = 1 - \mathcal{O}(\varepsilon).$$

We compute values of $X, Y$ for $q$. Using (2.23) and (2.25),

$$\xi = a(-L^-) + v(-L^-)z + q_1(a(-L^-) + v(-L^-)z, b(-L^-) - u(-L^-)z)$$

$$= v(-L^-)z + q_1(a(-L^-) + v(-L^-)z, b(-L^-) - u(-L^-)z)$$

$$- q_1(a(-L^-), b(-L^-))$$

$$= (1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z.$$

Similarly, we have

$$\eta = b(-L^-) - u(-L^-)z + q_2(a(-L^-) + v(-L^-)z, b(-L^-) - u(-L^-)z)$$

$$= \varepsilon - u(-L^-)z + q_2(a(-L^-) + v(-L^-)z, b(-L^-) - u(-L^-)z)$$

$$- q_2(a(-L^-), b(-L^-))$$

$$= \varepsilon + (\mathcal{O}(\varepsilon) + zh_\eta(z))z.$$

The functions $h_\xi$ and $h_\eta$ are analytic function on $|z| < (K_0 + 1)\mu$ and we have $h_\xi(z) = \mathcal{O}(1)$ and $h_\eta(z) = \mathcal{O}_z(1)$. Substituting $\xi, \eta$ above into the first item in (2.13), we obtain

$$X = (1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z - \mu\phi(\theta; \mu)$$

$$- \mu W^u(\varepsilon - \mu \psi(\theta; \mu) + (\mathcal{O}(\varepsilon) + zh_\eta(z))z, \theta; \mu)$$

$$= (1 + \mathcal{O}(\varepsilon) + zh_\xi(z))z - \mu\phi(\theta; \mu) - \mu W^u(\varepsilon - \mu \psi(\theta; \mu), \theta; \mu)$$

$$- \mu W^u(\varepsilon - \mu \psi(\theta; \mu) + (\mathcal{O}(\varepsilon) + zh_\eta(z))z, \theta; \mu)$$

$$+ \mu W^u(\varepsilon - \mu \psi(\theta; \mu), \theta; \mu).$$

This implies

$$X = (1 + \mathcal{O}_{\theta, \mu}(\varepsilon) + z\hat{h}(z, \theta; \mu))z - \mu\mathcal{O}_{\theta, \mu}(1).$$

where $\hat{h}(z, \theta; \mu)$ is analytic in $z, \theta, \mu$ and satisfies $\hat{h} = \mathcal{O}_{z, \theta, \mu}(1)$. Now substitute

$$X = \mu X, \quad z = \mu Z$$
into the above and note that $|Z| < K_0 + 1$. We obtain the claimed formula for $X$.

For the $Y$ component, we substitute $\xi, \eta$ above into the second item in (2.13) to obtain

$$Y = \varepsilon (O(\varepsilon) + zh_{\eta}(z))z - \mu \psi(\theta; \mu)$$
$$- \mu W^{s'}(1 + O(\varepsilon) + zh_{\xi}(z))z - \mu \phi(\theta; \mu, \theta; \mu).$$

Let $Y = \mu Y$, $z = \mu Z$, and note that $|Z| < K_0 + 1$. We obtain the claimed formula for $Y$. □

**Corollary 2.1.** On $\Sigma^-$, we have

$$Z = (1 + O_{\theta, p}(\varepsilon))X + O_{\theta, p}(1) + O_{X, \theta, p}(\mu).$$

**Proof:** We start with (2.27). This equality is invertible and we have

(2.28) $$z = (1 + O_{\theta, \mu}(\varepsilon)) + W\tilde{h}(W, \theta; \mu)W$$

where

$$W = X + \mu O_{\theta, \mu}(1)$$

and $\tilde{h}(W, \theta; \mu)$ is analytic in $W, \theta$ and $\mu$ and satisfy $\tilde{h} = O_{W, \theta, \mu}(1)$. Writing (2.28) in terms of $Z$ and $X$, we have

$$Z = (1 + O_{\theta, p}(\varepsilon) + \mu O_{X, \theta, p}(1))(X + O_{\theta, p}(1))$$
$$= (1 + O_{\theta, p}(\varepsilon))X + O_{\theta, p}(1) + O_{X, \theta, p}(\mu).$$

We also have

**Corollary 2.2.** On $\Sigma^-$, we have

$$Y = \mu^{-1}\varepsilon + O_{X, \theta, p}(1).$$

**Proof:** We first regard $Y$ as a function of $Z, \theta, p$ using the formula for $Y$ in Proposition 2.4 and then regard $Z$ as a function of $X, \theta, p$ using Corollary 2.1. □

**Remark:** Terms in the form of $\mu O_{X, \theta, p}(1)$ are not equivalent to the terms in the form of $O_{X, \theta, p}(\mu)$. A term of the form $\mu O_{X, \theta, p}(1)$ has $C^3$-norm bounded above by $K\mu$ while a term in the form of $O_{X, \theta, p}(\mu)$ has a $C^3$-norm bounded above by $K(\varepsilon)\mu$. In estimates in Sect. 2.3B and 2.3C, we always have the former, not the latter.

**C. Switching coordinates on $\Sigma^+$**

On $\Sigma^+$ we need to write $X, Y$ in terms of $Z$.

**Proposition 2.5.** On $\Sigma^+$ we have

$$X = \mu^{-1}\varepsilon + O_{Z, \theta, p}(1)$$
$$Y = (1 + O_{\theta, p}(\varepsilon) + \mu O_{Z, \theta, p}(1))Z - O_{\theta, p}(1)$$
Proof: On $\Sigma^+$, $s = L^+$. We have
\begin{align}
(2.29) & \quad a(L^+) = \varepsilon + Q_1(\varepsilon, 0) = \varepsilon + \mathcal{O}(\varepsilon^2), \quad b(L^+) = Q_2(\varepsilon, 0) = \mathcal{O}(\varepsilon^2), \\
(2.30) & \quad u(L^+) = -1 + \mathcal{O}(\varepsilon), \quad v(L^+) = \mathcal{O}(\varepsilon).
\end{align}
and

Let $(z, \theta) \in \Sigma^+$. We compute the values of $X, Y$ for this point. Using (2.16) and (2.1), we have
\begin{align*}
\xi &= a(L^+) + v(L^+)z + q_1(a(L^+) + v(L^+)z, b(L^+) - u(L^+)z) \\
&= \varepsilon + \mathcal{O}(\varepsilon)z + q_1(a(L^+) + v(L^+)z, b(L^+) - u(L^+)z) \\
&= \varepsilon + (\mathcal{O}(\varepsilon) + zk\xi)z.
\end{align*}
Similarly, we have
\begin{align*}
\eta &= b(L^+) - u(L^+)z + q_2(a(L^+) + v(L^+)z, b(L^+) - u(L^+)z) \\
&= -u(L^+)z + q_2(a(L^+) + v(L^+)z, b(L^+) - u(L^+)z) \\
&= (1 + \mathcal{O}(\varepsilon) + zk\eta)z.
\end{align*}

We now write $X, Y$ in terms of $z$ using (2.13). The rest of the proof is similar to that of Proposition 2.4. □

We have

Corollary 2.3. If $L^+$ is sufficiently large, then $\mathcal{Y} > 1$ on $\Sigma^+$.

Proof: This follows directly from the definition of $\Sigma^+$. □

3. Explicit Computation of $\mathcal{M}$ and $\mathcal{N}$

In this section we explicitly compute the flow-induced maps $\mathcal{M} : \Sigma^- \to \Sigma^+$ and $\mathcal{N} : \Sigma^+ \to \Sigma^-$. The map $\mathcal{M} : \Sigma^- \to \Sigma^+$ is computed in Sect. 3.1. In Sect. 3.2 we study the time-$t$ map of equation (2.15). The map $\mathcal{N} : \Sigma^+ \to \Sigma^-$ is computed in Sect. 3.3.

3.1. Computing $\mathcal{M} : \Sigma^- \to \Sigma^+$. Recall that $s = -L^-$ on $\Sigma^-$. Let $q_0 = (-L^-, Z_0, \theta_0) \in \Sigma^-$ and let $(s(t), Z(t), \theta(t))$ be the solution of equation (2.22) initiated at $(-L^-, Z_0, \theta_0)$. Let $\hat{t}$ be the time such that $s(\hat{t}) = L^+$. By definition $\mathcal{M}(q_0) = (L^+, Z(\hat{t}), \theta(\hat{t}))$. In what follows, we write
\begin{align*}
A_L &= \int_{-L^-}^{L^+} v(s)h(a(s), b(s))e^{-\int_0^s E(\tau)d\tau}ds \\
C_L &= \int_{-L^-}^{L^+} v(s)h(a(s), b(s))\cos(\omega s)e^{-\int_0^s E(\tau)d\tau}ds \\
S_L &= \int_{-L^-}^{L^+} v(s)h(a(s), b(s))\sin(\omega s)e^{-\int_0^s E(\tau)d\tau}ds.
\end{align*}
\(A, C, S\) in (H2) are from \(A_L, C_L\) and \(S_L\) by letting \(L^\pm \to \infty\) respectively. We also write
\[
P_L = e^{\int_{-L^+}^{L^-} E(s) ds}, \quad P_L^+ = e^{\int_0^{L^+} E(s) ds}.
\]
Note that for \(P_L\) we integrate from \(s = -L^-\) to \(s = L^+\), while for \(P_L^+\) the integration starts from \(s = 0\). We have

**Lemma 3.1.**
\[
P_L \sim \varepsilon^{\frac{\alpha}{\beta} - \frac{\beta}{\alpha}} << 1, \quad P_L^+ \sim \varepsilon^{-\frac{\alpha}{\beta}} > 1.
\]

**Proof:** Both estimates follows directly from Lemma 2.2 and the fact that, by definition,
\[
\varepsilon \sim e^{-\alpha L^+} \sim e^{-\beta L^-}.
\]

**Proposition 3.1.** Let \((X_0, \theta_0) \in \Sigma^-\) and denote \((\hat{Z}, \hat{\theta}) = \mathcal{M}(X_0, \theta_0)\). We have
\[
\hat{\theta} = \theta_0 + \omega(L^+ + L^-) + O_{X_0, \theta_0, p}(\mu)
\]
(3.1)
\[
\hat{Z} = P_L^+[p A_L + C_L \cos(\theta_0 + \omega L^-) + S_L \sin(\theta_0 + \omega L^-)]
\]
\[
+ P_L[(1 + O_{\theta_0, p}(\varepsilon))X_0 + O_{\theta_0, p}(1) + O_{X_0, \theta_0, p}(\mu)].
\]

**Proof:** Using the third item of (2.22), we have
\[
\theta(t) = \theta_0 + \omega t.
\]
Integrating the first items of (2.22), for \(t \in [-2L^-, 2L^+]\) we have
\[
s(t) = -L^- + t + O_{t, Z_0, \theta_0, p}(\mu).
\]
Inverting the last equality we obtain
\[
t(s) = s + L^- + O_{s, Z_0, \theta_0, p}(\mu).
\]
Substituting \(\theta(t)\) and \(t(s)\) into the second item of (2.22), we obtain
\[
\frac{dZ}{ds} = E(s)Z - (u(s) + v(s))(\rho \mathbb{H}(s, 0) + \sin(\theta_0 + \omega L^- + \omega s))
\]
\[
+ O_{s, Z_0, \theta_0, p}(\mu).
\]
(3.2)

We note that in the above, \((s, Z_0, \theta_0, p)\) is such that \(s \in [-2L^-, 2L^+]\), \((Z_0, \theta_0) \in \Sigma^-\), and \(p = \ln \mu \in (-\infty, \ln \mu_0)\). From (3.2) we obtain
\[
Z(s) = P_s \cdot (Z_0 - \Phi_s(\theta_0) + O_{s, Z_0, \theta_0, p}(\mu))
\]
(3.3)

where
\[
P_s = e^{\int_{-L^-}^s E(\tau) d\tau};
\]
(3.4)
\[
\Phi_s(\theta) = \int_{-L^-}^s (u(\tau) + v(\tau))(\rho \mathbb{H}(\tau, 0) + \sin(\theta + \omega L^- + \omega \tau))
\]
\[
e^{-\int_{-L^-}^s E(\tau) d\tau} d\tau.
\]
From (3.3), it follows that
\[
\begin{align*}
\dot{\theta} &= \theta_0 + \omega(L^- + L^+) + \mathcal{O}_{Z_0, \theta_0, p}(\mu) \\
\dot{Z} &= P_L(Z_0 - \Phi_L(\theta_0) + \mathcal{O}_{Z_0, \theta_0, p}(\mu)).
\end{align*}
\] (3.5)

We want to write the right-hand side of (3.5) in \((X_0, \theta_0)\). Using Corollary 2.1, we have
\[
\begin{align*}
\dot{\theta} &= \theta_0 + \omega(L^- + L^+) + \mathcal{O}_{X_0, \theta_0, p}(\mu) \\
\dot{Z} &= P_L ((1 + \mathcal{O}_{\theta_0, p}(\varepsilon))X_0 - \Phi_L(\theta_0) + \mathcal{O}_{\theta_0, p}(1) + \mathcal{O}_{X_0, \theta_0, p}(\mu)) .
\end{align*}
\] (3.6)

Let \(K_2\) be such that
\[
|(1 + \mathcal{O}_{\theta_0, p}(\varepsilon))X_0 + \mathcal{O}_{\theta_0, p}(1) + \mathcal{O}_{X_0, \theta_0, p}(\mu)| < K_2
\]
where the functions on the left hand are from the second line of (3.6). By letting
\[
K_0(\varepsilon) = \max_{\theta \in S^1, \, s \in [-2L^-, 2L^+]} 2|P_s(K_2 - \Phi_s(\theta))|,
\] (3.7)
we conclude from (3.3) that all solutions of equation (2.22) initiated inside of \(\Sigma^-\) will stay inside of
\[
\{(s, Z, \theta) : s \in [-2L^-, 2L^+], |Z| < K_0(\varepsilon)\}
\]
before reaching \(s = L^+\). To finish the proof of this proposition it now suffices for us to observe that
\[
P_L \Phi_L(\theta) = e^{\int_{0}^{L^+} E(\tau) d\tau} \int_{-L^-}^{L^+} (u(s) + v(s))(\rho h(a(s), b(s)) \\
+ \sin(\theta + \omega L^- + \omega s))e^{-\int_{0}^{s} E(\tau) d\tau} ds
\]
\[
= P_L^+ \cdot (\rho A_L + C_L \cos(\theta + \omega L^-) + S_L \sin(\theta + \omega L^-)).
\]
This proves proposition 3.1. \qed

Remark: In (3.1) for \(\dot{Z}\), the first term with \(P_L^+\) in front dominates the second term because \(P_L^+ >> 1 >> P_L\). The inclusion \(\mathcal{M}(\Sigma^-) \subset \Sigma^+\) follows directly from (3.1).

3.2. On the time-t map of equation (2.15). We compute \(N : \Sigma^+ \to \Sigma^-\) in two steps. The first step is to compute the time-t map of equation (2.15) inside \(\mathcal{U}_\varepsilon\). This is contained in Sect. 3.2. The second step is to compute the time it takes for a solution of equation (2.15) initiated in \(\Sigma^+\) to reach \(\Sigma^-\). This is contained in Sect. 3.3. These computations are technically involved because, in order to apply the theory of rank one maps to the return map \(N \circ \mathcal{M}\), we need to control the \(C^2\)-norms of the error terms contributed by the functions \(\mu \mathcal{F}\) and \(\mu \mathcal{G}\) in equation (2.15).

We start with the first step. Let \(W(\Sigma^+)\) be a small open neighborhood surrounding \(\Sigma^+\) in the space \((X, Y, \theta)\). In this subsection we let \((X_0, Y_0, \theta_0) \in W(\Sigma^+)\) and regard \(p = \ln \mu \in (-\infty, \ln \mu_0)\) as the parameter of equation (2.15). We study the time-t map of equation (2.15) assuming that up to time \(t\), all solutions initiated from \(W(\Sigma^+)\) are completely contained inside of \(\mathcal{U}_\varepsilon\). Recall that in equation (2.15),
\[
|F(X, Y, \theta; \mu)| < K, \quad |G(X, Y, \theta; \mu)| < K
\]
are analytic on
\[ D = \{ (X, Y, \theta, \mu) : \mu \in (0, \mu_0), (X, Y, \theta) \in U \} \]
where
\[ U = \{ (X, Y, \theta) : |(X, Y)| < 2\varepsilon \mu^{-1}, \theta \in S^1 \}. \]

For \( q_0 = (X_0, Y_0, \theta_0) \in W(\Sigma^+), \) let
\[ q(t, q_0; \mu) = (X(t, q_0; \mu), Y(t, q_0; \mu), \theta(t, q_0; \mu)) \]
be the solution of equation (2.15) initiated from \( q_0 \) at \( t = 0. \) Using (2.15), we have
\[
\begin{align*}
X(t, q_0; \mu) &= X_0 e^{\int_0^t (-\alpha + \mu \mathbb{E}(q(s, q_0; \mu); \mu)) ds} \\
Y(t, q_0; \mu) &= Y_0 e^{\int_0^t (\beta + \mu \mathbb{G}(q(s, q_0; \mu); \mu)) ds} \\
\theta(t, q_0; \mu) &= \theta_0 + \omega t.
\end{align*}
\]

We now introduce two functions \( U(t, q_0; \mu), V(t, q_0; \mu) \) and rewrite (3.8) as
\[
\begin{align*}
X(t, q_0; \mu) &= X_0 e^{(-\alpha + U(t, q_0; \mu)) t} \\
Y(t, q_0; \mu) &= Y_0 e^{(\beta + V(t, q_0; \mu)) t} \\
\theta(t, q_0; \mu) &= \theta_0 + \omega t.
\end{align*}
\]

We have from (3.9) that
\[
\begin{align*}
U(t, q_0; \mu) &= t^{-1} \ln \frac{X(t, q_0; \mu)}{X_0} + \alpha \\
V(t, q_0; \mu) &= t^{-1} \ln \frac{Y(t, q_0; \mu)}{Y_0} - \beta.
\end{align*}
\]

We also have
\[
\begin{align*}
U(t, q_0; \mu) &= t^{-1} \int_0^t \mu \mathbb{E}(q(s, q_0; \mu); \mu) ds \\
V(t, q_0; \mu) &= t^{-1} \int_0^t \mu \mathbb{G}(q(s, q_0; \mu); \mu) ds.
\end{align*}
\]

In the next proposition we regard \( U = U(t, q_0; \mu), V = V(t, q_0; \mu) \) as functions of \( t, q_0 \) and \( p, \) which we rewrite as \( U_{t, q_0, p}, V_{t, q_0, p} \) respectively. We define the domain of these two functions as follows: we let
\[ D_{t, q_0, p} = \{ q_0 \in W(\Sigma^+), p \in (-\infty, \ln \mu_0), t \in (1, T(q_0, p)) \} \]
where the upper bound \( T(q_0, p) \) for \( t \) is designed to keep the solution inside of \( U. \) We have

**Proposition 3.2.** Regard \( U_{t, q_0, p}, V_{t, q_0, p} \) as functions of \( t, q_0 \) and \( p. \) (Recall that \( p = \ln \mu \). Then there exists \( K > 0 \) so that
\[ \| U_{t, q_0, p} \|_{C^3(D_{t, q_0, p})} < K\mu, \quad \| V_{t, q_0, p} \|_{C^3(D_{t, q_0, p})} < K\mu. \]
Proof: Let \( \mathbb{F} = \mathbb{F}(X, Y, \theta; \mu) \), \( \mathbb{G} = \mathbb{G}(X, Y, \theta; \mu) \) be as in equation (2.15). For a combination \( Z = X^{d_1} Y^{d_2} \mu^{d_3} \) of powers of the variables \( X, Y \) and \( \mu \), let \( \partial^k_Z \) denote the corresponding partial derivative operator, where \( k = d_1 + d_2 + d_3 \) is the order. There exists \( K_3 > 0 \) such that for every \( Z \) of order \( k \) and \( i \geq 0 \) satisfying \( 0 \leq i + k \leq 3 \), we have

\[
|\partial^k_Z (\partial_Y \mathbb{F}) \cdot Z| < K_3, \quad |\partial^k_Z (\partial_Y \mathbb{G}) \cdot Z| < K_3
\]

on \( \mathcal{D}_{t,0,p} \). This is because the \( C^3 \)-norms of \( \mathbb{F}(X, Y, \theta; \mu) \) and \( \mathbb{G}(X, Y, \theta; \mu) \) are bounded on \( U \times [0, \mu_0] \) and because \( \mathbb{F}(X, Y, \theta; \mu) = \mathbb{F}(\mu X, \mu Y, \theta; \mu) \) and \( \mathbb{G}(X, Y, \theta; \mu) = \mathbb{G}(\mu X, \mu Y, \theta; \mu) \).

\( C^0 \)-estimates: Using (3.12) with \( i = k = 0 \) and (3.11), we have

\[
\|U\|_{C^0(\mathcal{D}_{t,0,p})}, \quad \|V\|_{C^0(\mathcal{D}_{t,0,p})} < K_3 \mu.
\]

\( C^1 \)-estimates: We now estimate the first derivatives.

On \( \partial_{\varphi_0} U \) and \( \partial_{\varphi_0} V \). Using \( \theta(t) = \theta_0 + \omega t \), we have \( \partial_{\varphi_0} \theta = 0 \). Using (3.11), we have

\[
\partial_{\varphi_0} U = \mu t^{-1} \int_0^t (\partial_X \mathbb{F} \cdot \partial_{\varphi_0} X + \partial_Y \mathbb{F} \cdot \partial_{\varphi_0} Y) \, ds;
\]

(3.14)

\[
\partial_{\varphi_0} V = \mu t^{-1} \int_0^t (\partial_X \mathbb{G} \cdot \partial_{\varphi_0} X + \partial_Y \mathbb{G} \cdot \partial_{\varphi_0} Y) \, ds.
\]

To make these formulas useful, we need to write \( \partial_{\varphi_0} X \) and \( \partial_{\varphi_0} Y \) in terms of \( \partial_{\varphi_0} U \) and \( \partial_{\varphi_0} V \). For this purpose we use (3.10). We have

\[
\partial_{\varphi_0} X = t X \partial_{\varphi_0} U
\]

(3.15)

\[
\partial_{\varphi_0} Y = t Y \partial_{\varphi_0} V + \frac{Y}{Y_0}.
\]

Combining (3.14) and (3.15) we obtain

\[
\partial_{\varphi_0} U = \mu t^{-1} \int_0^t (\partial_X \mathbb{F} \cdot X \cdot s \partial_{\varphi_0} U + \partial_Y \mathbb{F} \cdot Y \cdot s \partial_{\varphi_0} V) \, ds
\]

\[
+ \mu t^{-1} \int_0^t \partial_Y \mathbb{F} \cdot \frac{Y}{Y_0} \, ds;
\]

(3.16)

\[
\partial_{\varphi_0} V = \mu t^{-1} \int_0^t (\partial_X \mathbb{G} \cdot X \cdot s \partial_{\varphi_0} U + \partial_Y \mathbb{G} \cdot Y \cdot s \partial_{\varphi_0} V) \, ds
\]

\[
+ \mu t^{-1} \int_0^t \partial_Y \mathbb{G} \cdot \frac{Y}{Y_0} \, ds.
\]

Using (3.12),

\[
|\partial_X \mathbb{F} \cdot X|, \quad |\partial_X \mathbb{G} \cdot X|, \quad |\partial_Y \mathbb{F} \cdot Y|, \quad |\partial_Y \mathbb{G} \cdot Y| < K_3.
\]
Using (3.16), we have
\[ |\partial_Y U| \leq K\mu t^{-1} \int_0^t (|s\partial_Y U| + |s\partial_Y V|) \, ds + K\mu; \]
\[ |\partial_Y V| \leq K\mu t^{-1} \int_0^t (|s\partial_Y U| + |s\partial_Y V|) \, ds + K\mu, \] (3.17)
from which it follows that
\[ |\partial_Y U|, \ |\partial_Y V| < K\mu. \]

On \( \partial_X U \) and \( \partial_X V \). Mimic the proof above.

On \( \partial_\theta U \) and \( \partial_\theta V \). We follow the same lines of computation. Since \( \partial_\theta \theta = 1 \), we have
\[ \partial_\theta U = \mu t^{-1} \int_0^t (\partial_X F \cdot \partial_\theta X + \partial_Y F \cdot \partial_\theta Y + \partial_\theta F) \, ds; \]
\[ \partial_\theta V = \mu t^{-1} \int_0^t (\partial_X G \cdot \partial_\theta X + \partial_Y G \cdot \partial_\theta Y + \partial_\theta G) \, ds. \]

Corresponding to (3.15), we have
\[ \partial_\theta X = tX \partial_\theta U, \ \partial_\theta Y = tY \partial_\theta V. \]

Argue as above, we conclude that
\[ |\partial_\theta U|, \ |\partial_\theta V| < K\mu. \]

On \( \partial_p U \) and \( \partial_p V \). We follow similar lines of computation. Note that we have
\[ \partial_p \mu = \mu, \ \partial_p F = \mu \partial_\mu F, \]
and so on. Starting with (3.11), we have
\[ \partial_p U = \mu t^{-1} \int_0^t F ds + \mu t^{-1} \int_0^t (\partial_X F \cdot \partial_p X + \partial_Y F \cdot \partial_p Y + \mu \partial_\mu F) \, ds; \]
(3.18)
\[ \partial_p V = \mu t^{-1} \int_0^t G ds + \mu t^{-1} \int_0^t (\partial_X G \cdot \partial_p X + \partial_Y G \cdot \partial_p Y + \mu \partial_\mu G) \, ds \]
and from (3.10) we have
\[ \partial_p X = tX \partial_p U \]
\[ \partial_p Y = tY \partial_p V. \]

Now argue as above.

On \( \partial_t U \) and \( \partial_t V \). The partial derivatives of \( U \) and \( V \) with respect to \( t \) are easier to estimate because when differentiating with respect to \( t \) using (3.11), no derivatives are involved on the right-hand side so the corresponding estimates are obtained directly from \( C^0 \)-estimates. We have
\[ |\partial_t U|, \ |\partial_t V| < K\mu. \]

This finishes the desired estimates for the first derivatives.
Therefore, \( \partial_{Y_0 Y_0} U \) first. From the first line of (3.14), we have
\[
\partial_{Y_0 Y_0} U = \mu^{-1} \int_0^t \left( \partial^2_{XX} \mathcal{F} \cdot (\partial_{Y_0} \mathcal{X})^2 + 2 \partial^2_{XY} \mathcal{F} \cdot (\partial_{Y_0} \mathcal{X})(\partial_{Y_0} \mathcal{Y}) + \partial_{YY}(\mathcal{F} \cdot \partial_{Y_0} \mathcal{Y})^2 \right) ds \\
+ \mu^{-1} \int_0^t \left( \partial_{XX} \mathcal{F} \cdot \partial^2_{Y_0 Y_0} \mathcal{X} + \partial_{YY} \mathcal{F} \cdot \partial^2_{Y_0 Y_0} \mathcal{Y} \right) ds.
\]
Using (3.15), we have
\[
\partial^2_{Y_0 Y_0} \mathcal{X} = t \partial_{Y_0} \mathcal{X} \cdot \partial_{Y_0} U + t \mathcal{X} \partial^2_{Y_0 Y_0} U \\
\partial^2_{Y_0 Y_0} \mathcal{Y} = t \partial_{Y_0} \mathcal{Y} \cdot \partial_{Y_0} V + t \mathcal{Y} \cdot \partial_{Y_0 Y_0} V + \partial_{Y_0} \mathcal{Y} - \frac{\mathcal{Y}}{\mathcal{Y}_0}.
\]
Therefore, \( \partial_{Y_0 Y_0} U \) is given by
\[
(3.21) \\
\partial^2_{Y_0 Y_0} U = \mu^{-1} \int_0^t \left( \partial^2_{XX} \mathcal{F} \cdot (\partial_{Y_0} \mathcal{X})^2 + 2 \partial^2_{XY} \mathcal{F} \cdot (\partial_{Y_0} \mathcal{X})(\partial_{Y_0} \mathcal{Y}) + \partial_{YY}(\mathcal{F} \cdot \partial_{Y_0} \mathcal{Y})^2 \right) ds \\
+ \mu^{-1} \int_0^t \left( \partial_{XX} \mathcal{F} \cdot \partial_{Y_0} \mathcal{X} \cdot s \partial_{Y_0} U + \partial_{YY} \mathcal{F} \cdot \partial_{Y_0} \mathcal{Y} \cdot s \partial_{Y_0} V \right) ds \\
+ \mu^{-1} \int_0^t \partial_{YY} \mathcal{F} \cdot \left( \partial_{Y_0} \mathcal{Y} - \frac{\mathcal{Y}}{\mathcal{Y}_0} \right) ds \\
+ \mu^{-1} \int_0^t \left( \partial_{XX} \mathcal{F} \cdot \partial^2_{Y_0 Y_0} U + \partial_{YY} \mathcal{F} \cdot \mathcal{Y} \cdot s \partial^2_{Y_0 Y_0} V \right) ds.
\]
To estimate the first three lines of this formula for \( \partial_{Y_0 Y_0} U \), we use (3.15) for \( \partial_{Y_0} \mathcal{X} \)
and \( \partial_{Y_0} \mathcal{Y} \). Using the first derivative estimates and using (3.12) repeatedly, we easily
bound these lines by \( K \mu \). Note that we also need \( \mathcal{Y}_0 > 1 \) (See Corollary 2.3) for the
third line. Together with an analogous formula for \( \partial^2_{Y_0 Y_0} V \), in which we replace \( \mathcal{F} \) by
\( \mathcal{G} \), we conclude that
\[ |\partial^2_{Y_0 Y_0} U|, \ |\partial^2_{Y_0 Y_0} V| < K \mu. \]

All other second derivatives are estimated similarly. Here we skip the details to
avoid repetitive computations.

**C^3-estimates:** Third derivatives are estimated in the same spirit, only the formulas
get longer. Since the formulas for a given third derivative depend on previous
computations of relevant second derivatives, here we estimate \( \partial^3_{Y_0 Y_0 Y_0} U \) and \( \partial^3_{Y_0 Y_0 Y_0} V \) as a
representative example. Of all the third derivatives, these are the most tedious to
compute.

To compute \( \partial^3_{Y_0 Y_0 Y_0} U \) we apply \( \partial_p \) to (3.21). The explicit factor \( \mu \) written in front of
all integrals generates a collection of terms that is identical to the right-hand side of
(3.21). The size of each of these terms is obviously controlled by \( K \mu \) by our previous
estimates of second derivatives.

The remaining terms are produced by applying \( \partial_p \) to the functions inside of the
integrals in (3.21). The terms produced from the first three integrals are estimated
using the $C^2$-estimates. Estimate (3.12) is used repeatedly. It is critically important that potentially problematic terms in the form of powers of $Y$ and $X$, introduced by using the likes of (3.15), (3.19) and (3.20), are always matched perfectly with corresponding partial derivatives with respect to $F$ or $G$. Apply $\partial_p$ to the fourth integral we obtain an integral term of the form

$$ (I) = \mu t^{-1} \int_0^t \left( \partial_X F \cdot X + s \partial_X^3 Y_{\alpha p} U + \partial_Y F \cdot Y + s \partial_Y^3 Y_{\alpha p} V \right) ds, $$

and a collection of other terms that can be treated the same way as the terms produced by differentiating the first three integrals. We again have

$$ |(I)| \leq K \mu t^{-1} \int_0^t \left( |s \partial_X^3 Y_{\alpha p} U| + |s \partial_Y^3 Y_{\alpha p} V| \right) ds. $$

Combined with an analogous estimates for $|\partial_X^3 Y_{\alpha p} U|$, $|\partial_Y^3 Y_{\alpha p} V|$, we obtain

$$ |\partial_X^3 Y_{\alpha p} U|, |\partial_Y^3 Y_{\alpha p} V| < K \mu. $$

This completes our proof of Proposition 3.2. \hfill \Box

**Remark:** By combining Proposition 3.2 and (3.9), we can now write the time-$t$ map from $W(\Sigma^\pm)$ to $\mathcal{U}_e$ as

$$ X(t, X_0, Y_0, \theta_0; \mu) = X_0 e^{-(\alpha + \Omega_t, X_0, Y_0, \theta_0, \rho(\mu)) t}, $$

$$ Y(t, X_0, Y_0, \theta_0; \mu) = Y_0 e^{(\beta + \Omega_t, X_0, Y_0, \theta_0, \rho(\mu)) t}, $$

$$ \theta(t, X_0, Y_0, \theta_0; \mu) = \theta_0 + \omega t. $$

### 3.3. Estimates on $T(Z_0, \theta_0, p)$

For $q_0 = (Z_0, \theta_0) \in \Sigma^+$, let $q(t, q_0; \mu)$ be the solution of equation (2.15) initiated at $q_0$ and let $T$ be the time this solution reaches $\Sigma^-$. In this subsection we regard $T$ as a function of $Z_0, \theta_0, p$ and we obtain a well-controlled formula for $T$ that is explicit in the variables $Z_0, \theta_0$ and $p$. Since the images of $\mathcal{M}$ are given in coordinates $(Z, \theta)$ through (3.1), we must write the initial conditions for $\mathcal{N}$ in coordinates $(Z, \theta)$ on $\Sigma^+$ to facilitate the intended composition of $\mathcal{N}$ and $\mathcal{M}$.

Estimates for $T(Z_0, \theta_0, p)$ are complicated partly because, as a function of $Z_0, \theta_0$, it is implicitly defined through equations written in coordinates $X, Y, \theta$ on $\Sigma^\pm$. The computational process would therefore have to involve (3.22) and the coordinate transformations on $\Sigma^\pm$ presented in Sect. 2.3B and 2.3C. Before getting into the desired quantitative estimates, we explain how to obtain $T(Z_0, \theta_0, p)$ as a function of $Z_0, \theta_0$ and $p$ in a conceptual way. Using (3.9), we obtain

$$ X(T, X_0, Y_0, \theta_0; \mu) = X_0 e^{-(\alpha + U(T, X_0, Y_0, \theta_0, p)) T}, $$

$$ Y(T, X_0, Y_0, \theta_0; \mu) = Y_0 e^{(\beta + V(T, X_0, Y_0, \theta_0, p)) T}, $$

$$ \theta(T, X_0, Y_0, \theta_0; \mu) = \theta_0 + \omega T. $$
We know that in (3.23), $X_0, Y_0, \theta_0$ are not independent variables. They are related through

\begin{align*}
X_0 &= \mu^{-1} \varepsilon + O_{Z_0, \theta_0, p}(1) \\
Y_0 &= (1 + O_{\theta_0, p}(\varepsilon) + \mu O_{Z_0, \theta_0, p}(1)) Z_0 - O_{\theta_0, p}(1)
\end{align*}

by Proposition 2.5. Let us denote

\[ X(T) = X(T, X_0, Y_0, \theta_0; \mu), \quad Y(T) = Y(T, X_0, Y_0, \theta_0; \mu), \quad \theta(T) = \theta_0 + \omega T. \]

By definition, $X(T), Y(T), \theta(T)$ are also related through Corollary 2.2. For the benefit of a clear exposition, let us write the conclusion of Corollary 2.2 as

\[ Y = \varepsilon \mu^{-1} + f(X, \theta, p) \]

where

\[ f(X, \theta, p) = O_{X, \theta, p}(1). \]

We have

\begin{equation}
Y(T) = \varepsilon \mu^{-1} + f(X(T), \theta(T), p).
\end{equation}

We use the second item of (3.23) to implicitly define $T(Z_0, \theta_0, p)$. We write this equation as

\begin{equation}
Y(T) = Y_0 e^{(\beta + V(T, X_0, Y_0, \theta_0; p)) T}.
\end{equation}

The right-hand side of (3.26) is relatively simple: we only need to substitute for $X_0, Y_0$ using (3.24). The left-hand side of (3.26) is conceptually more complicated. We need to

(i) first write $Y(T)$ as a function of $X(T), \theta(T)$ and $p$ using (3.25);
(ii) then substitute $X(T), \theta(T)$ by using the the first and the third items of (3.23), writing now $Y(T)$ in terms of $T, X_0, Y_0, \theta_0$ and $p$;
(iii) finally, use (3.23) to again write $X_0, Y_0$ in terms of $Z_0, \theta_0$.

After all these substitutions, we can regard (3.26) as the equation that implicitly defines $T(Z_0, \theta_0, p)$. We use this equation as the basis for the computation of $T(Z_0, \theta_0, p)$.

**Proposition 3.3.** As a function of $Z_0, \theta_0, p$,

\[ \|T - \frac{1}{\beta} \ln \mu^{-1}\|_{C^3} < K. \]

**Proof:** The proof of this proposition is a little long because of the complicated composition process explained above.

**C^0-estimates:** First we have

**Lemma 3.2.** There exist constants $K_4 < K_5$ independent of $\varepsilon$ so that for all $q_0 \in \Sigma^+$, $K_4 \ln \mu^{-1} < T(q_0; \mu) < K_5 \ln \mu^{-1}$.  

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Proof of Lemma 3.2: From

\[ Y(T) = Y_0 e^{(\beta + V(T))T} \]

we obtain

\[ T = \frac{1}{\beta + V(T)} \ln \frac{Y(T)}{Y_0}. \]

Since \((X(T), Y(T), \theta(T))\) is on \(\Sigma^\circ\), Proposition 2.4 implies that,

\[ Y(T) \approx \mu^{-1} \varepsilon \]

and the desired estimates follows from \(|V(T)| < K\mu\) and \(1 < Y_0 < K(\varepsilon)\).

We also have

Lemma 3.3. \(\mu^{-1} e^{-\alpha T} < 1\).

Proof of Lemma 3.3: We substitute

\[ T = \frac{1}{\beta + V(T)} \ln \frac{Y(T)}{Y_0} \]

into the first item in (3.23) to obtain

\[ X(T) = \left( \frac{Y_0}{Y(T)} \right)^{\frac{\alpha - U(T)}{\beta + V(T)}} X_0. \]

We then use \(Y(T) \approx \varepsilon \mu^{-1}, X_0 \approx \varepsilon \mu^{-1}, |U(T)|, |V(T)| < K\mu\) and \(\alpha > \beta\) to conclude that \(X(T) \ll \varepsilon\). We have

\[ \frac{1}{10} \varepsilon \mu^{-1} e^{-\alpha T} < X_0 e^{(\alpha + U(T))T} = X(T) \ll \varepsilon. \]

For the first inequality we use \(X_0 \approx \varepsilon \mu^{-1}\) and \(|U(T)T| < K\mu \ln \mu^{-1} \ll 1\). This proves the lemma.

\(C^1\)-estimates: In this paragraph, the variables associated with the stated \(C^1\)-norms are \((Z_0, \theta_0, p)\), where \((Z_0, \theta_0) \in \Sigma^\circ\) and \(p \in (-\infty, \ln \mu_0)\).

Lemma 3.4. There exists constants \(K_7, K_8\) independent of \(\varepsilon\) so that

\[ \|X(T)\|_{C^1}, \|\theta(T)\|_{C^1} < K_7 + K_8 \|T\|_{C^1}. \]

Proof of Lemma 3.4: The bound on \(\theta(T)\) is trivial because \(\theta(T) = \theta_0 + \omega T\). For \(X(T)\) we have

\[ X(T) = X_0 e^{(-\alpha + U(T))T} \]

\[ = \varepsilon \mu^{-1} e^{(-\alpha + U(T, X_0, Y_0, \theta_0, p))T} + O_{Z_0, \theta_0, p}(1) e^{(-\alpha + U(T, X_0, Y_0, \theta_0, p))T} \]

Note that for the second equality, (3.24) is used for \(X_0\). We regard \(X_0, Y_0\) inside of \(U(T, X_0, Y_0, \theta_0, p)\) as functions of \(Z_0, \theta_0, p\) defined by (3.24). The desired estimate follows from using Proposition 3.2 for \(U\) and (3.24) for \(X_0, Y_0\). We also use Lemma 3.3.

We are now ready to prove

Lemma 3.5. \(\|T - \frac{1}{\beta} \ln \mu^{-1}\|_{C^1} < K\).
Proposition 3.4. The flow-induce map □ using Proposition 3.2, Proposition 3.3, and (3.24).

Proof: To regard \( \partial \) estimate

Proof of Lemma 3.5: Using (3.25), we write (3.26) as

\[
\mu^{-1}(\varepsilon + \mu f(\mathbf{X}(T), \theta(T), p)) = \mathbb{Y}_0 e^{(\beta + V(T))T}.
\]

Solving for \( T \), we obtain

\[
T - \frac{1}{\beta} \ln \mu^{-1} = -\frac{V(T)}{\beta(\beta + V(T))} \ln \mu^{-1} - \frac{1}{\beta + V(T)} \ln \mathbb{Y}_0
\]

(3.27)

+ \frac{1}{\beta + V(T)} \ln(\varepsilon + \mu f(\mathbf{X}(T), \theta(T), p)).

Again \( V(T) = V(T, \mathbf{X}_0, \mathbb{Y}_0, \theta_0, p) \), and \( \mathbf{X}_0, \mathbb{Y}_0 \) are written in terms of \( Z_0, \theta_0, p \) in (3.27) using (3.24). Using Proposition 3.2, we have

\[
\|T - \frac{1}{\beta} \ln \mu^{-1}\|_{C^0} < K.
\]

First derivative estimate of \( T \) are estimated by directly differentiating (3.27). We estimate \( \partial_{Z_0} T \) as a representative example. Differentiating (3.27), we have

\[
\partial_{Z_0} T = (I) + (II) \partial_{Z_0} T,
\]

where (I) is a collection of terms that do not depend on \( \partial_{Z_0} T \) and (II) is a function of \( Z_0, \theta_0 \) and \( p \). Using Proposition 3.2 for \( V(T) \), (3.24) for \( \mathbf{X}_0 \) and \( \mathbb{Y}_0 \) and Lemma 3.4 for \( \partial_{Z_0} \mathbf{X}_0(T) \) and \( \partial_{Z_0} \theta(T) \), we have (i) \( |(I)| < K \), and (ii) \( |(II)| << 1 \). □

Higher derivative estimates: With the first derivatives controlled by Lemmas 3.4 and 3.5, we estimate the second derivatives by first proving a version of Lemma 3.4 and then proving a version of Lemma 3.5 for the \( C^2 \)-norms. We then do the same for the \( C^3 \)-norms. This finishes our proof of Proposition 3.3. □

Computing \( \mathcal{N} : \Sigma^+ \to \Sigma^- \). We are ready to derive a formula for the induced map \( \mathcal{N}_p : \Sigma^+ \to \Sigma^- \). For \( (Z_0, \theta_0) \in \Sigma^+ \), we write \((\mathbf{X}_1, \theta_1) = \mathcal{N}_p(Z_0, \theta_0)\). We start with \( U, V \) in (3.23).

Lemma 3.6. On \( \Sigma^+ \times (-\infty, \ln \mu_0) \), we have

\[
U(T, \mathbf{X}_0, \mathbb{Y}_0, \theta_0, p), \quad V(T, \mathbf{X}_0, \mathbb{Y}_0, \theta_0, p) = \mu \mathcal{O}_{Z_0, \theta_0, p}(1).
\]

Proof: To regard \( U \) and \( V \) as functions of \((Z_0, \theta_0, p)\), we use Proposition 3.3 for \( T(Z_0, \theta_0, p) \) and (3.24) for \( \mathbf{X}_0, \mathbb{Y}_0 \). This lemma follows by applying the chain rule and using Proposition 3.2, Proposition 3.3, and (3.24). □

Proposition 3.4. The flow-induce map \( \mathcal{N}_p : \Sigma^+ \to \Sigma^- \) is given by

\[
\mathbf{X}_1 = \left( \frac{1}{\varepsilon + \mu \mathcal{O}_{Z_0, \theta_0, p}(1)} \right)^{\frac{1}{\alpha}} \left( \frac{1}{1 + \mathcal{O}_{Z_0, \theta_0, p}(\varepsilon) + \mu \mathcal{O}_{Z_0, \theta_0, p}(1)]Z_0 - \mathcal{O}_{\theta_0, p}(1)} \right)^{\frac{1}{\beta}}
\]

(3.28)

\[
\theta_1 = \theta_0 + \frac{\omega}{\beta} \ln \frac{(1 + \mathcal{O}_{Z_0, \theta_0, p}(\varepsilon) + \mu \mathcal{O}_{Z_0, \theta_0, p}(1)]Z_0 - \mathcal{O}_{\theta_0, p}(1)}{1 + \mathcal{O}_{\theta_0, p}(\varepsilon) + \mu \mathcal{O}_{Z_0, \theta_0, p}(1)Z_0 - \mathcal{O}_{\theta_0, p}(1)}
\]

where

\[
\bar{\alpha} = \alpha + \mu \mathcal{O}_{Z_0, \theta_0, p}(1), \quad \bar{\beta} = \beta + \mu \mathcal{O}_{Z_0, \theta_0, p}(1).
\]
Proof: Using (3.25), (3.26) and Lemma 3.6, we have

\[
T = \frac{1}{\beta + \mu O_{Z_0,\theta_0,p}(1)} \ln \frac{Y(T)}{Y_0} = \frac{1}{\beta + \mu O_{Z_0,\theta_0,p}(1)} \ln \left(\frac{\varepsilon + \mu f(X(T),\theta(T),p)}{Y_0}\right)^{\mu^{-1}}.
\]

By combining the \(C^3\)-version of Lemma 3.4, Proposition 3.3, and using the fact that 
\(f(X,\theta,p) = O_{X,\theta,p}(1)\), we get

\[f(X(T),\theta(T),p) = O_{Z_0,\theta_0,p}(1).\]

Now (3.29) gives,

\[
\frac{1}{\beta + \mu O_{Z_0,\theta_0,p}(1)} \ln \frac{\mu^{-1}(\varepsilon + \mu O_{Z_0,\theta_0,p}(1))}{1 + O_{\theta_0,p}(\varepsilon) + \mu O_{Z_0,\theta_0,p}(1)} Z_0 - O_{\theta_0,p}(1).
\]

Here we use (3.24) for \(Y_0\).

The desired formula for \(\theta_1\) now follows from \(\theta_1 = \theta_0 + \omega T\). For \(X_1\) we use

\[X_1 = \mu^{-1}(\varepsilon + \mu O_{Z_0,\theta_0,p}(1)) e^{-(\alpha + \mu O_{Z_0,\theta_0,p}(1))T}\]

and substitute for \(T\) using (3.30).

4. Proof of Theorem 1

In Sect. 4.1 we compute \(F = \mathcal{N} \circ \mathcal{M}\) by using the conclusions of Propositions 3.4 and 3.1. In Sect. 4.2 we apply the theory of [WY1] to \(F\) to prove the existence of rank one chaos as claimed in Theorem 1.

4.1. The flow-induced map \(F = \mathcal{N} \circ \mathcal{M}\). Let \(q_0 \in \Sigma^-\) be represented by \((X_0,\theta_0)\) and let \((X_1,\theta_1) = \mathcal{N} \circ \mathcal{M}(X_0,\theta_0)\). We regard \(p = \ln \mu\) as a parameter and write \(F_p : (X_0,\theta_0) \to (X_1,\theta_1)\). We now combine (3.28) and (3.1) to compute \(F_p\).

Proposition 4.1. The map \(F_p : \Sigma^- \to \Sigma^-\) is given by

\[
X_1 = (\mu(\varepsilon + O_{X_0,\theta_0,p}(\mu))^{-1})^{\frac{\hat{\alpha}}{\hat{\beta}}} (\mathcal{Y}(X_0,\theta_0,p))^{\frac{\hat{\beta}}{\hat{\alpha}}}
\]

\[
\theta_1 = \theta_0 + \omega(L^- + L^+) + O_{X_0,\theta_0,p}(\mu) + \frac{\omega}{\beta} \ln \frac{(\varepsilon + O_{X_0,\theta_0,p}(\mu))^{\mu^{-1}}}{\mathcal{Y}(X_0,\theta_0,p)}
\]

where

\[
\mathcal{Y}(X_0,\theta_0,p) = (1 + O_{\theta_0,p}(\varepsilon) + O_{X_0,\theta_0,p}(\mu)) Z(X_0,\theta_0,p) - O_{\theta_0,p}(1) - O_{X_0,\theta_0,p}(\mu)
\]

\[
Z(X_0,\theta_0,p) = P_L^+ [\rho A_L + C_L \cos(\theta_0 + \omega L^-) + S_L \sin(\theta_0 + \omega L^-)]
\]

\[
+ P_L [(1 + O_{\theta_0,p}(\varepsilon)) X_0 + O_{\theta_0,p}(1) + O_{X_0,\theta_0,p}(\mu)]
\]

\[
\hat{\alpha} = \alpha + O_{X_0,\theta_0,p}(\mu);
\]

\[
\hat{\beta} = \beta + O_{X_0,\theta_0,p}(\mu).
\]
Proof: Let us first go through (3.28), replacing $Z_0, \theta_0$ by $\hat{Z}, \hat{\theta}$. From (3.28) we have

$$X_1 = \left(\frac{\mu}{\varepsilon + \mu O_{\hat{Z}, \hat{\theta}, p}(1)}\right)^{\frac{\hat{\theta}}{\hat{Z}}} \left(1 + O_{\hat{Z}, \hat{\theta}, p}(\varepsilon) + \mu O_{\hat{Z}, \hat{\theta}, p}(1)\right)^{\frac{\hat{\theta}}{\hat{Z}}}\left(1 - O_{\hat{Z}, \hat{\theta}, p}(1)\right)^{\frac{\hat{\theta}}{\hat{Z}}}$$

(4.3)

$$\theta_1 = \hat{\theta} + \frac{\omega}{\beta} \ln \left(\frac{\varepsilon + \mu O_{\hat{Z}, \hat{\theta}, p}(\varepsilon)}{1 + O_{\hat{Z}, \hat{\theta}, p}(\varepsilon) + \mu O_{\hat{Z}, \hat{\theta}, p}(1)\hat{Z} - O_{\hat{Z}, \hat{\theta}, p}(1)}\right).$$

We now substitute using (3.1). This is to say that all $\hat{Z}, \hat{\theta}$ in the above are written in terms of $X_0, \theta_0, p$ through (3.1). Using (3.1), we see that the $C^3$-norms of $\hat{Z}$ and $\hat{\theta}$ are $< K(\varepsilon)$. It then follows by the chain rule that

$$\mu O_{\hat{Z}, \hat{\theta}, p}(1) = O_{X_0, \theta_0, p}(\mu).$$

(4.4)

We also have

$$O_{\hat{Z}, \hat{\theta}, p}(\varepsilon) = O_{\theta_0, p}(1) + O_{X_0, \theta_0, p}(\mu)$$

(4.5)

and

$$O_{\theta, p}(\varepsilon) = O_{\theta_0, p}(\varepsilon) + O_{X_0, \theta_0, p}(\mu).$$

(4.6)

We obtain the result of Proposition (4.1) by re-writing (4.3) using (4.4)-(4.6).

4.2. Proof of Theorem 1. For the purpose of fitting the family $F_p$ obtained in Proposition 4.1 into the setting of [WY1] and [WY2], we first write $F_p$ as a two-parameter family $F_{a, b}$ of 2D maps. Both $a$ and $b$ are derived from $\mu = e^p$ as follows. Let $\mu_0 > 0$ be sufficiently small. Define $\gamma : (0, \mu_0) \to \mathbb{R}$ via $\gamma(\mu) = \frac{\omega}{\beta} \ln \mu^{-1}$. For $n \in \mathbb{Z}^+$ satisfying $n > (2\pi \beta)^{-1} \omega \ln \mu_0^{-1}$, let $\mu_n \in (0, \mu_0)$ be such that $\gamma(\mu_n) = n$. Notice that $\mu_n \to 0$ monotonically. Set $b_n = \mu_n$. For $\mu \in (\mu_n + 1, \mu_n]$ and $a \in [0, 2\pi) = S^1$, we define

$$\mu(n, a) = \gamma^{-1}(\gamma(\mu_n) + a) = \mu_n e^{-\frac{\beta}{\omega} a}$$

and

$$p(n, a) = \ln \mu(n, a) = \ln \mu_n - \frac{\beta}{\omega} a.$$

Define

$$F_{a, b_n} = F_{p(n, a)}.$$

Proposition 4.2. We have

$$\|F_{a, b_n}(X_0, \theta) - (0, F_a(X_0, \theta))\|_{C^3([\Sigma^{-} \times [0, 2\pi]])} \to 0$$

as $b_n \to 0$ where

$$F_a(X, \theta) = \theta + \omega(L^- + L^+) + a + \frac{\omega}{\beta} \ln \frac{\varepsilon P_a^+ \rho A_L}{(1 + O_{\theta, a}(\varepsilon))}$$

$$- \frac{\omega}{\beta} \ln(1 + \frac{S_L}{\rho A_L} \sin(\theta + \omega L^-) + \frac{C_L}{\rho A_L} \cos(\theta + \omega L^-) + \frac{P_L}{P_a^+ \rho A_L}(X + O_{\theta, a}(1)))$$

(4.8)
is obtained by dropping all $O_{X_0, \theta_0, p}(\mu)$ terms and replacing $\frac{\omega}{\beta} \ln \mu^{-1}$ by $a$ in the line for $\theta_1$ in Proposition 4.1. We also change all the old references to $p$ to new references to $a$.

**Proof:** The only term with the potential to cause trouble in the formula for $\theta_1$ has the form

$$\frac{\omega}{\beta + O_{X_0, \theta_0, p}(\mu)} \ln \mu^{-1},$$

which we rewrite as

$$\frac{\omega}{\beta} \ln \mu^{-1} + \frac{\omega \cdot O_{X_0, \theta_0, p}(\mu)}{\beta(\beta + O_{X_0, \theta_0, p}(\mu))} \ln \mu^{-1}.$$ 

Observe that the $C^3$-norm of the second term $\to 0$ as $b_n \to 0$ and the first term may be computed modulo $2\pi$ and is therefore equal to $a$. We may certainly change references to $p$ to new references to $a$. Viewing $\mu$ as a function of $a$, the $C^3$-norm of $X_1$ is bounded by

$$K(\varepsilon) \mu^{\frac{a}{\beta} - 1},$$

which decays to 0 as $b_n \to 0$ provided that (H1)(ii) holds.

**□**

The setting of [WY1] and [WY2]: We are now ready to apply the theory of rank one maps in [WY1]-[WY3]. The study of Wang and Young started with a rather long and technical definition of admissible family of 1D maps. In order to prove Theorem 1, we need to first prove that

$$(4.9)$$

$$F_a(0, \theta) = \theta + \omega(L^- + L^+) + a + \frac{\omega}{\beta} \ln \frac{\varepsilon P_L^+ \rho A_L}{(1 + O_{\theta, a}(\varepsilon))}$$

$$- \frac{\omega}{\beta} \ln (1 + \frac{S_L}{\rho A_L} \sin(\theta + \omega L^-) + \frac{C_L}{\rho A_L} \cos(\theta + \omega L^-) + \frac{P_L}{P_L^+ \rho A_L} O_{\theta, a}(1))$$

is an admissible family of 1D maps. For this we appeal to a proposition, previously proved in a slightly restricted context in [WY4].

**Proposition 4.3.** Let $\Psi(\theta) : S^1 \to \mathbb{R}$ be a $C^3$ function with non-degenerate critical points and $\phi(\theta, a) : S^1 \times [a_0, a_1] \to \mathbb{R}$ be such that

$$\|\phi(\theta, a)\|_{C^3} < \frac{1}{100}.$$ 

We define a one parameter family of circle maps $\{f_a : a \in [0, 2\pi]\}$ by

$$f_a(\theta) = \theta + \phi(\theta, a) + a + K\Psi(\theta).$$

Then there exists $K_8$, determined by $\Phi(\theta)$ alone, such that if $K > K_8$, then $\{f_a\}$ is an admissible family of 1D maps.

We refer the reader to [WY3] for a precise definition of admissible family of 1D maps. A special case of this proposition, in which we set $\phi(\theta, a) = 0$, was first proved in [WY4]. That proof can be easily extended to prove Proposition 4.3. See also Proposition 2.1 in [WY5] and Appendix C in [LWY].
To apply Proposition 4.3 to $F_a(0, \theta)$ above, we let
\[ \Psi(\theta) = -\ln(1 + \frac{S_L}{\rho A_L} \sin(\theta + \omega L^-) + \frac{C_L}{\rho A_L} \cos(\theta + \omega L^-)), \]
and
\[ \phi(\theta, a) = F_a(0, \theta) - a - \theta - \omega(L^- + L^+) - \frac{\omega}{\beta} \ln(\varepsilon P_L^+ \rho A_L) - \frac{\omega}{\beta} \Psi(\theta). \]

To fit into Proposition 4.3 we regard 
\[ a + \omega(L^- + L^+) + \frac{\omega}{\beta} \ln(\varepsilon P_L^+ \rho A_L) \]
as $a$ in Proposition 4.3. To verify that $\Psi(\theta)$ is a well-defined function with non-degenerate critical points, it suffices to observe that by the way $\rho$ is defined, we have
\[ \frac{1}{4} < \frac{1}{\rho A_L} \sqrt{C_L^2 + S_L^2} < \frac{1}{2} \]
provided that $L^\pm$ are sufficiently large. To verify the assumptions on $\phi(\theta, a)$ we let $\varepsilon \omega \ll 1$. Here we also need
\[ \frac{P_L}{P_L^+ \rho A_L} < < 1, \]
an estimate follows from Lemma 3.1.

We now move to the 2D part of the setting of [WY1] and [WY2]. A 2-parameter $C^3$ family $\mathcal{F}_{a,b}(X, \theta)$ of 2D maps defined on $\mathcal{A} = I \times S^1$ is an admissible rank one family if

(C1) there exists a $C^2$-function $F_a(X, \theta)$ in $(a, X, \theta)$ so that
\[ \|\mathcal{F}_{a,b}(X, \theta) - (0, F_a(X, \theta))\|_{C^3} \to 0 \]
as $b \to 0$;

(C2) the 1-parameter family of 1D maps $f_a(\theta) = F_a(0, \theta)$ is an admissible 1D family; and

(C3) at the critical points of the 1D map $f_a(\theta)$, we have
\[ \frac{\partial}{\partial X} F_a(X, \theta) \neq 0. \]

The following is the main result of [WY1] and [WY2] for a given admissible rank one family $\mathcal{F}_{a,b}$ of 2D maps.

**Proposition 4.4.** There exists $b_0 > 0$, such that for all $|b| < b_0$, there exists a positive measure set $\Delta_b$ of values of $a$ such that for $a \in \Delta_b$, $\mathcal{F}_{a,b}$ admits SRB measures on $\mathcal{A}$.

In addition, if there exists $C$ independent of $b$ such that

(C4) For all $(X, \theta), (X', \theta') \in \mathcal{A},$
\[ \left| \frac{\det DF_{a,b}(X, \theta)}{\det DF_{a,b}(X', \theta')} \right| < C. \]

Then we also have from [WY1]...
**Proposition 4.5.** Lebesgue almost every point in $A$ is generic with respect to the SRB measure of Proposition 4.4 for all $a \in \Delta_b$.

**Proof of Theorem 1:** We base our proof of Theorem 1 on Proposition 4.4 and 4.5. It suffices for us to show that the two-parameter family $\mathcal{F}_{a,b_n}$ defined by $\mathcal{F}_{a,b_n} = \mathcal{F}_{p(n,a)}$ is an admissible family of 2D maps. This is now straightforward. (C1) follows from Proposition 4.2 assuming (H1)(ii). (C2) follows from (4.9) and Proposition 4.3. (C3) follows directly from (4.8). (C4) follows from a direct computation using the formulas for $\mathcal{F}$ in Proposition 4.1. This concludes the proof of Theorem 1. □

**References**


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