This is a homework problem. Solution for (a) is attached. (b) is assigned as homework.

Problem (Rockets): A rocket is launched upwards with an initial velocity \( v_0 \). We know that the gravitational constant is \( G = 6.67 \times 10^{-11} \text{N m}^{-2} \text{kg}^{-1} \text{s}^{-2} \), the mass of earth is \( M = 5.98 \times 10^{24} \text{kg} \) and the radius of the earth is \( r_0 = 6357 \text{km} \). Let us ignore the air resistance and consider the rocket as under the earth’s gravitational influence alone.

(a) (Escape velocity) Find the initial velocity \( v_0 \) for which the rocket will never fall back.

(b) (Trajectory) Compute the height of the rockets as a function of time.
(I) Variables: Let $r$ be the distance of the rocket to the center of the earth, $t$ be the time. The function we are looking for is $r(t)$.

(II) Equation: By Newton’s law of gravitations,

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$ 

This is the differential equation dominating the motion of the rockets.

(III) Find the general solution: Observe that

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dr}{dt} \frac{dv}{dr} = v \frac{dv}{dr}.$$ 

We have

$$v \frac{dv}{dr} = -\frac{GM}{r^2},$$

from which it follows that

$$\frac{1}{2}v^2 - \frac{GM}{r} = h$$ \quad (1)
where $h$ is an arbitrary constant (constant of energy). Note that at $r = r_0 (= 6357 \text{ km})$, $v = v_0$, we have

$$h_0 = \frac{1}{2}v_0^2 - \frac{GM}{r_0}.$$  

(IV) Solution for (a): From (1),

$$v^2 = 2(h_0 + \frac{GM}{r}).$$

For the rocket to fall back, there should be a time such that $v = 0$. So we must have

$$h_0 + \frac{GM}{r} = 0.$$  

To have a solution $r > 0$, it is necessary for $h_0 < 0$.

So to find the speed for the rocket to escape we should make $h_0 \geq 0$, which implies

$$v_0 > \sqrt{\frac{2GM}{r_0}} \approx 11.2 \text{ km/sec.}$$