

CHAOTIC BEHAVIOR IN NONAUTONOMOUS EQUATIONS WITHOUT ANY TIME PERIODICITY

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ABSTRACT. We investigate chaotic behavior of ordinary differential equations with a homoclinic orbit to a dissipative saddle point under a general time dependent forcing *without any periodicity* in time. We study Poincaré return maps in extended phase space, introducing a characteristic function that generalizes the classic Melnikov function. We then show that the dynamics of the solutions of these equations are largely determined by asymptotic behavior of this new characteristic function. We prove the existence of a spectrum of dynamical scenarios including (i) an attracting integral manifold; (ii) intersections of the stable and the unstable manifolds of the perturbed saddle point; and (iii) new dynamical structures that generalize Smale's horseshoes for time-periodic equations. In particular, intersections of the stable and unstable manifold of the perturbed saddle are neither necessary nor sufficient for chaotic dynamics to emerge. These results are also applied to Duffing's equation with a time dependent forcing.

1. Introduction

In this paper, we study the complicated dynamics of ordinary differential equations with a homoclinic orbit to a dissipative saddle under a general time dependent forcing with application to Duffing's equation. *We do not assume any time periodicity for the forcing functions.* To simplify our presentation, we study nonautonomous ordinary differential equations in \mathbb{R}^2 . The higher dimensional problem will be addressed in an upcoming paper.

We derive analytically Poincaré return maps induced by the perturbed equations in extended phase space. These return maps are defined on an infinitely long strip $\Sigma = \mathbb{R} \times I$ where \mathbb{R} represents the direction of time. Depending on the forcing functions, they are either fully or partially defined on Σ . When the forcing function is *time-periodic* and the stable and the unstable manifold of the saddle point do *not* intersect, these return maps are reduced to annulus maps that resemble the Hénon maps and the dissipative standard maps. When the forcing function has no periodicity in time, they are a *new class of maps* no longer related to these classical families but can nevertheless exhibit complicated dynamical behavior.

A. Brief summary of results. Let $(x, y) \in \mathbb{R}^2$ be the phase variable and t be the time. We start with an autonomous system

$$(1.1) \quad \frac{dx}{dt} = -\alpha x + f(x, y), \quad \frac{dy}{dt} = \beta y + g(x, y)$$

where $f(x, y), g(x, y)$ are the higher order terms. We assume that equation (1.1) has a homoclinic orbit to the dissipative saddle $(x, y) = (0, 0)$. Precise conditions on (1.1) will be given in Section 2. To the right hand side of equation (1.1) we add a time-dependent forcing to form a nonautonomous

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equation

$$(1.2) \quad \frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y, t), \quad \frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y, t)$$

where μ is a small parameter representing the magnitude of the forcing, $P(x, y, t)$ and $Q(x, y, t)$ are higher order terms at $(x, y) = (0, 0)$.

We derive analytically the Poincaré return map \mathcal{R} induced by equation (1.2) in extended phase space, introducing in Section 2 a characteristic function $\mathcal{W}(t)$ that is a natural extension of the classical Melnikov function (See (2.4)). This function measures the size of the separation of the stable manifold W^s and the unstable manifold W^u of $(0, 0)$. We also extend the concept of Smale's horseshoe to non-periodic equations (See Sect. 2E,F).

Let $\mathcal{W}(t)$ be the characteristic function given by (2.4) for equation (1.2). We denote

$$M = \sup_{t \in \mathbb{R}} \mathcal{W}(t), \quad m = \inf_{t \in \mathbb{R}} \mathcal{W}(t)$$

and let

$$m^\pm = \liminf_{t \rightarrow \pm\infty} \mathcal{W}(t), \quad M^\pm = \limsup_{t \rightarrow \pm\infty} \mathcal{W}(t).$$

Our results can be summarized as follows:

Main Theorem. *Assume that equation (1.1) has a homoclinic orbit to the dissipative saddle point $(x, y) = (0, 0)$ and α, β satisfy certain non-resonant condition. Then the return map \mathcal{R} induced by equation (1.2) in extended phase space is such that, for each of the items from (i)-(vi) below, there is a respective $\mu_0 > 0$ so that for all $0 < \mu < \mu_0$, we have the following.*

(i) **(Integral Manifold)** *If $m > 0$ and*

$$\sup_{t \in \mathbb{R}} \frac{1}{\beta} \frac{|\mathcal{W}'(t)|}{|\mathcal{W}(t)|} < 1,$$

then \mathcal{R} has an attracting invariant curve.

- (ii) **(Intersection and Full Horseshoe)** *If $m^\pm < 0 < M^\pm$, then \mathcal{R} admits a full horseshoe.*
 (iii) **(Intersection and Half Horseshoe)** *If $M^\pm, m^- > 0$ and $m^+ < 0$ (or $M^\pm, m^+ > 0$ and $m^- < 0$), then \mathcal{R} admits a half horseshoe.*
 (iv) **(Trivial Dynamics)** *If either $M^+ < 0$ or $M^- < 0$, then \mathcal{R} has trivial dynamics.*
 (v) **(Non-intersection and Half Horseshoe)** *If $m^\pm, M^\pm > 0$ and there exist L^+ , and sequences $a_k, b_k \rightarrow \infty$ with $0 < a_{k+1} - a_k, b_{k+1} - b_k < L^+$ for all $k \geq 0$ such that*

$$\lim_{k \rightarrow \infty} \mathcal{W}(a_k) = m^+, \quad \lim_{k \rightarrow \infty} \mathcal{W}(b_k) = M^+,$$

and

$$M^+ > m^+ e^{3\beta L^+},$$

then \mathcal{R} admits a half horseshoe.

- (vi) **(Non-intersection and Full Horseshoe)** *If the conditions in (v) also hold for m^-, M^- , then \mathcal{R} admits a full horseshoe.*
 (vii) **(An Application)** *The above phenomena appear in forced Duffing's equations.*

We note that the conditions in (i) mean that the stable and unstable manifolds do not intersect and the rate of change of the Melnikov function is relatively small comparing to the magnitude of the function. The existence of an attracting invariant curve for the return map \mathcal{R} gives an attracting integral manifold for equation (1.2) in a neighborhood of the homoclinic orbit in the extended phase space. This result can be viewed as an extension of the classic result of Levinson on the integral manifold nearby a periodic orbit of an autonomous differential equation under a time periodic forcing.

The numbers m^- and M^- are intrinsic limit values of the Melnikov function $\mathcal{W}(t)$ in the negative t -direction, and m^+ and M^+ are their correspondences in the positive t -direction. In principle, there is nothing *a priori* to relate the limit values of the Melnikov function $\mathcal{W}(t)$ in the positive t -direction to the limit values of the same function in the negative t -direction. Any conceivable combination for m^\pm, M^\pm, m , and M satisfying $m \leq m^- \leq M^- \leq M$ and $m \leq m^+ \leq M^+ \leq M$ is possible. See the example in Section 3. On the other hand, if equation (1.2) is time-periodic, then $m^\pm = m$, $M^\pm = M$. The same are true for equations that are almost periodic in time. Therefore, the freedom in getting different combinations of the intrinsic limit values are due mainly to equations that are not even almost periodic in time. Associated with this new freedom of choice is a set of new structures that are not permitted in almost-periodically perturbed equations.

The condition in (ii), $m^\pm < 0 < M^\pm$, implies that the stable manifold W^s and the unstable manifold W^u of $(0,0)$ intersect infinitely many times near both $+\infty$ and $-\infty$, while the condition in (iii) yields that the stable manifold and the unstable manifold intersect infinitely many times only in one direction. In both cases, the systems have chaotic behavior. However, unlike in the case of time periodic equations, non-empty intersections of the stable and unstable manifolds of the perturbed saddle point do not necessarily imply complicated dynamics. One such example is when $M^- < 0$, $m^+ < 0 < M^+$. W^s and W^u intersect infinitely many times near $+\infty$, but there is no complicated dynamics (as stated in (iv)).

Not only the non-empty intersections of the stable manifold and the unstable manifold of $(0,0)$ are not sufficient, but also they are not necessary for chaotic dynamics to emerge. This is stated in (v) above. Assuming $W^s \cap W^u = \emptyset$, the perturbed equations admit chaotic dynamics provided that the characteristic function oscillates with a non-trivial amplitude near infinity and the condition $M^+ > m^+ e^{3\beta L^+}$ holds. This condition can be achieved in two ways. The first is by making m^+ small. This implies that, even though W^u and W^s are eventually separated as $t \rightarrow +\infty$, their persistently getting close would generate enough expansion to create complicated structure. The second way is to make L^+ small, namely to have high frequency oscillation of the Melnikov function between m^+ and M^+ as $t \rightarrow +\infty$.

B. Relation to the existing literature. The study of complicated dynamics of ordinary differential equations under periodic perturbations has a long and rich history that dates back to Poincaré and Birkhoff. The complicated behavior induced by the presence of homoclinic intersections of the stable and the unstable manifold of a saddle fixed point was first observed by Poincaré [P]. “Irregular noise” was detected in a simple circuit by electrical engineer van der Pol. The van der Pol type of equations were extensively studied by Cartwright and Littlewood. Later, Levinson [L1] proved that a linearized version of the periodically perturbed van der Pol equation has infinitely many periodic solutions of different period. The existence of solutions corresponding to full Bernoulli shift for equations with homoclinic intersections was first described by Birkhoff [B], proved later by Smale [Sm1, Sm2] in a geometric form, and was systematically studied by Alekseev [A] with applications to Sitnikov’s three body problem [Sit]. Since then, the study of chaotic behavior has flourished spectacularly and the literature on this subject is vast. The Melnikov method [M] and the method of Lyapunov-Schmidt [CHM-P] have been developed for the purpose of verifying the existence of transversal homoclinic points, thus that of the Smale horseshoe, for ordinary differential equations under periodic perturbations. See, for example, [Shi], [Hol], [HM], [Le], [CH], [GH], [BP], [SSTC1, SSTC2], and the references therein.

Consider a planar differential equation with a homoclinic orbit to a saddle point. Under a periodic perturbation, the stable and unstable manifolds either intersect transversally/tangentially or no longer intersect, see Fig. 1. When they intersect transversally, the Smale-Birkhoff homoclinic theorem gives the existence of horseshoes. In the case of tangential intersection, the dynamics can be subtle and complicated, see Gavrilov and Shilnikov [GS] and Newhouse [N]. In [CHM-P], Chow, Hale,

and Mallet-Paret showed that there can be subharmonic solutions when the stable and unstable manifolds no longer intersect.

Recently, Wang and Ott [WO] and Wang and Oksasoglu [WOk], motivated by [AS], studied the complicated dynamics of periodically perturbed differential equations through a Poincaré return map in extended phase space. They explicitly computed the Poincaré return maps around a given homoclinic solution. In [WO], they showed that the return maps for Fig. 1(b) are *rank one maps*, a class of non-uniformly hyperbolic maps that has been studied extensively by Wang and Young in recent years [WY1]-[WY3] based on the theory of Benedicks and Carleson on strongly dissipative Hénon map [BC]. Consequently, they proved the existence of strange attractors with SRB measures for a positive measure set of forcing parameters for Fig. 1(b). In [WOk], Wang and Oksasoglu proved that the return maps for Fig. 1(a) are families of *infinitely wrapped horseshoe maps*. They applied many theories developed in recent years on non-uniformly hyperbolic maps, including the Newhouse's theory [N], [PT], the theory of SRB measures [Si], [R], [Bo] and the theory of Hénon-like attractors [BC], [MV], [BY], to the analysis of periodically perturbed differential equations through the return maps derived. They also obtained a comprehensive description on the dynamics of homoclinic tangles of Fig. 1(a) beyond Smale's horseshoes.

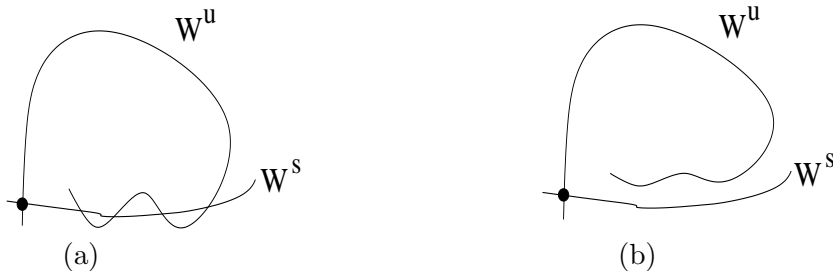


Fig. 1 Homoclinic tangles and separated invariant manifolds

Quasi-periodic ordinary differential equations arise naturally in applications when force of multiple frequencies are taken into account. Krylov and Bogoliubov [KB] and Mitropolsky [BM] studied quasiperiodic ordinary differential equations arising from the study of nonlinear oscillations. Under the assumption that the averaged equation has an asymptotically stable equilibrium point, they proved the existence of quasiperiodic integral manifolds, which gives the existence of asymptotically stable quasiperiodic orbits for a class of equations. Their results were later extended by Diliberto [Di1, Di2], Hufford [Hu], Marcus [Ma], Hale [H], and others. There has also been a substantial literature purposed on extending the Smale horseshoe from periodically forced to quasi-periodically and almost periodically forced equations by Scheurle [Sch], Palmer [Pa2], Stoffer [St1, St2], Palmer and Stoffer [PS], Meyer and Sell [MS], Yagasaki [Ya], Shilnikov [Shi1] and Wiggins [Wi].

Roughly speaking, there are two basic approaches to obtain a horseshoe for differential equations under a periodic perturbation. With the first approach, one computes explicitly the Melnikov function to verify the existence of a transversal homoclinic intersection for the time-period map of the given equation. Horseshoes then follow from Smale's geometric construction. See for example [GH]. The second approach is based on analytic shadowing, see [Pa0] and [Pa2]. This method does not rely on Smale's geometric construction. It consists of three steps:

- (a) to prove, using analytic shadowing, that if there is a finite collection of homoclinic solutions for the perturbed equation satisfying exponential dichotomy, then there is a collection of solutions corresponding to full Bernoulli shift using these homoclinic solutions as generating symbols;
- (b) to prove that for the perturbed equation, a homoclinic solution satisfies exponential dichotomy if and only if it corresponds to transversal homoclinic point of the time-period map; and
- (c) to again compute the Melnikov function to verify the existence of transversal homoclinic points for the time-period map.

Both approaches have been extended to the study of quasi-periodically perturbed equations. The extension for the first approach can be found in [Wi]. This extension is based on the following observation that goes back to Shilnikov [Shi1]: if the saddle point is replaced by a normally hyperbolic tori, and the stable and the unstable manifolds of this tori intersect transversally, then Smale's geometric construction of horseshoe remain valid but, in this case, each of the point of the horseshoe is replaced by a tori. In [Wi], Wiggins observed that by introducing one angular variable for each forcing frequency, the quasi-periodically perturbed equations become autonomous, and the time-1 map fits Shilnikov's setting. He also proved that if the Melnikov function in extended phase variables satisfies certain non-degenerate conditions, then there exists a transversal intersection. The second way is along the line of Palmer's approach [Pa2], [PS], [Sch], [MS]. This approach again focused on using homoclinic solutions satisfying exponential dichotomy as generating symbols to create solutions of full Bernoulli shift by analytic shadowing.

In [LS], Lerman and Shilnikov studied the complicated dynamics for differential equations under time dependent perturbations without assuming any periodicity in time. Their approach is similar to Palmer's. They started with a collection of homoclinic solutions satisfying the exponential dichotomy, using these solutions as generating symbols to construct solutions corresponding to full Bernoulli shift. Exponential dichotomy alone, however, are not sufficient in this case so they added a new set of technical assumptions to ensure certain uniformity. What they obtained is therefore a slightly weaker version corresponding to (a) in Palmer's approach. On the other hand, the authors of [LS] did not address the correspondences of (b) and (c) in Palmer's approach.

In this paper, we derive analytically the Poincaré return maps induced by the perturbed equations in the extended phase space. The smooth linearization theorem plays a crucial role in estimating the return maps. Instead of zooming into the neighborhood of a collection of homoclinic solutions, as in the way adopted in Palmer's approach and in [LS], we zoom out, deriving a return map in extended phase space that catches dynamical activities of ALL solutions stayed in the neighborhood of the unperturbed homoclinic loop in extended phase space. An advantage of this approach is that it can be used to obtain complicated dynamics not only when there is a homoclinic point but also when the stable and unstable manifolds do not intersect. To the best of our knowledge, this return map was not rigorously derived from a given non-autonomous equation with dissipations before, not even for the periodically perturbed case before [WO] and [WOk].

The study of the Poincaré return map around a homoclinic orbit for autonomous differential equations goes back to Shilnikov. In [Shi1, Shi2, Shi3, Shi4, Shi5], Shilnikov developed an approach (Shilnikov problem) to estimate the return map through which the chaotic behavior was obtained. This method was extensively used and extended by others to study the homoclinic bifurcation, see [Deng1, Deng2], [DS]. In [AS], Afraimovich and Shilnikov showed that the Poincaré return map can also be used to study periodically perturbed equations if the form of the return map is known. Here the basic issue is how to estimate the return map for differential equations with time-dependent perturbations in extended phase space. In the case considered in this paper, one needs to also overcome the difficulty caused by noncompactness in the direction of time.

This paper is organized as follows. In Section 2 we give the precise statement of results. In Section 3 we demonstrate how to apply our theorems to a nonautonomously forced Duffing's equation. In Section 4 we derive the return maps induced in extended phase space. In Section 5 we use the return maps to prove the theorems stated in Section 2. Two technical proofs are placed in appendices.

2. Statement of Results

A. Equations of study. Let $(x, y) \in \mathbb{R}^2$ be the phase variable and t be the time. We start with an autonomous system

$$(2.1) \quad \frac{dx}{dt} = -\alpha x + f(x, y), \quad \frac{dy}{dt} = \beta y + g(x, y)$$

where $f(x, y)$ and $g(x, y)$ are C^N functions for sufficient large N satisfying $f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0$. First, we assume

(H) (i) α and β satisfy the nonresonant conditions up to order N : there are no nonnegative integers m and n with $2 \leq m + n \leq N$ such that

$$\alpha = m\alpha + n\beta \quad \text{or} \quad \beta = m\alpha + n\beta;$$

(ii) $0 < \beta < \alpha$.

A sufficient condition for (H)(i) is that α and β are rationally independent. (H)(ii) assumes that the saddle point $(0, 0)$ is dissipative. We also assume that the positive x -side of the local stable manifold and the positive y -side of the local unstable manifolds of $(0, 0)$ are included as a part of a homoclinic solution, which we denote as

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$

To the right hand side of equation (2.1) we add a time-dependent forcing to form a nonautonomous equation

$$(2.2) \quad \frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y, t), \quad \frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y, t)$$

where μ is a small parameter representing the magnitude of the forcing. Let U be an open neighborhood in (x, y) -plane that contains the closure of ℓ , and $\mathcal{U} = U \times \mathbb{R}$. We assume that C^N norms of $P(x, y, t), Q(x, y, t)$ are uniformly bounded by a constant that is independent of μ on \mathcal{U} . We also assume that $P(x, y, t)$ and $Q(x, y, t)$ are high order terms at $(x, y) = (0, 0)$ for all t , namely, we assume $P(0, 0, t) = Q(0, 0, t) = \partial_x P(0, 0, t) = \partial_x Q(0, 0, t) = \partial_y P(0, 0, t) = \partial_y Q(0, 0, t) = 0$ for all $t \in \mathbb{R}$. We study equation (2.2) on $\mathcal{U} = U \times \mathbb{R}$ in the extended phase space (x, y, t) .

B. Return maps in extended phase space. We study the dynamics of equation (2.2) through the iteration of a return map we now introduce in extended phase space. U in the space of (x, y) is constructed by taking the union of a small neighborhood B_ε (a ball at $(0, 0)$ with radius ε) of $(0, 0)$ and a small neighborhood D around ℓ outside of $B_{\frac{1}{4}\varepsilon}$. See Fig. 2.

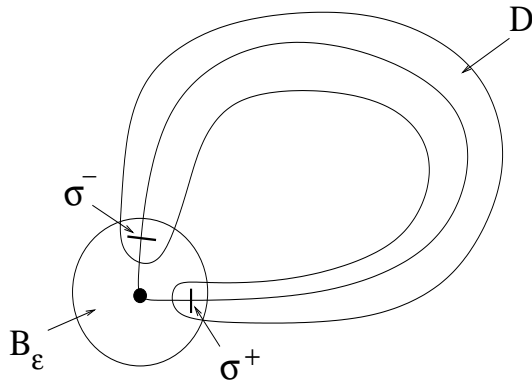


Fig. 2 B_ε, D and σ^\pm .

Let $\sigma^\pm \in B_\varepsilon \cap D$ be the two line segments depicted in Fig. 2, both perpendicular to the homoclinic solution ℓ . In the extended phase space (x, y, t) we denote

$$\mathcal{B}_\varepsilon = B_\varepsilon \times \mathbb{R}, \quad \mathcal{D} = D \times \mathbb{R}$$

and let

$$\Sigma^\pm = \sigma^\pm \times \mathbb{R}.$$

Let $\mathcal{N} : \Sigma^+ \rightarrow \Sigma^-$ be induced by the solutions on \mathcal{B}_ε and $\mathcal{M} : \Sigma^- \rightarrow \Sigma^+$ be induced by the solutions on \mathcal{D} . We first compute \mathcal{M} and \mathcal{N} separately, then compose \mathcal{N} and \mathcal{M} to obtain an explicit formula for the return map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \rightarrow \Sigma^-$.

C. Objects of study. We introduce a function, which we call the Melnikov function for equation (2.2), as follows: Let

$$(u(t), v(t)) = \left| \frac{d}{dt} \ell(t) \right|^{-1} \frac{d}{dt} \ell(t)$$

be the unit tangent vector of ℓ at $\ell(t)$ and let

$$(2.3) \quad \begin{aligned} E(t) = & v^2(t)(-\alpha + \partial_x f(a(t), b(t))) + u^2(t)(\beta + \partial_y g(a(t), b(t))) \\ & - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))). \end{aligned}$$

The quantity $E(t)$ measures the rate of expansion of the solutions of equation (2.1) in the direction normal to ℓ at $\ell(t)$. The **Melnikov function** $\mathcal{W}(t)$ for equation (2.2) is defined as

$$(2.4) \quad \mathcal{W}(t) = \int_{-\infty}^{\infty} (v(s)P(a(s), b(s), s+t) - u(s)Q(a(s), b(s), s+t))e^{-\int_0^s E(\tau) d\tau} ds.$$

Observe that $E(t) \rightarrow \beta$ as $t \rightarrow +\infty$ and $E(t) \rightarrow -\alpha$ as $t \rightarrow -\infty$. Since P and Q are uniformly bounded on \mathcal{U} , $\mathcal{W}(t)$ is uniformly bounded for all t . Also observe that, as a normally hyperbolic set, the line $(x, y) = (0, 0)$ in extended phase space has a 2-dimensional stable manifold which we denote by W^s and a 2-dimensional unstable manifold which we denote by W^u . Let

$$M = \sup_{t \in \mathbb{R}} \mathcal{W}(t), \quad m = \inf_{t \in \mathbb{R}} \mathcal{W}(t).$$

Theorem 2.1. (a) Assume that $m < 0 < M$. Then there exist $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, $W^s \cap W^u \neq \emptyset$.

(b) Assume $m > 0$. Then there exists $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, $W^s \cap W^u = \emptyset$.

(c) Assume $M < 0$. Then there exists $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, $W^s \cap W^u = \emptyset$.

For case (a) of Theorem 2.1, it is expected that the return map $\mathcal{R} : \Sigma^- \rightarrow \Sigma^-$ is only partially defined on Σ^- . After following the entire length of the homoclinic loop of the unperturbed equation, part of Σ^- would hit Σ^+ on one side of the local stable manifold of $(x, y) = (0, 0)$ where they return to Σ^- ; and the rest would hit the other side where they sneak out of \mathcal{U} , see Fig. 3(a). For case (b) of Theorem 2.1, all solutions starting from Σ^- will eventually return to Σ^- . In this case the return map \mathcal{R} is well defined on Σ^- , see Fig 3(b). For case (c), all solutions starting from Σ^- will hit the wrong side of the local stable manifold on Σ^+ and there will be no return to Σ^- , see Fig. 3(c).

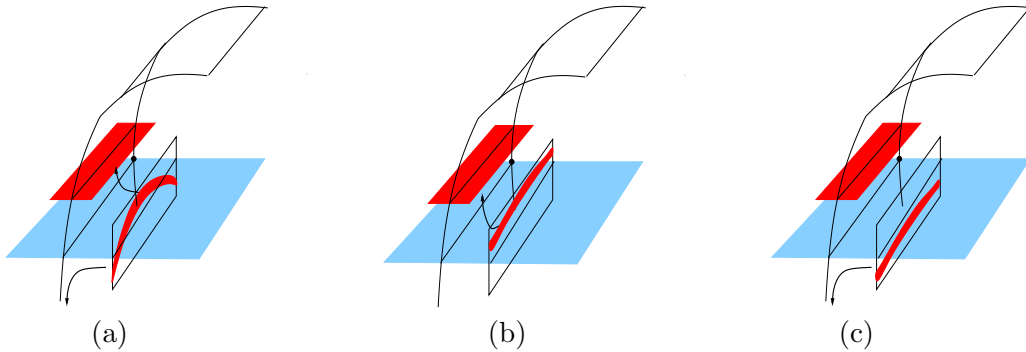


Fig. 3 Three cases for the return map \mathcal{R} .

We will study the geometrical and dynamical structure of all solutions that stay inside of U in phase space (x, y) for all times. Among the three scenarios of Theorem 2.1, there is obviously nothing interesting about (c). So we will only consider scenarios (a) and (b). Let W be the subset of Σ^- on which the return map \mathcal{R} is defined (For scenarios (b), $W = \Sigma^-$). Let

$$\Omega = \{p \in W : \mathcal{R}^n(p) \in W, \text{ for all } n \geq 1\}$$

and

$$\Lambda = \bigcap_{n \geq 1} \mathcal{R}^n(\Omega).$$

Then, the solutions initiated in Ω are all solutions that stay forever in U in forward times and the solutions initiated in Λ are all solutions that stays forever in U in both forward and backward times. The *ultimate objective* of our study is to understand the geometrical and dynamical structure of Λ for the return map \mathcal{R} derived from equation (2.2).

D. Stable dynamics. The structure of Λ could sometimes be simple and sometimes be complicated. We start with the simple case of an attracting invariant curve.

Theorem 2.2. *Assume that $m > 0$ and*

$$(2.5) \quad \sup_{t \in \mathcal{R}} \frac{1}{\beta} \left| \frac{\mathcal{W}'(t)}{\mathcal{W}(t)} \right| < 1.$$

Then there exists $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, the return map $\mathcal{R} : \Sigma^- \rightarrow \Sigma^-$ admits an invariant curve h defined by a C^N function $h(t) : \mathbb{R} \rightarrow \sigma^-$. Furthermore, h is globally attracting in the sense that, for every $p \in \Sigma^-$, there exists a point $p_0 \in h$ such that

$$\lim_{n \rightarrow +\infty} |\mathcal{R}^n(p) - \mathcal{R}^n(p_0)| \rightarrow 0.$$

Remark: (1) All solutions started from h form a 2D integral manifold for equation (2.2) in extended phase space, which we denote as S , and there exists an open neighborhood $U_S \supset S$ so that all solutions initiated in U_S are attracted to that of S . This result may be viewed as an extension of the classic result of Levinson [L2] on the integral manifold nearby a periodic orbit of an autonomous differential equation under a time periodic forcing.

(2) If $P(x, y, t), Q(x, y, t)$ are periodic of period T in t , then we can regard the time as an angular variable and the return map \mathcal{R} as an annulus map. In this case the invariant curve h of Theorem 2.2 is reduced to an invariant circle and \mathcal{R} induces a circle diffeomorphism on h , the rotation number of which depends sensitively on μ . For a set of μ with positive Lebesgue density at $\mu = 0$, S is an invariant tori of quasi-periodic solutions, see [WO]. If $P(x, y, t), Q(x, y, t)$ are quasi-periodic in t , then the invariant curve h is also quasi-periodic in t , see [LW].

E. Existence of topological horseshoes While we could regard the dynamics presented in Theorem 2.2 as *simple*, we now consider the case where the structure of Λ is *complicated*. We start with periodically perturbed equations, for which the return maps \mathcal{R} are reduced to annulus maps that resemble the Hénon maps and the dissipative standard maps. Many theories developed in recent years on non-uniformly hyperbolic maps can be applied to these return maps, and for the corresponding equations, Λ assumes a variety of complicated structures ranging from horseshoes to strange attractors with SRB measures, see [WO] and [WOk]. However, when the forcing function has no periodicity, return maps \mathcal{R} do not admit similar reduction. They are a new class of maps no longer related to these classical families, and there is no previous theory that is ready to be applied concerning chaotic dynamics.

To study chaotic behavior in non-autonomously forced equations without any time periodicity, we start from extending the concept of horseshoe for \mathcal{R} . First some geometric terms. We call the direction of t in Σ^- the horizontal direction and the direction of σ^- (transversal to the homoclinic

solution ℓ in the original phase space) the vertical direction. In Σ^- , a vertical curve is a non-self-intersecting, continuous curve that connects the two horizontal boundaries of Σ^- . We call a region that is bounded by two non-intersecting vertical curves a *vertical strip*, which we denote as V . The two defining vertical curves for a given vertical strip V are the vertical boundaries of V . We call a non-self-intersecting continuous curve connecting the two vertical boundaries of V a *fully extended horizontal curve* in V . Let V_1, V_2 be two non-intersecting vertical strips. We say that $\mathcal{R}(V_1)$ *crosses* V_2 *horizontally* if for every fully extended horizontal curve h of V_1 , there is a subsegment \tilde{h} of h so that $\mathcal{R}(\tilde{h})$ is a fully extended horizontal curve in V_2 .

Definition 2.1 (Topological horseshoe). *We say that \mathcal{R} admits a topological horseshoe of k -branches, $k \leq \infty$, if there exists a bi-infinite sequence of non-intersecting vertical strips $\{V_n\}_{n=-\infty}^{\infty}$ lined up monotonically from $t = -\infty$ to $t = +\infty$ in Σ^- , such that*

- (1) *For every n , there exists a $\hat{n}_1 > n$, such that $\mathcal{R}(V_n)$ crosses $V_{\hat{n}_1}, V_{\hat{n}_1+1}, \dots, V_{\hat{n}_2+k}$ horizontally.*
- (2) *For every n , there exists a $\hat{n}_2 < n$, such that $\mathcal{R}(V_{\hat{n}_2-k}), \dots, \mathcal{R}(V_{\hat{n}_2})$ crosses V_n horizontally.*

Observe that, if \mathcal{R} admits a horseshoe of k -branches, then inside every vertical strip V_n , there exists a 2D Cantor set Λ_n formed by the intersections of a k -Cantor set of vertical curves and a k -Cantor set of fully extended horizontal curves, so that $\Lambda_n \subset \Lambda$ for all n . If \mathcal{R} is from a time-periodic equation, then the vertical strips $\{V_n\}_{n=-\infty}^{+\infty}$ can be arranged periodically in the t -direction in Σ^- , and the structure in Definition 2.1 is a topological horseshoe defined on an annulus through proper quotient in t -direction. In this sense definition 2.1 is a rather natural generalization of the classical horseshoe. If \mathcal{R} admits a topological horseshoe according to Definition 2.1, then the set Λ for \mathcal{R} is *complicated* and we have a *chaotic* attractor for equation (2.2).

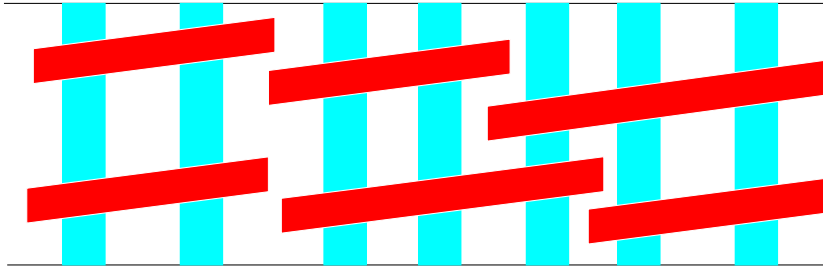


Fig. 4 A topological horseshoe of 2-branches

We are now ready to state our first theorem on the existence of complicated dynamics. Let $\mathcal{W}(t)$ be the Melnikov function in (2.4). Recall that

$$m^\pm = \liminf_{t \rightarrow \pm\infty} \mathcal{W}(t); \quad M^\pm = \limsup_{t \rightarrow \pm\infty} \mathcal{W}(t).$$

Theorem 2.3. *Assume that*

$$m^-, m^+ < 0 < M^-, M^+.$$

Then there exists a $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, the return map \mathcal{R} for equation (2.2) admits a topological horseshoe of infinitely many branches.

F. Homoclinic intersection and half horseshoes. m^- and M^- are intrinsic limit values of $\mathcal{W}(t)$ in the negative t -direction, and m^+ and M^+ are their correspondences in the positive t -direction. In principle, there is nothing *a priori* to relate the limit values of the Melnikov function $\mathcal{W}(t)$ in the positive t -direction to the limit values of the same function in the negative t -direction. Any conceivable combination for m^\pm, M^\pm, m and M satisfying $m \leq m^- \leq M^- \leq M, m \leq m^+ \leq M^+ \leq M$ is possible. See the example in Section 3. On the other hand, if equation (2.2) is time-periodic,

then $m^\pm = m$, $M^\pm = M$. The same are true for equations that are almost periodic in time. Therefore the freedom in getting different combinations of these intrinsic limit values is due mainly to equations that are not almost periodic in time. Associated to this new freedom of choice is new scenarios that are not permitted in almost-periodically perturbed equations.

Let us exclude the case that one of the values of m^\pm , M^\pm , m and M is zero in our narrative throughout. By combining Theorem 2.1(a) and Theorem 2.3, it follows that, for equations that are almost periodic in time, non-empty intersection of W^s and W^u implies the existence of complicated dynamics in Λ . This is not the case in general.

Theorem 2.4. *If either M^- or M^+ is negative, then $\Lambda = \emptyset$.*

It follows from Theorem 2.4 that Λ could be trivial even if W^s and W^u are allowed to intersect infinitely many times. This is the case if, for an instance, $m^-, M^-, m^+ < 0 < M^+$. To avoid the trivial case of Theorem 2.4, let us assume that $M^-, M^+ > 0$ in the rest of this section. Excluding the case $m^-, m^+ < 0$, which is covered by Theorem 2.3, there are three remaining possibilities:

$$(i) \ m^+ > 0, \ m^- < 0; \quad (ii) \ m^+ < 0, \ m^- > 0; \quad (iii) \ m^-, \ m^+ > 0.$$

Each we now consider separately.

Case (i) $m^+ > 0$, $m^- < 0$: In this case we need a weaker version of horseshoe associated to the persistent intersection of W^u and W^s as $t \rightarrow -\infty$.

Definition 2.2 (Half horseshoe). *We say that \mathcal{R} admits a half horseshoe of k -branches, $k \leq \infty$, if there exists a sequence of non-intersecting vertical strips $\{V_n\}_{n=-\infty}^0$ lined up monotonically from $t = -\infty$ to $t < K < +\infty$ in Σ^- , such that*

- (1) *For every $n \leq 0$, there exists $\hat{n} < n$, such that $\mathcal{R}(V_{\hat{n}-k}), \dots, \mathcal{R}(V_{\hat{n}})$ crosses V_n horizontally.*
- (2) *The set $\mathcal{R}(V_0) \subset \Omega$.*

Recall that Ω is the subset of Σ^- , where \mathcal{R}^n is well-defined for all $n > 0$. Observe that, if Λ admits a half horseshoe of Definition 2.2, then there is a k -Cantor set Γ_0 of fully extended horizontal curves in V_0 so that $\Gamma_0 \subset \Lambda$. In addition, for every V_n such that $\mathcal{R}^i(V_n)$ horizontally crossing V_0 for some $i > 0$, there is a vertical strip $\hat{V}_n \subset V_n$, so that a k -Cantor set of fully extended horizontal curves in \hat{V}_n is in Λ . We have

Theorem 2.5. *Assume*

$$m^- < 0 < m^+, M^\pm.$$

Then, there exists a $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, the return map \mathcal{R} in extended phase space admits a half horseshoe of infinitely many branches of Definition 2.2.

Case (ii) $m^- > 0$, $m^+ < 0$: In this case we need a different definition of half horseshoe that is associated to the persistent intersection of W^u and W^s as $t \rightarrow +\infty$.

Definition 2.3 (Half horseshoe). *We also say that \mathcal{R} admits a half horseshoe of k -branches, $k \leq \infty$, if there exists a sequence of non-intersecting vertical strips $\{V_n\}_{n=0}^\infty$ lined up monotonically from $t > K$ to $t = +\infty$ in Σ^- , such that*

- (1) *For every $n \geq 0$, there exists a $\hat{n} > n$, such that $\mathcal{R}(V_n)$ crosses $V_{\hat{n}}, \dots, V_{\hat{n}+k}$ horizontally.*
- (2) *Every vertical curve of V_0 contains a point, all inverse images of which under \mathcal{R} are in Σ^- .*

Observe that if \mathcal{R} admits a half horseshoe of Definition 2.3, then there is a k -Cantor set of vertical curves in V_0 , each of which contains at least one point that is inside of Λ . In addition, for every V_n such that $\mathcal{R}^i(V_0)$ horizontally crossing V_n for some $i > 0$, there is also a k -Cantor set of vertical curves in V_n , each of which contains at least one point in Λ .

We have

Theorem 2.6. *Assume*

$$m^+ < 0 < m^-, M^\pm.$$

Then, there exists a $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, the return map \mathcal{R} in extended phase space admits a half horseshoe of infinitely many branches of Definition 2.3.

Case (iii) $m^-, m^+ > 0$: Though $m^\pm > 0$ does not rule out completely the possibility that W^u intersects W^s (remember $m \leq m^\pm$ and it is still possible for m to be negative), it has ruled out the non-trivial impact of such intersections. The structure of Λ might be simple, as in the case of Theorem 2.2, but it could also be complicated. This case is considered in next paragraph.

G. Horseshoes not related to $W^u \cap W^s$. Complicated dynamical structures, in principle, are caused by expansions imposed by the solutions of equation (2.2) in extended phase space. So far in the above, the expansions responsible for the existence of full and half horseshoes are created by the persistent intersections of W^u and W^s at $t = \pm\infty$. In this subsection we offer one more theorem on the existence of complicated dynamics without the presence of intersections of W^u and W^s .

We need a technical phrase before precisely stating our next theorem. By definition, we know that there exists a sequence $a_k \rightarrow +\infty$ so that $\mathcal{W}(a_k) \rightarrow M^+$.

Definition 2.4. *Let $L^+ > 0$ be a constant. We say that M^+ is **densely approached by an L^+ -sequence** if there exists a monotone sequence $a_k \rightarrow +\infty$ satisfying $a_{k+1} - a_k < L^+$ for all k , so that $\mathcal{W}(a_k) \rightarrow M^+$.*

Corresponding definition for m^+, M^- and m^- are similar.

Theorem 2.7. *Assume that $M^\pm, m^\pm > 0$. If both M^+ and m^+ are densely approached by L^+ -sequences and m^+, M^+ , and L^+ satisfy*

$$(2.6) \quad M^+ > m^+ e^{3\beta L^+},$$

then there exists $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, \mathcal{R} admits a half horseshoe of 2-branches of Definition 2.3.

Remarks: (1) For inequality (2.6) to hold we must have $M^+ > m^+$ so the Melnikov function must oscillate with a non-trivial amplitude as $t \rightarrow +\infty$. This inequality is then achieved in two ways. The first is by making m^+ small. This implies that, even though W^u and W^s are eventually separated as $t \rightarrow +\infty$, their persistently getting close would generate enough expansion to create complicated structure in Λ . The second way to make inequality (2.6) hold is to make L^+ small. This is to have high frequency oscillations of the Melnikov function between m^+ and M^+ as $t \rightarrow +\infty$.

(2) We also have slightly different versions of Theorem 2.7 according to various combinations of limit behavior of $\mathcal{W}(t)$ as $t \rightarrow \pm\infty$. For instance, if we further assume that m^- and M^- are also densely approached by L^+ -sequences and

$$M^- > m^- e^{3\beta L^+},$$

then \mathcal{R} would admit a full horseshoe of 2-branches of Definition 2.1. Full horseshoe is also admitted if we assume, instead of $m^- > 0$ in Theorem 2.7, $m^- < 0 < M^-$, and so on.

3. Applications to Nonautonomous Duffing's Equations

In this section, we apply Theorems 2.1-2.7 to a nonautonomously forced Duffing's equation without any time periodicity. We start with the autonomous Duffing's equation

$$(3.1) \quad \frac{d^2 q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^3 = 0$$

where $\lambda > 0$ and γ are parameters. By letting $p = \frac{dq}{dt}$, we can write equation (3.1) as a system of first order equations in terms of p and q . We first borrow a result on (3.1) from [HR].

Proposition 3.1 (Dissipative homoclinic saddle). *There exists $\lambda_0 > 0$ sufficiently small, such that for $\lambda \in [0, \lambda_0)$, there exists a $\gamma_\lambda, |\gamma_\lambda| < 10\lambda$ such that for $\gamma = \gamma_\lambda$;*

(i) *equation (3.1) has a homoclinic solution to $(q, p) = (0, 0)$, which we denote as*

$$\ell_\lambda = \{\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t)), t \in \mathbb{R}\};$$

(ii) *for any given $L > 0$, there exists a $K(L)$ independent of λ , such that for all $t \in [-L, L]$,*

$$|\ell_\lambda(t) - \ell_0(t)| < K(L)\lambda$$

where

$$\ell_0(t) = \left(\frac{2\sqrt{2}e^t}{(1+e^{2t})}, \frac{2\sqrt{2}(e^t - e^{3t})}{(1+e^{2t})^2} \right)$$

is a homoclinic solution of equation (3.1) for $\lambda = \gamma = 0$.

Let

$$(3.2) \quad \alpha = \frac{1}{2}(\sqrt{\lambda^2 + 4} + \lambda), \quad \beta = \frac{1}{2}(\sqrt{\lambda^2 + 4} - \lambda).$$

Then $-\alpha, \beta$ are the two eigenvalues of the linearized part at $(0, 0)$. Since $-\alpha + \beta = -\lambda < 0$, $(0, 0)$ is a dissipative saddle provided that $\lambda > 0$.

We fix λ throughout of this section and let γ_λ be as in Proposition 3.1. We consider a class of C^N function

$$\Phi_{c_1, c_2} : \mathbb{R} \rightarrow [\min\{c_1, c_2\}, \max\{c_1, c_2\}]$$

such that

$$(3.3) \quad \lim_{t \rightarrow -\infty} \Phi_{c_1, c_2}(t) = c_1, \quad \lim_{t \rightarrow +\infty} \Phi_{c_1, c_2}(t) = c_2$$

where c_1, c_2 are two real numbers. We assume that the C^N -norm of Φ_{c_1, c_2} is uniformly bounded by a constant, which we denote as $\|\Phi_{c_1, c_2}\|$. We add an external forcing $\mu q^2 \Phi_{\eta^-, \eta^+}(t) \sin \omega t$ to equation (3.1) and perturb its damping by $\mu \Phi_{\tau^-, \tau^+}(t) q^2$ to have

$$(3.4) \quad \frac{d^2 q}{dt^2} + (\lambda - (\gamma_\lambda + \mu \Phi_{\tau^-, \tau^+}(t)) q^2) \frac{dq}{dt} - q + q^3 = \mu q^2 \Phi_{\eta^-, \eta^+}(t) \sin \omega t$$

where $\tau^\pm, \eta^\pm, \mu, \omega$ are forcing parameters. As we will see momentarily, our admitting of four arbitrary constants τ^\pm, η^\pm are to create arbitrary combinations of m^\pm, M^\pm for the Melnikov functions of the corresponding equations. The use of sine function is to assure that the limit values of $\mathcal{W}(t)$ are densely approached by L -sequences where L is determined by the forcing frequency ω .

We re-write equation (3.4) as

$$(3.5) \quad \begin{aligned} \frac{dq}{dt} &= p \\ \frac{dp}{dt} &= -(\lambda - \gamma_\lambda q^2) p + q - q^3 + \mu (\Phi_{\tau^-, \tau^+}(t) q^2 p + q^2 \Phi_{\eta^-, \eta^+}(t) \sin \omega t). \end{aligned}$$

To put the linear part of equation (3.5) in a canonical form, we introduce new variables (x, y) so that

$$(3.6) \quad q = x + \alpha y, \quad p = -\alpha x + y,$$

where $\alpha = \frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4})$ is as in (3.2). In reverse we have

$$(3.7) \quad x = \frac{1}{1 + \alpha^2}(q - \alpha p) \quad y = \frac{1}{1 + \alpha^2}(\alpha q + p).$$

The new equations in (x, y) are

$$(3.8) \quad \begin{aligned} \frac{dx}{dt} &= -\alpha x + f(x, y) + \mu(A(x, y)\Phi_{\tau^-, \tau^+}(t) + C(x, y)\Phi_{\eta^-, \eta^+}(t) \sin \omega t), \\ \frac{dy}{dt} &= \beta y + g(x, y) + \mu(B(x, y)\Phi_{\tau^-, \tau^+}(t) + D(x, y)\Phi_{\eta^-, \eta^+}(t) \sin \omega t), \end{aligned}$$

where $\beta = \alpha^{-1}$ is again as in (3.2), and

$$\begin{aligned} f(x, y) &= \frac{\alpha}{1 + \alpha^2} (\gamma_\lambda(x + \alpha y)^2(y - \alpha x) + (x + \alpha y)^3), \\ g(x, y) &= \frac{-1}{1 + \alpha^2} (\gamma_\lambda(x + \alpha y)^2(y - \alpha x) + (x + \alpha y)^3); \\ A(x, y) &= \frac{\alpha}{1 + \alpha^2} (x + \alpha y)^2(y - \alpha x), \quad B(x, y) = \frac{-1}{1 + \alpha^2} (x + \alpha y)^2(y - \alpha x); \\ C(x, y) &= \frac{\alpha}{1 + \alpha^2} (x + \alpha y)^2, \quad D(x, y) = \frac{-1}{1 + \alpha^2} (x + \alpha y)^2. \end{aligned}$$

Observe that the functions f, g, A, B, C, D are all functions of x, y of order at least two at $(x, y) = (0, 0)$.

Let ℓ_λ be the homoclinic solution of equation (3.1) from Proposition 3.1. In the coordinate (x, y) , let us write $\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t))$. Let

$$(u_\lambda(t), v_\lambda(t)) = \frac{1}{\left| \frac{d}{dt} \ell_\lambda(t) \right|} \frac{d}{dt} \ell_\lambda(t)$$

be the tangent vector of ℓ_λ at $\ell_\lambda(t)$, and

$$(3.9) \quad \begin{aligned} E_\lambda(t) &= v_\lambda^2(t)(-\alpha + \partial_x f(a_\lambda(t), b_\lambda(t))) + u_\lambda^2(t)(\beta + \partial_y g(a_\lambda(t), b_\lambda(t))) \\ &\quad - u_\lambda(t)v_\lambda(t)(\partial_y f(a_\lambda(t), b_\lambda(t)) + \partial_x g(a_\lambda(t), b_\lambda(t))). \end{aligned}$$

Then the Melnikov function for equation (3.5) is

$$(3.10) \quad \mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t)$$

where

$$(3.11) \quad \begin{aligned} \mathcal{W}_1(t) &= \int_{-\infty}^{+\infty} (v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s))\Phi_{\tau^-, \tau^+}(s+t)e^{-\int_0^s E_\lambda(\tau)d\tau} ds, \\ \mathcal{W}_2(t) &= \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s))\Phi_{\eta^-, \eta^+}(s+t) \sin \omega(s+t)e^{-\int_0^s E_\lambda(\tau)d\tau} ds, \end{aligned}$$

in which $A_\lambda(t) = A(a_\lambda(t), b_\lambda(t))$ and $B_\lambda(t), C_\lambda(t), D_\lambda(t)$ are similar.

Denote

$$(3.12) \quad \begin{aligned} J &= \int_{-\infty}^{+\infty} (v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s))e^{-\int_0^s E_\lambda(\tau)d\tau} ds, \\ J_s &= \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s)) \sin(\omega s)e^{-\int_0^s E_\lambda(\tau)d\tau} ds, \\ J_c &= \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s)) \cos(\omega s)e^{-\int_0^s E_\lambda(\tau)d\tau} ds. \end{aligned}$$

Recall that $m^\pm = \liminf_{t \rightarrow \pm\infty} \mathcal{W}(t)$, $M^\pm = \limsup_{t \rightarrow \pm\infty} \mathcal{W}(t)$. We have

Proposition 3.2. *Let $R > 0$ be fixed and assume $\omega \in (0, R)$. Then there exists λ_0 sufficiently small, depending on R , such that for all $\lambda \in (0, \lambda_0)$,*

$$(a) \quad J > 0, \quad J_s^2 + J_c^2 \neq 0;$$

- (b) $m^\pm = J\tau^\pm - \sqrt{J_s^2 + J_c^2}\eta^\pm$, $M^\pm = J\tau^\pm + \sqrt{J_s^2 + J_c^2}\eta^\pm$; and
(c) m^\pm , M^\pm are all densely approached by L -sequences where $L = \frac{4\pi}{\omega}$.

Proof: (a) is Proposition 5.2 in [W]. See [W] for a detailed proof.

For (b) we first recall that $\lim_{t \rightarrow +\infty} \Phi_{\tau^-, \tau^+}(t) = \tau^+$. So for all $\varepsilon > 0$, there exists t_0 sufficiently large so that

$$|\Phi_{\tau^-, \tau^+}(t) - \tau^+| < \varepsilon$$

for all $t \geq t_0$. Write

$$\begin{aligned} \mathcal{W}_1(t) &= \int_{-\infty}^{-t+t_0} (v(s)A_\lambda(s) - u(s)B_\lambda(s))\Phi_{\tau^-, \tau^+}(s+t)e^{-\int_0^s E_\lambda(\tau)d\tau} ds \\ &\quad + \int_{-t+t_0}^{+\infty} (v(s)A_\lambda(s) - u(s)B_\lambda(s))\Phi_{\tau^-, \tau^+}(s+t)e^{-\int_0^s E_\lambda(\tau)d\tau} ds. \end{aligned}$$

As $t \rightarrow +\infty$, $-t+t_0 \rightarrow -\infty$, and the first integral goes to zero. For the second integral we observe that, for all $s \geq -t+t_0$, $|\Phi_{\tau^-, \tau^+}(t+s) - \tau^+| < \varepsilon$. It then follows that the value of the second integral is $K\varepsilon$ -close to the value of the integral in which we replace $\Phi_{\tau^-, \tau^+}(s+t)$ in the same integral by τ^+ . Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{t \rightarrow +\infty} \mathcal{W}_1(t) = \tau^+ J.$$

The limit for $\mathcal{W}_1(t)$ as $t \rightarrow -\infty$ is similarly obtained. In summary we have

$$(3.13) \quad \lim_{t \rightarrow \pm\infty} \mathcal{W}_1(t) = \tau^\pm J.$$

For $\mathcal{W}_2(t)$ we have

$$(3.14) \quad \mathcal{W}_2(t) = \cos(\omega t)J_s(t) + \sin(\omega t)J_c(t)$$

where

$$\begin{aligned} J_s(t) &= \int_{-\infty}^{+\infty} (v(s)C_\lambda(s) - u(s)D_\lambda(s))\Phi_{\eta^-, \eta^+}(s+t) \sin(\omega s) e^{-\int_0^s E_\lambda(\tau)d\tau} ds, \\ J_c(t) &= \int_{-\infty}^{+\infty} (v(s)C_\lambda(s) - u(s)D_\lambda(s))\Phi_{\eta^-, \eta^+}(s+t) \cos(\omega s) e^{-\int_0^s E_\lambda(\tau)d\tau} ds. \end{aligned}$$

First we prove

$$(3.15) \quad \lim_{t \rightarrow \pm\infty} J_s(t) = \eta^\pm J_s, \quad \lim_{t \rightarrow \pm\infty} J_c(t) = \eta^\pm J_c.$$

These limit values are obtained in the same way as those of $\mathcal{W}_1(t)$. (3.14) and (3.15) are then combined to give

$$(3.16) \quad \liminf_{t \rightarrow \pm\infty} \mathcal{W}_2(t) = -\eta^\pm \sqrt{J_s^2 + J_c^2}; \quad \limsup_{t \rightarrow \pm\infty} \mathcal{W}_2(t) = \eta^\pm \sqrt{J_s^2 + J_c^2}.$$

(b) now follows from (3.13), (3.14), and (3.16).

(c) also follows from (3.13), (3.14), and (3.16). □

We also have the following estimates on $m = \inf_{t \in \mathbb{R}} \mathcal{W}(t)$ and $M = \sup_{t \in \mathbb{R}} \mathcal{W}(t)$.

Proposition 3.3. *Let $\mathcal{W}(t)$ be as in (3.10). We have*

$$\begin{aligned} m &\geq \frac{1}{2} \left((\tau^+ + \tau^-)J - (\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} \right) - K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|), \\ M &\leq \frac{1}{2} \left((\tau^+ + \tau^-)J + (\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} \right) + K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|). \end{aligned}$$

Proof: By definition

$$(3.17) \quad |\mathcal{W}(t) - \tau^\pm J - \eta^\pm (J_s \cos \omega t + J_c \sin \omega t)| \leq K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|)$$

where

$$K = \int_{-\infty}^{+\infty} (|v(s)A_\lambda(s) - u(s)B_\lambda(s)| + |v(s)C_\lambda(s) - u(s)D_\lambda(s)|) e^{-\int_0^s E_\lambda(\tau) d\tau} ds.$$

Estimates on m and M follows directly from (3.17). \square

We are ready to discuss the applications of Theorems 2.1-2.7 to equation (3.4). First let us fix λ sufficiently small. We also assume that λ is such that the non-resonance condition (H)(i) hold for α and β of (3.2). Let γ_λ be the value of γ given in Proposition 3.1(a).

Applications of Theorem 2.1: For the assumptions of Theorem 2.1(a) to hold it suffices for us to have

$$\min(m^+, m^-) < 0 < \max(M^+, M^-).$$

By Proposition 3.2(b), this is

$$(3.18) \quad \begin{aligned} \min(J\tau^- - \sqrt{J_s^2 + J_c^2}\eta^-, J\tau^+ - \sqrt{J_s^2 + J_c^2}\eta^+) &< 0, \\ \max(J\tau^- + \sqrt{J_s^2 + J_c^2}\eta^-, J\tau^+ + \sqrt{J_s^2 + J_c^2}\eta^+) &> 0. \end{aligned}$$

We derive from (3.18) three particular cases. Note that if $c_1 = c_2$, then $\Phi_{c_1, c_2}(t)$ is a constant function by definition.

Case (i). $\eta^+ = \eta^- = 0$. In this case equation (3.4) becomes

$$(3.19) \quad \frac{d^2q}{dt^2} + (\lambda - (\gamma_\lambda + \mu\Phi_{\tau^-, \tau^+}(t))q^2) \frac{dq}{dt} - q + q^3 = 0.$$

For this equation to have homoclinic solutions, it suffice to have

$$\tau^+ \cdot \tau^- < 0$$

from (3.18). This is an example for which the nonautonomous perturbations create homoclinic intersections but not complicated structures.

Case (ii). $\tau^+ = \tau^- = 0$. In this case the condition for Theorem 2.1(a) derived from (3.18) is $\eta^\pm \neq 0$. Equation (3.4) becomes

$$(3.20) \quad \frac{d^2q}{dt^2} + (\lambda - \gamma_\lambda q^2) \frac{dq}{dt} - q + q^3 = \mu q^2 \Phi_{\eta^-, \eta^+}(t) \sin \omega t.$$

As far as $\eta^- \neq \eta^+$, this nonautonomous equation has no time periodicity.

Case (iii). $\tau^- = \tau^+ := \rho$, and we assume $\eta^- = \eta^+ = 1$. In this case equation (3.4) becomes

$$(3.21) \quad \frac{d^2q}{dt^2} + (\lambda - (\gamma_\lambda + \mu\rho)q^2) \frac{dq}{dt} - q + q^3 = \mu q^2 \sin \omega t,$$

and the condition for Theorem 2.1(a) from (3.18) is

$$|\rho| < \frac{\sqrt{J_s^2 + J_c^2}}{J}.$$

This is exactly the sufficient condition for the existence of homoclinic intersections derived from the traditional Melnikov method for equation (3.21).

For the assumptions of Theorem 2.1(b) and (c) we use estimates in Proposition 3.3. We obtain a sufficient condition for Theorem 2.1(b) as

$$(\tau^+ + \tau^-)J > (\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} + 2K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|)$$

and a sufficient condition for Theorem 2.1(c) as

$$(\tau^+ + \tau^-)J < -(\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2} - 2K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|).$$

Various conditions for equations (3.19)-(3.21) are derived accordingly. We skip the details.

Applications of Theorem 2.2: First we apply Theorem 2.2 to equation (3.19) for which $\eta^+ = \eta^- = 0$. We have

Proposition 3.4. *The conditions of Theorem 2.2 hold for equation (3.19) if*

$$(3.22) \quad \frac{1}{2}(\tau^+ + \tau^-) > (K|\tau^+ - \tau^-| + \tilde{K}\beta^{-1}\|\Phi_{\tau^-, \tau^+}\|)J^{-1}.$$

Proof: From Proposition 3.3, we obtain

$$\tau^+ + \tau^- > 2K|\tau^+ - \tau^-|J^{-1}$$

as a sufficient condition for $m > 0$, and we have

$$(3.23) \quad m \geq \frac{1}{2}(\tau^+ + \tau^-)J - K|\tau^+ - \tau^-|.$$

To apply Theorem 2.2 we also need to verify

$$(3.24) \quad \frac{1}{\beta} \sup_{t \in \mathbb{R}} \left| \frac{\mathcal{W}'(t)}{\mathcal{W}(t)} \right| < 1.$$

We use (3.23) for m to bound $|\mathcal{W}(t)|$. We also denote

$$\tilde{K} = \int_{-\infty}^{+\infty} (|v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s)| + |v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s)|) e^{-\int_0^s E_\lambda(\tau) d\tau} ds.$$

For $\mathcal{W}'(t)$ we have

$$(3.25) \quad \begin{aligned} |\mathcal{W}'(t)| &= \left| \int_{-\infty}^{+\infty} (v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s))\Phi'_{\tau^-, \tau^+}(s+t) e^{-\int_0^s E_\lambda(\tau) d\tau} ds \right| \\ &< \tilde{K}\|\Phi_{\tau^-, \tau^+}\|. \end{aligned}$$

So for (3.24) to hold it suffices to have

$$\frac{\tilde{K}\|\Phi_{\tau^-, \tau^+}\|}{\frac{1}{2}(\tau^+ + \tau^-)J - K|\tau^+ - \tau^-|} < \beta,$$

from which we obtain

$$\frac{1}{2}(\tau^+ + \tau^-) > (K|\tau^+ - \tau^-| + \tilde{K}\beta^{-1}\|\Phi_{\tau^-, \tau^+}\|)J^{-1}.$$

□

It follows from Proposition 3.4 and Theorem 2.2 that equation (3.19) satisfying (3.22) has an attracting integral manifold of Levinson type in extended phase space.

We now derive a general condition for Theorem 2.2 to hold for equation (3.4).

Proposition 3.5. *The conditions of Theorem 2.2 hold for equation (3.4) if*

$$|\omega| < \frac{\beta \left((\tau^+ + \tau^-)J - (\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2} \right) - 2\beta K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|) - 2\tilde{K}\|\Phi\|}{(\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2} + 2K|\eta^+ - \eta^-|}.$$

where $\|\Phi\| = \|\Phi_{\eta^-, \eta^+}\| + \|\Phi_{\tau^-, \tau^+}\|$.

Proof: We estimate $\mathcal{W}'_2(t)$ as follows

$$\mathcal{W}'_2(t) = \omega(J_s(t) \sin \omega t + J_c(t) \cos \omega t) + J'_s(t) \cos \omega t + J'_c(t) \sin \omega t.$$

The first two terms are estimated similar to (3.17) and for the last two terms we use

$$(3.26) \quad |J'_s(t)|, |J'_c(t)| < \tilde{K} \|\Phi_{\eta^-, \eta^+}\|.$$

It follows that

$$|\mathcal{W}'_2(t)| < |\omega| \left(\frac{1}{2}(\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} + K|\eta^+ - \eta^-| \right) + \tilde{K} \|\Phi_{\eta^-, \eta^+}\|.$$

Combined with (3.25), which is basically an estimate for $\mathcal{W}'_1(t)$, we obtain

$$|\mathcal{W}'(t)| < |\omega| \left(\frac{1}{2}(\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} + K|\eta^+ - \eta^-| \right) + \tilde{K} \|\Phi\|.$$

To have (3.24) hold it suffices to have

$$\frac{|\omega| \left(\frac{1}{2}(\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} + K|\eta^+ - \eta^-| \right) + \tilde{K} \|\Phi\|}{\frac{1}{2} \left((\tau^+ + \tau^-)J - (\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} \right) - K(|\eta^+ - \eta^-| + |\tau^+ - \tau^-|)} \leq \beta,$$

from which it follows that

$$|\omega| < \frac{\beta \left((\tau^+ + \tau^-)J - (\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} \right) - 2\beta K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|) - 2\tilde{K} \|\Phi\|}{(\eta^+ + \eta^-) \sqrt{J_s^2 + J_c^2} + 2K|\eta^+ - \eta^-|}.$$

□

Qualitatively, Proposition 3.5 claims the following. Assume η^\pm are not zeros (this is to assume that there are persistent oscillations as $t \rightarrow \pm\infty$). Then for the stable scenario of Theorem 2.2 to happen we need two things. First we need to make $\tau^+ + \tau^-$ large. This is to pull all of the images of Σ^- to the correct side of the local stable manifold of $(x, y) = (0, 0)$ on Σ^+ (See Fig. 4(b)). Second we need to make $|\omega|$, the frequency of oscillations at $t = \pm\infty$, small. The second requirement is in line with Theorem 2.7, which claims that large frequency of oscillations at $t = \pm\infty$ tends to create complicated structures.

Applications of Theorems 2.3: The condition for Theorem 2.3 is

$$\max(m^+, m^-) < 0 < \min(M^+, M^-).$$

By using Proposition 3.1(b), this is transformed to

$$(3.27) \quad \begin{aligned} \max(J\tau^+ - \sqrt{J_s^2 + J_c^2}\eta^+, J\tau^- - \sqrt{J_s^2 + J_c^2}\eta^-) &< 0, \\ \min(J\tau^+ + \sqrt{J_s^2 + J_c^2}\eta^+, J\tau^- + \sqrt{J_s^2 + J_c^2}\eta^-) &> 0. \end{aligned}$$

It follows from (3.27) that (i) Theorem 2.3 can not be applied to equation (3.19), for the inequalities of (3.27) become self-conflicting if $\eta^+ = \eta^- = 0$, and (ii) Theorem 2.3 always applies to equation (3.20), for it is trivial for (3.27) to hold if $\tau^+ = \tau^- = 0$. A necessary condition for Theorem 2.3 to apply to equation (3.21) is, denoting $\tau^+ = \tau^- := \rho$,

$$|\rho| < \frac{\sqrt{J_s^2 + J_c^2}}{J}.$$

Applications of Theorem 2.4: We conclude that $\Lambda = \emptyset$ if either

$$M^- = J\tau^- + \sqrt{J_s^2 + J_c^2}\eta^- < 0,$$

or

$$M^+ = J\tau^+ + \sqrt{J_s^2 + J_c^2}\eta^+ < 0.$$

Applications of Theorems 2.5: The first condition for Theorem 2.5 is $m^- < 0 < m^+$. By using Proposition 3.2(b), this is to require

$$\tau^- < \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^-; \quad \tau^+ > \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+.$$

We also need $M^- > 0$, which requires

$$\tau^- > -\frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^-.$$

Putting these two together we obtain

$$|\tau^-| < \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^-; \quad \tau^+ > \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+.$$

These are the conditions for Theorem 2.5 to apply to equation (3.4). We note that these inequalities become self-conflicting if $\tau^- = \tau^+$, $\eta^+ = \eta^-$. This agrees with our previous observation that Theorem 2.5 only applies to equations that have no periodicity.

Applications of Theorems 2.6: The first condition for Theorem 2.6 is $m^+ < 0 < m^-$. By using Proposition 3.2(b), this is to require

$$\tau^+ < \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+; \quad \tau^- > \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^-.$$

We also need $M^+ > 0$, which requires

$$\tau^+ > -\frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+.$$

Putting these two together we obtain

$$|\tau^+| < \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+; \quad \tau^- > \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^-.$$

These are the conditions for Theorem 2.6 to apply to equation (3.4). We again note that these inequalities become self-conflicting if $\tau^- = \tau^+$, $\eta^+ = \eta^-$.

Applications of Theorem 2.7: In order to apply Theorem 2.7 to equation (3.4), we first need $m^\pm > 0$, which implies

$$(3.28) \quad \tau^+ > \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+; \quad \tau^- > \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^-.$$

We also need

$$M^+ > m^+ e^{3\beta L}.$$

By using the values of m^+ and M^+ from Proposition 3.2(b) and the value of L from Proposition 3.2(c), we obtain

$$\frac{\tau^+ + \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+}{\tau^+ - \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+} > e^{12\pi\beta\omega^{-1}}.$$

From this we have

$$(3.29) \quad \omega^{-1} < \frac{1}{12\pi\beta} \ln \frac{\tau^+ + \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+}{\tau^+ - \frac{\sqrt{J_s^2 + J_c^2}}{J} \eta^+},$$

which together with (3.28) are sufficient conditions for Theorem 2.7 to hold for equation (3.4).

4. Return maps in extended phase space

In this section, we study the Poincaré return map in the extended phase space. In Sect. 4.1 we introduce a coordinate change that linearizes equation (2.2) in B_ε . In Sect. 4.2 we derive a normal form for equation (2.2) around the entire length of the homoclinic loop ℓ out of $B_{\frac{1}{2}\varepsilon}$. The Poincaré sections Σ^\pm are introduced in precise terms in Sect. 4.3. We then compute the return maps $\mathcal{R} : \Sigma^- \rightarrow \Sigma^-$ based on the equations derived in Sects. 4.1 and 4.2.

Two small scales: Two small quantities $\mu \ll \varepsilon \ll 1$ represent two small scales of different magnitude. ε represents the size of a small neighborhood of $(x, y) = (0, 0)$ that makes the linearization of Sect. 4.1 valid. Associated to ε is the small neighborhood

$$B_\varepsilon = \{(x, y) : x^2 + y^2 < 4\varepsilon^2\}, \quad \mathcal{B}_\varepsilon = B_\varepsilon \times \mathbb{R},$$

and L^+ and $-L^-$, the respective times the homoclinic solution $\ell(t)$ enters $B_{\frac{1}{2}\varepsilon}$ in the positive and the negative directions of time. The quantities L^+ and L^- are related, both are completely determined by ε and $\ell(t)$. The parameter $\mu (\ll \varepsilon)$ controls the magnitude of the time periodic perturbation.

Notation: The letter K is used throughout to generically represent constants that are *independent* of μ . The precise value of K is allowed to change from line to line. In occasions, a specific constant is used in different places. There are also times we need to distinguish two K in the same line. We will then use subscripts to denote them as K_0, K_1, \dots . We will also make distinctions between constants that are dependent of ε and those do not by making such dependency explicit. A constant that depends on ε is written as $K(\varepsilon)$. A constant written as K is independent of ε .

4.1. Local linearization. Let X, Y be such that

$$(4.1) \quad \begin{aligned} x &= X + \mathbb{P}(X, Y) + \mu \tilde{\mathbb{P}}(X, Y, t), \\ y &= Y + \mathbb{Q}(X, Y) + \mu \tilde{\mathbb{Q}}(X, Y, t), \end{aligned}$$

where $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$ as functions of X and Y are C^r for $r \geq 2$ on $|(X, Y)| < 2\varepsilon$, $t \in \mathbb{R}$ and the values of these functions and their first derivatives with respect to X and Y at $(X, Y) = (0, 0)$ are all zero. As is explicitly indicated in (4.1), \mathbb{P} and \mathbb{Q} are independent of t and μ .

Proposition 4.1. *For each integer $r > 0$, there exists an integer $N_0 = N_0(k, \alpha, \beta) > r$ such that if f, g, P, Q are C^N for $N \geq N_0$ with uniformly bounded derivatives and $-\alpha, \beta$ satisfies the nonresonant conditions up to order N_0 , then there exists a C^r -coordinate transformation in the form of (4.1) defined on $B_\varepsilon \times \mathbb{R} \times [-\mu_0, \mu_0]$ that transforms equation (2.2) into*

$$\frac{dX}{dt} = -\alpha X, \quad \frac{dY}{dt} = \beta Y$$

where B_ε is a small neighborhood of $(X, Y) = (0, 0)$ and μ_0 is a positive constant. Moreover, the C^r -norms of $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$ as functions of X, Y, t are all uniformly bounded by a constant K that is independent of both ε and μ on $(X, Y) \in B_\varepsilon$, $t \in \mathbb{R}$.

Proposition 4.1 follows from the main theorem in Appendix A.

4.2. A standard form around homoclinic loop. In this subsection we derive a standard form for equation (2.2) around the homoclinic loop of equation (2.1) outside of $B_{\frac{1}{2}\varepsilon}$.

Let us regard t in $\ell(t) = (a(t), b(t))$ not as time, but as a parameter that parameterizes the curve ℓ in (x, y) -space. We replace t by s to write this homoclinic loop as $\ell(s) = (a(s), b(s))$. We have

$$(4.2) \quad \frac{da(s)}{ds} = -\alpha a(s) + f(a(s), b(s)), \quad \frac{db(s)}{ds} = \beta b(s) + g(a(s), b(s)).$$

By definition,

$$(4.3) \quad \begin{aligned} u(s) &= \frac{-\alpha a(s) + f(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}, \\ v(s) &= \frac{\beta b(s) + g(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}. \end{aligned}$$

Let

$$\mathbf{e}(s) = (v(s), -u(s)).$$

We now introduce new variables (s, z) such that

$$(x, y) = \ell(s) + z\mathbf{e}(s).$$

This is to say that

$$(4.4) \quad x = x(s, z) := a(s) + v(s)z, \quad y = y(s, z) := b(s) - u(s)z.$$

We derive the equations for (2.2) in new variables (s, z) defined through (4.4). Differentiating (4.4) we obtain

$$(4.5) \quad \begin{aligned} \frac{dx}{dt} &= (-\alpha a(s) + f(a(s), b(s)) + v'(s)z) \frac{ds}{dt} + v(s) \frac{dz}{dt}, \\ \frac{dy}{dt} &= (\beta b(s) + g(a(s), b(s)) - u'(s)z) \frac{ds}{dt} - u(s) \frac{dz}{dt}, \end{aligned}$$

where $u'(s) = \frac{du(s)}{ds}$, $v'(s) = \frac{dv(s)}{ds}$. Let us denote

$$\begin{aligned} F(s, z) &= -\alpha(a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)), \\ G(s, z) &= \beta(b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)), \\ P(s, z, t) &= P(a(s) + zv(s), b(s) - zu(s), t), \\ Q(s, z, t) &= Q(a(s) + zv(s), b(s) - zu(s), t). \end{aligned}$$

By using equation (2.2), we obtain from equation (4.5) the new equations for s, z as

$$\begin{aligned} \frac{ds}{dt} &= \frac{u(s)F(s, z) + v(s)G(s, z) + \mu(u(s)P(s, z, t) + v(s)Q(s, z, t))}{\sqrt{F(s, 0)^2 + G(s, 0)^2} + z(u(s)v'(s) - v(s)u'(s))}, \\ \frac{dz}{dt} &= v(s)F(s, z) - u(s)G(s, z) + \mu(v(s)P(s, z, t) - u(s)Q(s, z, t)). \end{aligned}$$

We re-write these equations as

$$(4.6) \quad \begin{aligned} \frac{ds}{dt} &= 1 + zw_1(s, z, t) + \frac{\mu(u(s)P(s, z, t) + v(s)Q(s, z, t))}{\sqrt{F(s, 0)^2 + G(s, 0)^2}}, \\ \frac{dz}{dt} &= E(s)z + z^2w_2(s, z) + \mu(v(s)P(s, z, t) - u(s)Q(s, z, t)), \end{aligned}$$

where

$$\begin{aligned} E(s) &= v^2(s)(-\alpha + \partial_x f(a(s), b(s))) + u^2(s)(\beta + \partial_y g(a(s), b(s))), \\ &\quad - u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s))). \end{aligned}$$

Also in the rest of this section we let $K_0(\varepsilon)$ be a given constant independent of μ and regard equation (4.6) as been defined on

$$\{s \in [-2L^-, 2L^+], t \in \mathbb{R} \mid |z| < K_0(\varepsilon)\mu\}.$$

The C^r -norms of $w_1(s, z, t)$ and $w_2(s, z)$ are bounded above by a constant $K(\varepsilon)$.

Finally we re-scale the variable z by letting

$$(4.7) \quad Z = \mu^{-1}z.$$

We arrive at the following equations

$$(4.8) \quad \begin{aligned} \frac{ds}{dt} &= 1 + \mu \tilde{w}_1(s, Z, t), \\ \frac{dZ}{dt} &= E(s)Z + \mu \tilde{w}_2(s, Z, t) + (v(s)P(s, 0, t) - u(s)Q(s, 0, t)), \end{aligned}$$

where (s, Z, t) is defined on

$$\mathbf{D} = \{(s, Z, t) : s \in [-2L^-, 2L^+], |Z| \leq K_0(\varepsilon), t \in \mathbb{R}\}.$$

Here we assume that μ is sufficiently small so that

$$\mu \ll \min_{s \in [-2L^-, 2L^+]} (F(s, 0)^2 + G(s, 0)^2).$$

Again, the C^r -norms of the functions \tilde{w}_1, \tilde{w}_2 are uniformly bounded by a constant $K(\varepsilon)$ on \mathbf{D} . Equation (4.8) is the one we need. Note that

$$P(s, 0, t) = P(a(s), b(s), t), \quad Q(s, 0, t) = Q(a(s), b(s), t).$$

4.3. Poincaré sections Σ^\pm . We define Σ^\pm inside of $\mathcal{B}_\varepsilon \cap \mathbf{D}$ by letting

$$\Sigma^- = \{(X, Y, t) : Y = \varepsilon, |X| < \mu, t \in \mathbb{R}\},$$

and

$$\Sigma^+ = \{(X, Y, \theta) : X = \varepsilon, |Y| < K_1(\varepsilon)\mu, t \in \mathbb{R}\}.$$

$K_1(\varepsilon)$ will be precisely defined momentarily.

Let $q \in \Sigma^+$ or Σ^- . We can also use (s, Z, t) -coordinate to represent q , for which the defining equations for Σ^\pm are not as direct. To compute the return maps, we need to first address two issues that are technical in nature. First, we need to derive the defining equations on Σ^\pm for (s, Z, t) . Second, we need to be able to change coordinates from (X, Y, t) to (s, Z, t) and vice versa on Σ^\pm . We start with some preparations in notation.

Notation: The intended formula for the return maps would inevitably contain terms that are explicit and terms that are implicit. Implicit terms are usually “error” terms, and the usefulness of a derived formula would depend completely on how well the error terms are controlled. We aim at C^r -control on all error terms. The derivations of the return maps involve a composition of maps and multiple coordinate changes. To facilitate our presentation, from this point on we adopt specific conventions for indicating controls on magnitude. For a given constant, we write $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\mu)$ to indicate that the magnitude of the constant is bounded by K , $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. For a function of a set V of variables on a specific domain, we write $\mathcal{O}_V(1)$, $\mathcal{O}_V(\varepsilon)$ or $\mathcal{O}_V(\mu)$ to indicate that the C^r -norm of the function on the specified domain is bounded by K , $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in the notation. For example, $\mathcal{O}_{Z,t}(\mu)$ represents a function of Z, t , the C^r -norm of which is bounded above by $K(\varepsilon)\mu$.

In what follows we let

$$\mathbb{X} = \mu^{-1}X, \quad \mathbb{Y} = \mu^{-1}Y.$$

Proposition 4.2. *Coordinate conversions on Σ^\pm are as follows:*

- (a) On Σ^+ , (i) $s = L^+ + \mathcal{O}_{Z,t}(\mu)$, (ii) $\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))Z + \mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu)$.
- (b) On Σ^- , (i) $s = -L^- + \mathcal{O}_{Z,t}(\mu)$, (ii) $Z = (1 + \mathcal{O}(\varepsilon))\mathbb{X} + \mathcal{O}_t(1) + \mathcal{O}_{\mathbb{X},t}(\mu)$.

Proposition 4.2 is proved in Appendix B.

4.4. **The map** $\mathcal{M} : \Sigma^- \rightarrow \Sigma^+$. Let

$$(4.9) \quad \mathcal{W}_L(t) = \int_{-L^-}^{L^+} (v(s)P(a(s), b(s), s+t) - u(s)Q(a(s), b(s), s+t))e^{-\int_0^s E(\tau)d\tau} ds.$$

We also write

$$(4.10) \quad P_L = e^{\int_{-L^-}^{L^+} E(s)ds}, \quad P_L^+ = e^{\int_0^{L^+} E(s)ds}.$$

Note that for P_L we integrate from $s = -L^-$ to $s = L^+$, while for P_L^+ the integration starts from $s = 0$. First we have

Lemma 4.1.

$$P_L \sim \varepsilon^{\frac{\alpha}{\beta} - \frac{\beta}{\alpha}} \ll 1, \quad P_L^+ \sim \varepsilon^{-\frac{\beta}{\alpha}} \gg 1.$$

Proof: By the definition of L^\pm we have

$$\varepsilon \sim e^{-\alpha L^+} \sim e^{-\beta L^-}.$$

We also have

$$P_L \sim e^{\beta L^+ - \alpha L^-}, \quad P_L^+ \sim e^{\beta L^+}.$$

Lemma 4.1 follows directly from these estimates. \square

For $q = (s^-, Z, t_0) \in \Sigma^-$, the value of s^- is uniquely determined by that of (Z, t_0) through Proposition 4.2(b)(i). So we can use (Z, t_0) for q . Let $(s(t), Z(t))$ be the solution of equation (4.8) initiated from (s^-, Z) at $t = t_0$, and \hat{t} be the time $(s(\hat{t}), Z(\hat{t}))$ hits Σ^+ . In what follows we write

$$s^+ = s(\hat{t}), \quad \hat{Z} = Z(\hat{t}).$$

Proposition 4.3. Denote $(\hat{Z}, \hat{t}) = \mathcal{M}(Z, t_0)$. We have

$$(4.11) \quad \begin{aligned} \hat{Z} &= P_L^+ \mathcal{W}_L(t_0 + L^-) + P_L Z + \mathcal{O}_{Z, t_0}(\mu), \\ \hat{t} &= t_0 + L^+ + L^- + \mathcal{O}_{Z, t_0}(\mu). \end{aligned}$$

Proof: We re-write equation (4.8) as

$$(4.12) \quad \begin{aligned} \frac{dZ}{ds} &= E(s)Z + (v(s)P(a(s), b(s), t) - u(s)Q(a(s), b(s), t)) + \mathcal{O}_{s, Z, t}(\mu), \\ \frac{dt}{ds} &= 1 + \mathcal{O}_{s, Z, t}(\mu) \end{aligned}$$

on $\mathbf{D} \times (0, R_\mu^{-1})$ where

$$\mathbf{D} = \{(s, Z, t) : s \in [-2L^-, 2L^+], |Z| < K_1(\varepsilon), t \in \mathbb{R}\}.$$

From the second item of (4.12) we obtain

$$t = t_0 + s + L^- + \mathcal{O}_{s, Z, t_0}(\mu),$$

from which the claim on \hat{t} follows. Substituting it into the first item of (4.12) we obtain

$$\frac{dZ}{ds} = E(s)Z + (v(s)P(a(s), b(s), t_0 + s + L^-) - u(s)Q(a(s), b(s), t_0 + s + L^-)) + \mathcal{O}_{s, Z, t_0}(\mu),$$

from which it follows that

$$\hat{Z} = P_L(Z + \Phi_L(t_0)) + \mathcal{O}_{Z, t_0}(\mu)$$

where P_L is as in (4.10) and

$$(4.13) \quad \Phi_L(t) = \int_{-L^-}^{L^+} (v(s)P(a(s), b(s), t + s + L^-) - u(s)Q(a(s), b(s), t + s + L^-)) \cdot e^{-\int_{-L^-}^s E(\tau)d\tau} ds.$$

We also observe that

$$P_L \Phi_L(t) = P_L^+ \cdot \mathcal{W}_L(t + L^-).$$

This proved the line for \hat{Z} .

Let

$$(4.14) \quad K_1(\varepsilon) = \max_{t \in \mathbb{R}, s \in [-2L^-, 2L^+]} P_s(2 + |\Phi_s(t)|)$$

where P_s and Φ_s are obtained by replacing L^+ with s in P_L and Φ_L . $K_1(\varepsilon)$ is the one we use for \mathbf{D} and Σ^+ . The solutions of (4.12) initiated on Σ^- will stay inside of \mathbf{D} before hitting Σ^+ . \square

4.5. The return map \mathcal{R} . First we compute $\mathcal{N} : \Sigma^+ \rightarrow \Sigma^-$. For $(\mathbb{X}, \mathbb{Y}, t) \in \Sigma^+$ we have $\mathbb{X} = \varepsilon \mu^{-1}$. Similarly, for $(\mathbb{X}, \mathbb{Y}, t) \in \Sigma^-$ we have $\mathbb{Y} = \varepsilon \mu^{-1}$. Denote a point on Σ^+ by using (\mathbb{Y}, t) and a point on Σ^- by using (\mathbb{X}, t) . For $(\mathbb{Y}, t) \in \Sigma^+$, let

$$(\tilde{\mathbb{X}}, \tilde{t}) = \mathcal{N}(\mathbb{Y}, t).$$

Proposition 4.4. *We have for $(\mathbb{Y}, t) \in \Sigma^+$,*

$$(4.15) \quad \begin{aligned} \tilde{\mathbb{X}} &= (\mu \varepsilon^{-1})^{\frac{\alpha}{\beta} - 1} \mathbb{Y}^{\frac{\alpha}{\beta}}, \\ \tilde{t} &= t + \frac{1}{\beta} \ln(\varepsilon \mu^{-1}) - \frac{1}{\beta} \ln \mathbb{Y}. \end{aligned}$$

Proof: Let T be the time it takes for the solution of the linear equation of Proposition 4.1 from $(\varepsilon, Y, t) \in \Sigma^+$ to get to $(\tilde{X}, \varepsilon, \tilde{t}) \in \Sigma^-$. We have

$$\tilde{X} = \varepsilon e^{-\alpha T}, \quad \varepsilon = Y e^{\beta T}, \quad \tilde{t} = t + T$$

from which (4.15) follows. \square

We are now ready to compute the return map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \rightarrow \Sigma^-$. We use (\mathbb{X}, t) to represent a point on Σ^- and denote $(\tilde{\mathbb{X}}, \tilde{t}) = \mathcal{R}(\mathbb{X}, t)$.

Proposition 4.5. *The map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \rightarrow \Sigma^-$ is given by*

$$(4.16) \quad \begin{aligned} \tilde{\mathbb{X}} &= (\mu \varepsilon^{-1})^{\frac{\alpha}{\beta} - 1} [(1 + \mathcal{O}(\varepsilon)) P_L^+ \mathbb{F}(\mathbb{X}, t)]^{\frac{\alpha}{\beta}}, \\ \tilde{t} &= t + (L^+ + L^-) + \frac{1}{\beta} \ln \mu^{-1} \varepsilon (1 + \mathcal{O}(\varepsilon)) P_L^+ + \mathcal{O}_{\mathbb{X}, t}(\mu) - \frac{1}{\beta} \ln \mathbb{F}(\mathbb{X}, t), \end{aligned}$$

where

$$(4.17) \quad \mathbb{F}(\mathbb{X}, t) = \mathcal{W}_L(t + L^-) + P_L(P_L^+)^{-1}(1 + \mathcal{O}(\varepsilon))\mathbb{X} + (P_L^+)^{-1}(1 + P_L)\mathcal{O}_t(1) + \mathcal{O}_{\mathbb{X}, t}(\mu),$$

and P_L, P_L^+ and $\mathcal{W}_L(t)$ are as in (4.9) and (4.10).

Proof: By using Proposition 4.3 and Proposition 4.2(b)(ii), we have

$$\begin{aligned} \hat{Z} &= P_L(1 + \mathcal{O}(\varepsilon))\mathbb{X} + P_L^+ \mathcal{W}_L(t + L^-) + P_L \mathcal{O}_t(1) + \mathcal{O}_{\mathbb{X}, t}(\mu), \\ \hat{t} &= t + (L^+ + L^-) + \mathcal{O}_{\mathbb{X}, t}(\mu). \end{aligned}$$

Let $\hat{\mathbb{Y}}$ be the \mathbb{Y} -coordinate for (\hat{Z}, \hat{t}) , we have from Proposition 4.2(a)(ii),

$$(4.18) \quad \hat{\mathbb{Y}} = (1 + \mathcal{O}(\varepsilon)) P_L^+ \mathbb{F}(\mathbb{X}, t)$$

where $\mathbb{F}(\mathbb{X}, t)$ is as in (4.17). We then obtain (4.16) by using (4.15). \square

5. Dynamics of the Return Maps

In Section 4 we derived a formula for the return map $\mathcal{R} : \Sigma^- \rightarrow \Sigma^-$ and the final result was presented in Proposition 4.5. Our proof of Theorems 2.1-2.7 will be based exclusively on the formulas of this proposition. Before moving on to these proofs, let us take a pause to make these formulas appear more transparent. Let $(t, \mathbb{X}) \in \Sigma^-$, and $(t_1, \mathbb{X}_1) = \mathcal{R}(t, \mathbb{X})$. We have from Proposition 4.5

$$(5.1) \quad \begin{aligned} t_1 &= t + \mathbf{a} - \frac{1}{\beta} \ln \mathbb{F}(t, \mathbb{X}, \mu) + \mathcal{O}_{\mathbb{X}, t}(\mu), \\ \mathbb{X}_1 &= \mathbf{b}[\mathbb{F}(t, \mathbb{X}, \mu)]^{\frac{\alpha}{\beta}}, \end{aligned}$$

where

$$(5.2) \quad \begin{aligned} \mathbf{a} &= \frac{1}{\beta} \ln \mu^{-1} + (L^+ + L^-) + \frac{1}{\beta} \ln(\varepsilon(1 + \mathcal{O}(\varepsilon))P_L^+), \\ \mathbf{b} &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1} [(1 + \mathcal{O}(\varepsilon))P_L^+]^{\frac{\alpha}{\beta}}, \end{aligned}$$

and

$$(5.3) \quad \mathbb{F}(t, \mathbb{X}, \mu) = \mathcal{W}(t) + \mathbf{k}\mathbb{X} + \mathbb{E}(t, \mu) + \mathcal{O}_{t, \mathbb{X}}(\mu),$$

in which $\mathcal{W}(t)$ is as in (2.4),

$$(5.4) \quad \mathbf{k} = P_L(P_L^+)^{-1}(1 + \mathcal{O}(\varepsilon)),$$

and

$$(5.5) \quad \mathbb{E}(t, \mu) = (P_L^+)^{-1}(1 + P_L)\mathcal{O}_t(1) + \mathcal{W}_L(t) - \mathcal{W}(t).$$

We note that, instead of $\mathcal{W}(t + L^-)$, we write $\mathcal{W}(t)$ in (5.3) and (5.5). This is achieved by a simple change of variable from $t \rightarrow t - L^-$. Nothing else is effected. We have

- (i) $\mathbf{a} \approx \frac{1}{\beta} \ln \mu^{-1}$. $\mathbf{a} \rightarrow +\infty$ as $\mu \rightarrow 0$.
- (ii) $\mathbf{b} \sim \mu^{\frac{\alpha}{\beta}-1}$. $\mathbf{b} \rightarrow 0$ as $\mu \rightarrow 0$ by (H)(ii).
- (iii) $\mathbf{k} \sim \varepsilon^{\frac{\alpha}{\beta}}$.
- (iv) $\mathbb{E}(t, \mu) \sim \varepsilon^{\frac{\beta}{\alpha}} \mathcal{O}_t(1)$.

We can think \mathcal{R} as a 2D family of maps unfolded from the 1D maps

$$(5.6) \quad f(t) = t + \mathbf{a} - \frac{1}{\beta} \ln(\mathcal{W}(t) + \mathbb{E}(t, 0)).$$

Since $\mathbf{k} \gg \mu$, the first derivative of $\mathbb{F}(t, \mathbb{X}, \mu)$ with respect to \mathbb{X} is approximately \mathbf{k} and the unfolding from $f(t)$ to \mathcal{R} in \mathbb{X} -direction is determined mainly by the linear term $\mathbf{k}\mathbb{X}$. Note that when ε is sufficiently small, $\mathbb{E}(t, \mu)$ is a C^r -small perturbation to $\mathcal{W}(t)$.

In what follows we also write $\mathbb{F}(t, \mathbb{X}, \mu)$ as $\mathbb{F}(t, \mathbb{X})$ and $\mathbb{E}(t, \mu)$ as $\mathbb{E}(t)$.

5.1. Proof of Theorem 2.1. (a) Assume $m < 0 < M$. We make ε sufficiently small so that

$$\sup_{t \in \mathbb{R}} |\mathbb{E}(t, \mu)| \ll \min\{|m|, M\}.$$

It then follows that there exist values of t so that

$$(5.7) \quad \mathbb{F}(t, 0, \mu) = \mathcal{W}(t) + \mathbb{E}(t, \mu) + \mathcal{O}_t(\mu) = 0$$

where $\mathbb{F}(t, \mathbb{X}, \mu)$ is as in (5.3). Observe that local unstable manifold of $(X, Y) = (0, 0)$ is defined by $\mathbb{X} = 0$ and the local stable manifold is defined by $\mathbb{Y} = 0$. Let $(\hat{t}(t, \mathbb{X}), \hat{\mathbb{Y}}(t, \mathbb{X})) = \mathcal{M}(t, \mathbb{X})$. For t satisfying (5.7), we have

$$\hat{\mathbb{Y}}(t, 0) = 0.$$

This proves $W^u \cap W^s \neq \emptyset$.

(b) Assume $m > 0$. We again make ε sufficiently small so that

$$\sup_{t \in \mathbb{R}} |\mathbb{E}(t)| \ll m.$$

Let

$$\Sigma^- = \{(t, \mathbb{X}) : t \in \mathbb{R}, \mathbb{X} \in [0, 1]\}.$$

Then for any given $(t, 0) \in \Sigma^-$, the \mathbb{Y} -coordinates of $\mathcal{R}^n(t, 0)$ will be positive for all $n \geq 1$ according to (5.1) for \mathcal{R} . This rules out the possibility for such orbit to be part of W^s .

(c) If $M < 0$, then all solutions starting from the line $\mathbb{X} = 0$ on Σ^- will hit Σ^+ with a negative \mathbb{Y} -coordinate. Such solutions will hit $\mathbb{Y} = -\varepsilon$ then get out of \mathcal{U}_ε . These solutions, together with the surface defined by $\mathbb{Y} = -\varepsilon$ and $\mathbb{X} = 0$, form a 2D surface in the extended phase space that prevents the possibility of any of these solutions to intersect W^s in the future. \square

5.2. Proof of Theorem 2.2. By assuming $m > 0$, we can again make ε sufficiently small so that \mathcal{R} is well-defined on $\Sigma^- = \{(t, \mathbb{X}), 0 \leq \mathbb{X} \leq 1\}$. For $q \in \Sigma^-$, let $\mathbf{v} = (\delta_t, \delta_{\mathbb{X}})$ be a tangent vector at q , and $s(\mathbf{v}) = \delta_{\mathbb{X}} \delta_t^{-1}$ be the slope of \mathbf{v} . Let $\mathcal{C}_h(q)$ be the collection of all \mathbf{v} satisfying $|s(\mathbf{v})| < \frac{1}{100}$, and $\mathcal{C}_v(q)$ be the collection of all \mathbf{v} satisfying $|s(\mathbf{v})| > 100$. Let $D\mathcal{R}$ be the Jacobian matrix of \mathcal{R} in (5.1). With the assumptions of Theorem 2.2, we have

- (i) $D\mathcal{R}(\mathcal{C}_h(q)) \subset \mathcal{C}_h(\mathcal{R}(q))$, and
- (ii) $D\mathcal{R}^{-1}(\mathcal{C}_v(q)) \subset \mathcal{C}_v(\mathcal{R}(q))$

for all $q \in \Sigma^-$. (i) and (ii) follows from direct computation using (5.1). To prove Theorem 2.2 we now apply the standard graph transformation argument over all Lipschitz curves of slope $\leq \frac{1}{100}$. We also obtain an associated stable foliation that verifies the rest of Theorem 2.2 because of the strong contractions in the vertical direction. In fact $\mathbb{X} = 0$ is an approximated normally hyperbolic invariant manifold. The theory of invariant manifolds developed in [BLZ] can be applied to this case. \square

5.3. Proof of Theorem 2.3. Assume that ε is sufficiently small so that

$$\sup_{t \in \mathbb{R}} |\mathbb{E}(t, \mu)| \ll \min\{|m^\pm|, M^\pm\}.$$

Let $\{a_k\}_{k=-\infty}^{+\infty}$ be a monotone bi-infinite sequence. $a_k \rightarrow \pm\infty$ as $k \rightarrow \pm\infty$ is such that

$$\lim_{k \rightarrow \pm\infty} \mathcal{W}(a_k) = M^\pm.$$

Similarly, let $\{b_k\}_{k=-\infty}^{+\infty}$, $b_k \rightarrow \pm\infty$ as $k \rightarrow \pm\infty$ be such that $\lim_{k \rightarrow \pm\infty} \mathcal{W}(b_k) = m^\pm$. Without loss of generality we assume

$$\mathcal{W}(a_k) > \frac{99}{100} M^+, \quad \mathcal{W}(b_k^+) < \frac{99}{100} m^+$$

for all $k \geq 0$; and

$$\mathcal{W}(a_k) > \frac{99}{100} M^-, \quad \mathcal{W}(b_k^+) < \frac{99}{100} m^-$$

for all $k < 0$. We also assume that

$$b_{k-1} < a_k < b_k$$

for all $k \in \mathbb{Z}$. Let $\Sigma^- = \{(t, \mathbb{X}) : 0 \leq \mathbb{X} \leq 1\}$. For $0 \leq \delta \leq 1$, denote

$$t_{k, \text{left}}(\delta) = \sup_{t < a_k} \{t : \mathbb{F}(t, \delta) = 0\},$$

$$t_{k, \text{right}}(\delta) = \inf_{t > a_k} \{t : \mathbb{F}(t, \delta) = 0\}.$$

From $b_{k-1} < a_k < b_k$, and the assumption that

$$m^\pm < 0 < M^\pm,$$

we have

$$(5.8) \quad b_{k-1} < t_{k, \text{left}}(\delta) < a_k < t_{k, \text{right}}(\delta) < b_k.$$

Let $\{V_k\}_{k=-\infty}^{+\infty}$ be such that

$$(5.9) \quad V_k = \{(t, \mathbb{X}) \in \Sigma^- : t \in [t_{k, \text{left}}(\mathbb{X}), t_{k, \text{right}}(\mathbb{X})]\}.$$

$\{V_k\}_{k=-\infty}^{+\infty}$ serve as the bi-infinite sequence of vertical strips of Definition 2.1: The two vertical boundaries of V_k are well-defined continuous curves defined by

$$\mathbb{F}(t, \mathbb{X}) = 0,$$

and none of these boundary curves are allowed to intersecting each other because of (5.8). For Definition 2.1 to fulfill, it suffices to observe that

$$t_1(t_{k, \text{right}}(\delta), \delta) = t_1(t_{k, \text{left}}(\delta), \delta) = +\infty$$

for all k . □

5.4. Proof of Theorem 2.4. The assumption $M^- < 0$ implies that there exists a $K_1 > -\infty$ such that $\mathcal{W}(t) < \frac{1}{2}M^-$ for all $t < K_1$. This in turn implies $\mathbb{Y}(t, \mathbb{X}) < 0$ for all $(t, \mathbb{X}) \in \Sigma^-$ satisfying $t < K_1$. Denote $\Sigma_{K_1} = \{t > K_1\}$ in Σ^- . Then W , hence Ω , is a subset of Σ_{K_1} . From (5.1) for \mathcal{R} , we then conclude that $\mathcal{R}^n(\Omega) \subset \Sigma_{K_1 + \frac{n}{2}\mathbf{a}}$. Therefore $\Lambda = \bigcap_n \mathcal{R}^n(\Omega) = \emptyset$.

The assumption $M^+ < 0$ would exclude Σ_K for some positive $K < +\infty$ from W . Since every orbit of \mathcal{R} are moving from left to right, eventually entering Σ_K , $\Omega = \emptyset$. □

5.5. Proof of Theorem 2.5. With the assumption that $m^- < 0 < M^-$, we assume the same sequences $a_k, b_k \rightarrow -\infty$ in the proof of Theorem 2.3 but only for $k \leq 0$. Repeating the same construction of vertical strips in the proof of Theorem 2.3 we obtain V_k for all $k \leq 0$, and item (1) in Definition 2.2 holds for all $\{V_k\}_{k \leq 0}$. On the other hand, the assumption $m^+ > 0$ implies that there exists a $K_1 > 0$ so that $\Sigma_{K_1} \subset \Omega$ where Σ_{K_1} is the part of Σ^- satisfying $t > K_1$. We make μ sufficiently small, hence \mathbf{a} sufficiently large, and verify $\mathcal{R}(V_0) \subset \Sigma_{K_1}$ by using (5.1). □

5.6. Proof of Theorem 2.6. With the assumption that $m^+ < 0 < M^+$, we assume the same sequences $a_k, b_k \rightarrow +\infty$ in the proof of Theorem 2.3 but only for $k \geq 0$. Repeating the same construction of vertical strips in Theorem 2.3 we obtain V_k for all $k \geq 0$, and item (1) of Definition 2.3 holds for $\{V_k\}_{k \geq 0}$.

The assumption $m^- > 0$ implies that there exists a $K_1 > -\infty$ so that so that $\mathcal{W}(t) > \frac{1}{2}m^-$ for all $t < K_1$. This implies that \mathcal{R} is well-defined on the part of Σ^- satisfying $t < K_1$. We can make K_1 sufficiently negative so that $(K_1, 0) \in \Sigma^-$ is on the left side of V_0 . We can also make μ sufficiently small so that $\mathcal{R}(K_1, 0)$ is on the right side of V_0 .

We show that every vertical curve in V_0 contain a point, all inverse images of which under \mathcal{R} are in Σ^- . Let l be a vertical curve in V_0 . It suffices to prove that, for every fixed $n > 0$, there exists a t_n , so that $p_n = \mathcal{R}^n(t_n, 0) \in l$. Any accumulation point of $\{p_n\}$ on l is a point of item (2) of Definition 2.3 for l . For the existence of t_n , we observe that for every $n > 0$, there exists a K_n , such that (i) the point $(K_n, 0) \in \Sigma^-$ is on the left side of V_0 , and (ii) the image of $\{(t, 0) \in \Sigma^- : t \in (-\infty, K_n)\}$ under \mathcal{R}^n is with one end at $t = -\infty$, and the other end on the right side of V_0 . □

5.7. Proof of Theorem 2.7. Again, let ε be sufficiently small so that

$$\sup_{t \in \mathbb{R}} |\mathbb{E}(t)| \ll \min\{m^\pm, M\}.$$

Let $\{a_k\}_{k \geq 0}, \{b_k\}_{k \geq 0}$, $a_k, b_k \rightarrow +\infty$ be monotone sequences such that $\lim_{k \rightarrow +\infty} \mathcal{W}(a_k) = M^+$, $\lim_{k \rightarrow +\infty} \mathcal{W}(b_k) = m^+$. We can assume without loss of generality that

$$b_k < a_k < b_{k+1}$$

for all $k > 0$, and further that $a_{k+1} - a_k, b_{k+1} - b_k < L^+$ for some fixed $L^+ > 0$ because of the assumption that both m^+ and M^+ are densely approached by L^+ -sequences. Let

$$\tilde{M} = \max\left\{M^+ - \frac{1}{10}(M^+ - m^+), \frac{99}{100}M^+\right\}$$

and

$$\tilde{m} = \min\left\{m^+ + \frac{1}{10}(M^+ - m^+), \frac{101}{100}m^+\right\}.$$

We can also assume that

$$\mathcal{W}(a_k) > \tilde{M}, \quad \mathcal{W}(b_k) < \tilde{m}$$

for all $k \geq 0$.

We construct V_k for all $k \geq 0$ as follows. For $0 \leq \delta \leq 1$, let

$$(5.10) \quad \begin{aligned} t_{k, \text{left}}(\delta) &= \sup_{t < a_k} \{t : \mathbb{F}(t, \delta) = \tilde{m}\}, \\ t_{k, \text{right}}(\delta) &= \inf_{t > a_k} \{t : \mathbb{F}(t, \delta) = \tilde{m}\}. \end{aligned}$$

We define

$$V_k = \cup_{0 \leq \delta \leq 1} \{(t, \delta) : t \in [t_{k, \text{left}}(\delta), t_{k, \text{right}}(\delta)]\}$$

for all $k \geq 0$.

For the claim that $\mathcal{R}(V_k)$ crosses two other vertical strips horizontally for all $k \geq 0$, it suffices for us to have

$$t_1(t_{k, \text{left}}(\delta), \delta) - t_1(a_k, \delta) > 2L^+.$$

From (5.1)

$$t_1(t_{k, \text{left}}(\delta), \delta) - t_1(a_k, \delta) = t_{k, \text{left}}(\delta) - a_k + \beta^{-1}(\ln \tilde{M} - \ln \tilde{m}) + \mathcal{O}(\mu).$$

So it suffices to have

$$\frac{\tilde{M}}{\tilde{m}} > e^{3\beta L^+}.$$

By $\tilde{M} > \frac{99}{100}M^+, \tilde{m} < \frac{101}{100}m^+$, it then suffices that

$$M^+ > \frac{101}{99}m^- e^{3\beta L^+}.$$

This is still slightly stronger than what is assumed in Theorem 2.7. However, we observe that in this proof the number $\frac{101}{99}$ is quite arbitrary, and can be taken arbitrarily close to 1.

The argument for the second item of Definition 2.3 is the same as in the proof of Theorem 2.6. \square

APPENDIX A. PROOF OF PROPOSITION 4.1

In this appendix, we give a result on smooth linearization for a nonautonomous differential equation around an equilibrium point. Consider a nonautonomous differential equation in \mathbb{R}^d

$$(A.1) \quad \frac{dx}{dt} = Ax + f(t, x, \mu),$$

where A is a $d \times d$ real matrix, f is a smooth function, and μ is a parameter.

For the matrix A , we assume that

Hypothesis A: A is hyperbolic, that is, A has no eigenvalues on the imaginary axis.

This condition implies that there exist an invariant splitting of the phase space $\mathbb{R}^d = E_u \oplus E_s$ with the associated projections Π_u and Π_s and positive constants α , β , and K such that

$$(A.2) \quad \begin{aligned} \|e^{At}\Pi_s\| &\leq Ke^{-\beta t} \quad \text{for } t \geq 0, \\ \|e^{At}\Pi_u\| &\leq Ke^{\beta t} \quad \text{for } t \leq 0, \\ \|e^{At}\| &\leq Ke^{\alpha|t|} \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

We assume that for the nonlinear term $f(t, x, \mu)$

Hypothesis B: There are an open neighborhood U of 0 in \mathbb{R}^d and $\mu_0 > 0$ such that

(i) $f : \mathbb{R} \times U \times [-\mu_0, \mu_0] \rightarrow \mathbb{R}^d$ is C^N for some integer $N \geq 2$ with uniformly bounded derivatives

$$\sup_{(t,x,\mu) \in \mathbb{R} \times U \times [-\mu_0, \mu_0]} \|D^k f(t, x, \mu)\| \leq K_1,$$

where K_1 is a positive constant;

(ii) $f(t, 0, \mu) = 0$ and $D_x f(t, 0, \mu) = 0$.

Theorem A.1. *Assume hypotheses (A) and (B) hold. For each integer $k > 0$, there exists an integer $N_0 = N_0(k, \alpha, \beta)$ such that if f is C^N for $N \geq N_0$ and the real parts of eigenvalues of A , $\lambda_1, \dots, \lambda_p$, satisfy the nonresonant conditions up to order N_0 ,*

$$\lambda_i \neq (\tau, \lambda), \quad \text{for all } 1 \leq i \leq p, \quad 2 \leq |m| \leq N_0.$$

where $(\tau, \lambda) = \sum_{j=1}^p \tau_j \lambda_j$, $(\tau_1, \dots, \tau_p) \in \mathbb{N}^p$, $\lambda = (\lambda_1, \dots, \lambda_p)$, and $|m| = \sum_{j=1}^p m_j$, then there is a C^k invertible transformation $x = H(t, y, \mu) = y + \tilde{H}(t, y, \mu)$ which transforms equation (A.1) to the linear equation

$$\frac{dy}{dt} = Ay$$

where $\tilde{H} : \mathbb{R} \times V \times [-\mu_0, \mu_0] \rightarrow \mathbb{R}^d$ is a C^k function with all bounded derivatives and $\tilde{H}(t, 0, \mu) = 0$ and $D_y \tilde{H}(t, 0, \mu) = 0$, V is an open neighborhood of 0.

In order to construct the transformation, we use the standard cut-off function to modify the nonlinearity $f(t, x, \mu)$.

Let $\sigma(s)$ be a C^∞ function from $(-\infty, \infty)$ to $[0, 1]$ with

$$\sigma(s) = 1 \quad \text{for } |s| \leq 1, \quad \sigma(s) = 0 \quad \text{for } |s| \geq 2,$$

$$\sup_{s \in \mathbb{R}} |\sigma'(s)| \leq 2.$$

Let ρ be a positive constant such that the ball $B(0, \rho) \subset U$. We consider a modification of $f(t, x, \mu)$. Let

$$\tilde{f}(t, x, \mu) = \sigma_\rho(|x|)f(t, x, \mu), \quad \text{where } \sigma_\rho(|x|) = \sigma\left(\frac{|x|}{\rho}\right).$$

An elementary calculation gives

- (i) $\tilde{f}(t, x, \mu) = f(t, x, \mu)$, for $|x| \leq \rho$;
- (ii) There exists a positive constant K_2 such that

$$\begin{aligned} \|\partial_x \tilde{f}(t, x, \mu)\| &\leq 10K_1\rho \quad \text{for all } (t, x, \mu) \in \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0]; \\ \sup_{(t,x,\mu) \in \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0]} \|D^k \tilde{f}(t, x, \mu)\| &\leq \tilde{K}_2 \quad \text{for } 2 \leq k \leq N. \end{aligned}$$

Let $x(t, \omega, x_0, \mu)$ denote the solution of

$$\frac{dx}{dt} = Ax + \tilde{f}(t + \omega, x, \mu), \quad x(0) = x_0.$$

Clearly, $x(t, \omega, x_0, \mu)$ exists for all $t \in \mathbb{R}$, $\omega \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$, and $\mu \in [-\mu_0, \mu_0]$ and satisfies

$$x(t, \omega, x_0, \mu) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s + \omega, x(s, \omega, x_0, \mu), \mu) ds.$$

Note that $x(t, \omega, x_0, \mu)$ forms a nonautonomous dynamical system. It together with $\theta^t \omega = t + \omega$ forms a cocycle. We will first consider the corresponding time discrete nonautonomous dynamical system $\phi(n, \omega, x, \mu) = x(n, \omega, x, \mu)$. We write its time-one map as

$$\varphi(\omega, x, \mu) := \Phi x + F(\omega, x, \mu)$$

where $\Phi = e^A$ and $F(\omega, x, \mu) = \int_0^1 e^{A(1-s)} f(s + \omega, x(s, \omega, x_0, \mu), \mu) ds$. Note that

$$(A.3) \quad \begin{aligned} F(\omega, 0, \mu) = 0 \quad \text{and} \quad D_x F(\omega, 0, \mu) = 0, \\ \sup_{(t, x, \mu) \in \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0]} \|D^k \tilde{f}(t, x, \mu)\| \leq \tilde{K}_3 \text{ for } 2 \leq k \leq N \end{aligned}$$

for some positive constant K_3 . We choose $\rho > 0$ such that

$$(A.4) \quad \rho \leq \min \left\{ \frac{\beta}{60K^2K_3}, \frac{1}{20Ke^\alpha K_3} \right\}.$$

This implies that $\psi(\omega, \mu)x := \Phi x + F(\omega, x, \mu)$ is a C^N diffeomorphism on \mathbb{R}^d . $\psi(\omega, \mu)$ generates a C^N nonautonomous dynamical system $\phi(n, \omega, x, \mu)$. We note that each sequence x_n satisfies $x_n = \phi(n, \omega, x_0, \mu)$ if and only if x_n satisfies

$$(A.5) \quad x_{n+1} = \Phi x_n + F(n + \omega, x_n, \mu).$$

For the time discrete nonautonomous dynamical system generated by $\varphi(\omega, x, \mu)$, we have the following theorem of linearization

Theorem A.2. *Assume hypotheses (A) and (B) hold. For each integer $k > 0$, there exists an integer $N_0 = N_0(k, \alpha, \beta)$ such that if φ is C^N for $N \geq N_0$ and the real parts of eigenvalues of A , $\lambda_1, \dots, \lambda_p$, satisfy the nonresonant conditions up to order N_0 ,*

$$\lambda_i \neq (\tau, \lambda), \quad \text{for all } 1 \leq i \leq p, \quad 2 \leq |m| \leq N_0.$$

then there is a C^k local diffeomorphism $x = h(\omega, y, \mu) = y + \tilde{h}(\omega, y, \mu)$ such that

$$h(\theta\omega, \cdot, \mu) \circ \varphi(\omega, x, \mu) = \Phi h(\omega, x, \mu)$$

where $\tilde{h} : \mathbb{R} \times V \times [-\mu_0, \mu_0] \rightarrow \mathbb{R}^d$ is a C^k function with all bounded derivatives and $\tilde{h}(t, 0, \mu) = 0$ and $D_y \tilde{h}(\omega, 0, \mu) = 0$, V is an open neighborhood of 0.

The proof of Theorem A.1 follows from Theorem A.2 by letting

$$H(\omega, x, \mu) = \int_0^1 e^{-As} h(\theta^s \omega, \phi(s, \omega, x, \mu), \mu) ds.$$

The proof of Theorem A.2 follows exactly the same lines presented in [LL] with some modification on the smoothness of the transformation in terms of parameters ω and μ . The smoothness of the transformation with respect to parameter follows from the ones for stable and unstable manifolds and the solution of the recurrent equation. The proofs follow the standard arguments. Here, we outline the main ingredients of the proof of Theorem A.2 in terms of lemmas and propositions. These lemmas and propositions are presented in the following subsections.

A.1. Stable and Unstable manifolds and Invariant Foliations. We first define for the nonautonomous dynamical system ϕ the stable set

$$W^s(\omega, \mu) = \{x_0 \in \mathbb{R}^d \mid x_n = \phi(n, \omega, x_0, \mu) \rightarrow 0, \text{ as } n \rightarrow +\infty\}$$

and the unstable set

$$W^u(\omega, \mu) = \{x_0 \in \mathbb{R}^d \mid x_n = \phi(n, \omega, x_0, \mu) \rightarrow 0, \text{ as } n \rightarrow -\infty\}.$$

Clearly, both $W^s(\omega, \mu)$ and $W^u(\omega, \mu)$ are invariant for the nonautonomous dynamical system $\phi(n, \omega, x, \mu)$. For each $x \in \mathbb{R}^d$, we may write it as

$$x = x^u + x^s, \quad \text{for some } x^u \in E_u, x^s \in E_s.$$

Lemma A.1. (*Unstable Manifold Theorem*) Assume that Hypotheses A and B hold. Then, there exists $\rho_0 > 0$ such that for each $0 < \rho \leq \rho_0$ the unstable set $W^u(\omega, \mu)$ is a C^N manifold given by

$$W^u(\omega, \mu) = \{\eta + h^u(\eta, \omega, \mu) \mid \eta \in E_u\}$$

where

$$h^u : E_u \times \mathbb{R} \times [-\mu_0, \mu_0] \rightarrow E_s$$

satisfies

(i) $h^u(\eta, \omega, \mu)$ is C^N in (η, ω, μ) with

$$\text{Lip } h^u(\cdot, \omega, \mu) < 1, \quad h^u(0, \omega, \mu) = 0, \quad Dh^u(0, \omega, \mu) = 0.$$

(ii) $|D^i h^u(\eta, \omega, \mu)| \leq K_i$ for each $0 \leq i \leq N$, where K_3 is a positive constant.

(iii) $W_{loc}^u(\omega, \mu) = \{x \in W^u(\omega, \mu) \mid x \in B(0, \rho)\}$ is a local unstable manifold for the original system.

Moreover, for each $x_0 \in W^u(\omega, \mu)$, the orbit $\mathbf{x} = \{x_n\}_{n \leq 0}$ of ϕ with the initial point x_0 satisfies

$$\sup_{n \leq 0} e^{-\frac{\beta}{5}n} |x_n| < \infty.$$

Lemma A.2. (*Stable Manifold Theorem*) Assume that Hypotheses A and B hold. Then, there exists $\rho_0 > 0$ such that for each $0 < \rho \leq \rho_0$ the stable set $W^s(\omega, \mu)$ is a C^N manifold given by

$$W^s(\omega, \mu) = \{\xi + h^s(\xi, \omega, \mu) \mid \xi \in E_s\}$$

where

$$h^s : E_s \times \mathbb{R} \times [-\mu_0, \mu_0] \rightarrow E_u$$

satisfies

(i) $h^s(\xi, \omega, \mu)$ is C^N in (ξ, ω, μ) with

$$\text{Lip } h^s(\cdot, \omega, \mu) < 1, \quad h^s(0, \omega, \mu) = 0, \quad Dh^s(0, \omega, \mu) = 0.$$

(ii) $|D^i h^s(\xi, \omega, \mu)| \leq K_i$ for each $0 \leq i \leq N$, where K_3 is a positive constant.

(iii) $W_{loc}^s(\omega, \mu) = \{x \in W^s(\omega, \mu) \mid x \in B(0, \rho)\}$ is a local stable manifold for the original system.

Moreover, for each $x_0 \in W^s(\omega, \mu)$, the orbit $\mathbf{x} = \{x_n\}_{n \geq 0}$ of ϕ with the initial point x_0 satisfies

$$\sup_{n \geq 0} e^{\frac{\beta}{5}n} |x_n| < \infty.$$

The next lemma is on the existence of a stable foliation. Let $\mathbf{x} = \{x_n\}_{n \geq 0}$ be a positive orbit of ϕ . Define the following set

$$\mathcal{M}^s(x_0, \omega, \mu) = \{y_0 \mid \sup_{n \leq 0} e^{-\frac{\beta}{5}n} |y_n - x_n| < \infty\},$$

where $\{y_n\}_{n \geq 0}$ is a positive orbit of ϕ .

Lemma A.3. Assume that Hypotheses A and B hold. Then, there exists $\rho_0 > 0$ such that for each $0 < \rho \leq \rho_0$ there is a mapping

$$g^s : E_s \times \mathbb{R}^d \times \mathbb{R} \times [-\mu_0, \mu_0] \rightarrow E_u$$

satisfying

(i) $g^s(\xi, x_0, \omega, \mu)$ is continuous in (ξ, x_0, ω, μ) and is Lipschitz continuous in ξ with

$$\text{Lip}_\xi g^s < 1.$$

(ii) $\mathcal{M}^s(x_0, \omega, \mu)$ is the graph of φ_s

$$\mathcal{M}^s(x_0, \omega, \mu) = \{x_0 + \xi + \varphi_s(\xi, x_0, \omega, \mu) \mid \xi \in E_s\}.$$

(iii) $W^u(\omega, \mu) \cap \mathcal{M}^s(x_0, \omega, \mu)$ contains a single point \bar{x}_0 and

$$|x_n - \bar{x}_n| \leq \frac{24}{15} K |x_0^s - \bar{x}_0^s| e^{-\frac{\beta(\omega)}{5}n}.$$

Furthermore, $\mathbb{R}^d = \cup_{x_0 \in W^u(\omega, \mu)} \mathcal{M}^s(x_0, \omega, \mu)$.

Remark: Each leaf $\mathcal{M}^s(x_0, \omega, \mu)$ is in fact a C^N submanifolds and is Hölder continuous in x_0 . $\mathcal{M}^s(0, \omega, \mu)$ is the stable manifold $W^s(\omega, \mu)$.

Similarly, we have a unstable foliation

Lemma A.4. *The same conditions of Lemma A.3 hold. Then there exists a mapping*

$$g^u : E_u \times \mathbb{R}^d \times \mathbb{R} \times [-\mu_0, \mu_0] \rightarrow E_s$$

such that

(i) $g^u(\eta, x_0, \omega, \mu)$ is continuous in (η, x_0, ω, μ) and is Lipschitz continuous in η with

$$\text{Lip}_\eta g^u < 1$$

(ii) $\mathcal{M}^u(x_0, \omega, \mu)$ is the graph of g^u

$$\mathcal{M}^u(x_0, \omega, \mu) = \{x_0 + \eta + g^u(\eta, x_0, \omega, \mu) \mid \eta \in E_u\}$$

(iii) $W^s(\omega, \mu) \cap \mathcal{M}^u(x_0, \omega, \mu)$ contains a single point \bar{x}_0 and

$$|x_n - \bar{x}_n| \leq \frac{24}{15} K |x_0^u - \bar{x}_0^u| e^{\frac{\beta(\omega)}{5} n}, \quad \text{for } n \leq 0.$$

Furthermore, $\mathbb{R}^d = \cup_{x_0 \in W^u(\omega, \mu)} \mathcal{M}^s(x_0, \omega, \mu)$.

A.2. Smooth Conjugacy. We first give a general result (Theorem A.3) on smooth conjugacy in a jet class. Then, we present the theorem of linearization.

Definition A.1. *Two nonautonomous dynamical systems $\phi(n, \omega, x, \mu)$ and $\psi(n, \omega, x, \mu)$ with a fixed point $x = 0$ are said to be C^k conjugate locally if there exists a C^k map $h : \mathbb{R} \times U \times [-\mu_0, \mu_0] \rightarrow \mathbb{R}^d$ for which $h(\omega, \cdot, \mu)$ is a diffeomorphism from the open neighborhood U of 0 to its image with $h(\omega, 0, \mu) = 0$ such that*

$$(A.6) \quad h(\theta\omega, \phi(1, \omega, x, \mu), \mu) = \psi(1, \omega, h(\omega, x, \mu), \mu) \quad \text{for } x \in U, \omega \in \mathbb{R}.$$

The conjugacy relationship (A.6) implies that $h(\theta^n \omega, \phi(n, \omega, x, \mu), \mu) = \psi(n, \omega, h(\omega, x, \mu), \mu)$ when the orbits stay in the corresponding domains.

The next result gives a smooth conjugacy in a jet class.

Theorem A.3. *Let ϕ be a nonautonomous dynamical system satisfying Hypothesis A and B. Then, for any integer $k > 0$, there exists a positive integer $N_0 = N_0(k, \alpha, \beta)$ such that if ϕ is C^N for $N \geq N_0$ and ψ is a C^N nonautonomous dynamical system in \mathbb{R}^d with $\phi - \psi = O(|x|^{N_0})$ uniformly in $(\omega, \mu) \in \mathbb{R} \times [-\mu_0, \mu_0]$, then ϕ and ψ are C^k locally conjugate.*

The proof of this theorem is based on the stable and unstable manifolds, invariant foliations, and the following lemmas and proposition. We first applying the unstable manifold theorem (Theorem A.1) and the stable manifold theorem (Theorem A.2) to ϕ . Then, ϕ has a C^N local unstable manifold

$$W_{loc}^u(\omega, \mu) = \{\eta + h^u(\eta, \omega, \mu) \mid \eta \in E_u, |\eta| \leq \rho\}$$

and a C^N local stable manifold

$$W_{loc}^s(\omega, \mu) = \{\xi + h^s(\xi, \omega, \mu) \mid \xi \in E_s, |\xi| \leq \rho\}$$

We may identify E_u and E_s with \mathbb{R}^{d_u} and \mathbb{R}^{d_s} , respectively. Next, we use the stable and unstable manifolds as new axes to rewrite ϕ . Consider the coordinate nonautonomous transformation defined on a ball $B_r(0)$

$$(A.7) \quad \begin{aligned} z &= \eta - h^s(\xi, \omega, \mu), \\ y &= \xi - h^u(\eta, \omega, \mu). \end{aligned}$$

It also follows from Lemma A.1 and A.2 that this transformation is a C^N nonautonomous diffeomorphism. By this transformation, the nonautonomous diffeomorphism $\phi(1, \omega, \mu)$ is locally conjugate to a C^N nonautonomous diffeomorphism

$$\hat{\phi}(1, \omega, \mu) : x = (y, z)^T \mapsto \Phi x + F_1(\omega, x, \mu)$$

where Φ is written as $\text{diag}(\Phi^s, \Phi^u)$, $F_1(\omega, x, \mu)$ satisfies $F_1(\omega, 0, \mu) = 0$, $DF_1(\omega, 0, \mu) = 0$. Furthermore,

$$F_1 = \begin{pmatrix} F_{1,s}(\omega, y, z, \mu) \\ F_{1,u}(\omega, y, z, \mu) \end{pmatrix}, \quad F_{1,s}(\omega, 0, z, \mu) = 0, \quad F_{1,u}(\omega, y, 0, \mu) = 0,$$

which implies that $y = 0$ is the stable manifold and $z = 0$ is the unstable manifold for the new nonautonomous dynamical system.

Similarly, $\psi(1, \omega, \mu)$ is C^N conjugate to

$$\hat{\psi}(1, \omega, \mu) : x = (y, z)^T \mapsto \Phi x + F_2(\omega, x, \mu)$$

where $F_2(\omega, x, \mu)$ satisfies (A.3), and $F_2(\omega, 0, \mu) = 0$, $DF_2(\omega, 0, \mu) = 0$. Furthermore,

$$F_2 = \begin{pmatrix} F_{2,s}(\omega, y, z, \mu) \\ F_{2,u}(\omega, y, z, \mu) \end{pmatrix}, \quad F_{2,s}(\omega, 0, z, \mu) = 0, \quad F_{2,u}(\omega, y, 0, \mu) = 0.$$

Let

$$R = \begin{pmatrix} R_s(\omega, y, z, \mu) \\ R_u(\omega, y, z, \mu) \end{pmatrix} = F_1 - F_2 = \begin{pmatrix} F_{1,s}(\omega, y, z, \mu) - F_{2,s}(\omega, y, z, \mu) \\ F_{1,u}(\omega, y, z, \mu) - F_{2,u}(\omega, y, z, \mu) \end{pmatrix}.$$

Then $R_s(\omega, 0, z, \mu) = 0$, $R_u(\omega, y, 0, \mu) = 0$. When $\phi - \psi = O(|x|^N)$, the order of the difference between the corresponding stable manifolds and order of the difference between unstable manifolds are $O(|x|^N)$, thus $\hat{\phi} - \hat{\psi} = O(|x|^N)$.

In the following, we will decompose R into two parts, one part is dominated by a power of y while another is dominated by a power of z .

Lemma A.5. *The function R can be written as $R = R_1 + R_2$, where $R_i(\omega, \cdot, \mu)$ are $C^{[N/2]}$ functions satisfying*

$$\Pi_s R_2(\omega, 0, z, \mu) = 0, \quad \Pi_u R_1(\omega, y, 0, \mu) = 0,$$

$$(A.8) \quad |R_1| \leq C_0 |y|^{[N/2]}, \quad |R_2| \leq C_0 |z|^{[N/2]}, \quad (\omega, x, \mu) \in \mathbb{R} \times U \times [-\mu_0, \mu_0]$$

if $\text{jet}_{x=0}^n R = 0$, here C_0 is a positive constant.

The proof of this lemma is based on Taylor expansion and follows the exact the lines of Lemma 4.3 in [LL].

Next, we use the cut-off procedure introduced at the beginning of this section to modify the functions, F_1 , F_2 , R_1 , and R_2 . We set for $i = 1, 2$

$$\tilde{F}_i(\omega, x, \mu) = \sigma_{\rho_1}(|x|) F_i(\omega, x, \mu),$$

$$\tilde{R}_i(\omega, x, \mu) = \sigma_{\rho_1}(|x|) R_i(\omega, x, \mu).$$

We choose $\rho_1 < \rho_0$. Note that \tilde{F}_1 and \tilde{F}_2 are C^N with bounded derivatives up to order N and \tilde{R}_1 and \tilde{R}_2 are $C^{[N/2]}$ with bounded derivatives. Moreover, the following hold:

$$(A.9) \quad |\tilde{R}_1| \leq C_0 |y|^{[N/2]} \sigma_{\rho_1}(|y|), \quad |\tilde{R}_2| \leq C_0 |z|^{[N/2]} \sigma_{\rho_1}(|z|), \quad \text{for } x \in \mathbb{R}^d,$$

if $\text{jet}_{x=0}^N R = 0$.

Let $\tilde{\phi} = \Phi x + \tilde{F}_1$ and $\tilde{\psi} = \Phi x + \tilde{F}_2$. Then

$$\hat{\phi} = \tilde{\phi}, \quad \hat{\psi} = \tilde{\psi}, \quad \text{for } |x| < \rho_1.$$

Let $\phi_\tau(m, \omega, x, \mu)$ and $\psi_\tau(m, \omega, x, \mu)$, $\tau \in [0, 1]$, be the families of nonautonomous dynamical systems whose time-one maps have the forms

$$\phi_\tau(1, \omega, x, \mu) = \tilde{\phi}(\omega, x, \mu) + \tau \tilde{R}_1$$

and

$$\psi_\tau(1, \omega, x, \mu) = \tilde{\psi}(\omega, x, \mu) + \tau \tilde{R}_2$$

respectively. Then we have the following estimates.

Lemma A.6. *For any given integer $k > 0$, there exist constants $M_k = M_k(\alpha, \beta)$ and d_k such that*

$$(A.10) \quad \max_{1 \leq |\alpha| \leq k} \sup_{(\omega, x, \mu) \in \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0]} \|D^\alpha \phi_\tau^{\pm 1}(n, \omega, x, \mu)\| \leq d_k M_k^{|\alpha|}, \quad n \in \mathbb{Z},$$

$$(A.11) \quad \max_{1 \leq |\alpha| \leq k} \sup_{(\omega, x, \mu) \in \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0]} \|D^\alpha \psi_\tau^{\pm 1}(n, \omega, x, \mu)\| \leq d_k M_k^{|\alpha|}, \quad n \in \mathbb{Z}.$$

The proof of this lemma is based on the discrete version of variation of constants formula and the discrete-time Gronwall-Bellman lemma, see for example, Lemma 4.4 [LL].

The next lemma states that one may reduce the problem of C^k conjugacy (A.6) to a problem of solving a linear recurrent functional equation. This idea is based on the so-called homotopy method.

Lemma A.7. *Let $\phi_\tau(n, \omega(\omega, x, \mu) \in \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0])$, $\tau \in [0, 1]$, $n \in \mathbb{Z}$, be a family of C^N nonautonomous dynamical systems in \mathbb{R}^d with fixed point 0 whose time-one mapping has the form*

$$\phi_\tau(1, \omega, \mu)x = \varphi_\tau(\omega, x, \mu) = \varphi(\omega, x, \mu) + \tau R(\omega, x, \mu).$$

If there exists a mapping $r : \mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0] \times [0, 1] \rightarrow \mathbb{R}^d$ such that

(i) *r is C^k and satisfies*

$$(A.12) \quad |r(\omega, x, \mu, \tau)| \leq a|x|^2, \quad \text{for } |x| \leq 1,$$

where a is a constant;

(ii) *r satisfies the following equation*

$$(A.13) \quad D\varphi_\tau(\omega, x, \mu)r(\omega, x, \mu, \tau) - r(\theta\omega, \varphi_\tau(\omega, x, \mu), \mu, \tau) = -R(\omega, x, \mu),$$

then ϕ_0 and ϕ_1 are C^k locally conjugate.

The proof of this lemma is exactly same as Lemma 4.5 in [LL]. The proof also gives that the conjugacy map is a near identity map of form $Id + O(|x|^2)$ uniformly in (ω, μ) .

Next, we construct formal solutions of linear functional equation (A.13) in terms of an infinite series. We will see later that they are convergent for R being R_1 and R_2 .

For simplicity, we set $(f_*R)(x) := (DfR) \circ f^{-1}(x)$ for a diffeomorphism $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a vector field R in \mathbb{R}^d . Notice that $f_*(g_*R) = (f \circ g)_*R$. Then, we have two formal solutions to equation (A.13).

Lemma A.8. *The functions*

$$(A.14) \quad r_1(\omega, x, \mu, \tau) = - \sum_{m=1}^{\infty} \left((\phi_\tau^{-1}(m, \omega, \mu))_* R(\theta^{m-1}\omega, \varphi_\tau^{-1}(\theta^{m-1}\omega, \cdot, \mu), \mu) \right)(x)$$

and

$$(A.15) \quad r_2(\omega, x, \mu, \tau) = \sum_{m=0}^{\infty} \left((\phi_\tau^{-1}(-m, \omega, \mu))_* R(\theta^{-m-1}\omega, \varphi_\tau^{-1}(\theta^{-m-1}\omega, \cdot, \mu), \mu) \right)(x)$$

are formal solutions of (A.13).

The proof of this lemma is straightforward. In the next lemma, we will see the convergence of the formal series solutions. Let

$$\bar{R}_1(\omega, x, \mu, \tau) = \tilde{R}_1(\omega, \phi_\tau^{-1}(1, \omega, x, \mu), \mu), \quad \bar{R}_2(\omega, x, \mu, \tau) = \tilde{R}_2(\omega, \psi_\tau^{-1}(1, \omega, x, \mu), \mu),$$

then \bar{R}_1 and \bar{R}_2 are $C^{[N/2]}$ and satisfy

$$(A.16) \quad |\bar{R}_1(\omega, x, s)| \leq \bar{C}_0 |y|^{N/2} \sigma_{\rho_1}(|y|), \quad \text{if } jet_{x=0}^N R = 0,$$

$$(A.17) \quad |\bar{R}_2(\omega, x, s)| \leq \bar{C}_0 |z|^{N/2} \sigma_{\rho_1}(|z|), \quad \text{if } jet_{x=0}^N R = 0$$

where \bar{C}_0 is a positive constant.

Lemma A.9. *For any given integer $k > 0$, there exists an integer $N_0 = N_0(k, \alpha, \beta)$ such that if $N \geq N_0$ and $\text{jet}_{x=0}^N R = 0$, then the functions r_1 and r_2 defined by*

$$(A.18) \quad r_1(\omega, x, \mu, \tau) = - \sum_{n=1}^{\infty} \left((\phi_{\tau}^{-1}(n, \omega, \mu))_* \bar{R}_1(\theta^{n-1}\omega, \cdot, \mu, \tau) \right)(x)$$

and

$$(A.19) \quad r_2(\omega, x, \mu, \tau) = \sum_{n=0}^{\infty} \left((\psi_{\tau}^{-1}(-n, \omega, \mu))_* \bar{R}_2(\theta^{-n-1}\omega, \cdot, \mu, \tau) \right)(x)$$

are C^k smooth from $\mathbb{R} \times \mathbb{R}^d \times [-\mu_0, \mu_0] \times [0, 1]$ to \mathbb{R}^d and satisfy (A.12).

As we did in the proof of Lemma 4.7 [LL], we use the results on invariant foliations to estimate $\Pi_s \phi_{\tau}(n, \omega, x, \mu)$ for $n \geq 0$ and $\Pi_u \phi_{\tau}(n, \omega, x, \mu)$ for $n \leq 0$. The proof of Lemma 4.7 gives that r_1 and r_2 are C^k in x and τ . Using Lemma A.5 and Lemma A.6 and the same dominated convergent series there, we have r_1 and r_2 are C^k in all variables.

Summarizing the above discussions gives

Proposition A.1. *For each $k \in \mathbb{N}$, there exists an integer $N_0 = N_0(k, \alpha, \beta)$ such that if ϕ and ψ are C^N for $N \geq N_0$ and (A.9) holds for $(\omega, \mu) \in \mathbb{R} \times [-\mu_0, \mu_0]$, then $\tilde{\phi}$ and $(\tilde{\phi} + \tilde{R}_1)$, $(\tilde{\phi} + \tilde{R}_1)$ and $(\tilde{\phi} + \tilde{R}_1 + \tilde{R}_2) = \tilde{\psi}$ are C^k locally conjugate;*

This proposition together with Lemma A.5 and the nonautonomous transformation (A.7) gives Theorem A.3.

The proof of Theorem A.2 follows from the slight modification of Corollary 8.2.10 in [AR] on normal form and Theorem A.3.

APPENDIX B. PROOF OF PROPOSITION 4.2

In this appendix we prove Proposition 4.2. We start with the defining equations for Σ^+ in (s, Z, t) .

Lemma B.1. *We have for $(s, Z, t) \in \Sigma^+$*

$$s = L^+ + \mathcal{O}_{Z,t}(\mu).$$

Proof: We have on Σ^+ ,

$$(B.1) \quad \begin{aligned} a(s) + v(s)z &= \varepsilon + \mathbb{P}(\varepsilon, Y) + \mu \tilde{\mathbb{P}}(\varepsilon, Y, t), \\ b(s) - u(s)z &= Y + \mathbb{Q}(\varepsilon, Y) + \mu \tilde{\mathbb{Q}}(\varepsilon, Y, t). \end{aligned}$$

By the definition

$$(B.2) \quad \begin{aligned} a(L^+) &= \varepsilon + \mathbb{P}(\varepsilon, 0), \\ b(L^+) &= \mathbb{Q}(\varepsilon, 0). \end{aligned}$$

Let

$$(B.3) \quad \begin{aligned} W_1 &= a(s) - a(L^+) + v(s)z - \mu \tilde{\mathbb{P}}(\varepsilon, 0, t), \\ W_2 &= b(s) - b(L^+) - u(s)z - \mu \tilde{\mathbb{Q}}(\varepsilon, 0, t). \end{aligned}$$

We have from (B.1) and (B.2),

$$\begin{aligned} W_1 &= \mathbb{P}(\varepsilon, Y) - \mathbb{P}(\varepsilon, 0) + \mu(\tilde{\mathbb{P}}(\varepsilon, Y, t) - \tilde{\mathbb{P}}(\varepsilon, 0, t)), \\ W_2 &= Y + \mathbb{Q}(\varepsilon, Y) - \mathbb{Q}(\varepsilon, 0) + \mu((\tilde{\mathbb{Q}}(\varepsilon, Y, t) - \tilde{\mathbb{Q}}(\varepsilon, 0, t))) \end{aligned}$$

which we rewrite as

$$(B.4) \quad \begin{aligned} W_1 &= (\mathcal{O}(\varepsilon) + \mu \mathcal{O}_t(1))Y + \mathcal{O}_{Y,t}(1)Y^2, \\ W_2 &= (1 + \mathcal{O}(\varepsilon) + \mu \mathcal{O}_t(1))Y + \mathcal{O}_{Y,t}(1)Y^2. \end{aligned}$$

We first obtain

$$(B.5) \quad Y = (1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2$$

by inverting the second line in (B.4). We then substitute it into the first line in (B.4) to obtain

$$\begin{aligned} W_1 &= (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))((1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2) \\ &\quad + \mathcal{O}_{Y,t}(1)((1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2)^2 \\ &= (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2. \end{aligned}$$

Consequently,

$$(B.6) \quad F(s, Z, t) := W_1 - (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2 = 0,$$

where W_1, W_2 as function of s, Z, t are defined by (B.3). To rewrite W_1 and W_2 , we let

$$(B.7) \quad \xi = s - L^+$$

and expand $a(s)$ in terms of ξ as

$$a(s) = a(L^+) + a'(L^+)\xi + \sum_{i=2}^{\infty} a_i(L^+)\xi^i.$$

Expansions for $b(s)$, $u(s)$, and $v(s)$ are similar. We have

$$(B.8) \quad \begin{aligned} W_1 &= a'(L^+)\xi + \sum_{i=2}^{\infty} a_i(L^+)\xi^i + v(L^+)z + (v'(L^+)\xi + \sum_{i=2}^{\infty} v_i(L^+)\xi^i)z \\ &\quad - \mu\tilde{\mathbb{P}}(\varepsilon, 0, t), \\ W_2 &= b'(L^+)\xi + \sum_{i=2}^{\infty} b_i(L^+)\xi^i - u(L^+)z - (u'(L^+)\xi + \sum_{i=2}^{\infty} u_i(L^+)\xi^i)z \\ &\quad - \mu\tilde{\mathbb{Q}}(\varepsilon, 0, t). \end{aligned}$$

We now put (B.8) for W_1 and W_2 back into equation (B.6) and replace z by μZ . We obtain

$$(a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+) + h(t, \xi)\xi)\xi = \mathcal{O}_{Z,t}(\mu)$$

where the C^r norm of $h(t, \xi)$ is bounded from above by $K(\varepsilon)$. Also note that $a'(L^+) \approx -\alpha\varepsilon$, $b'(L^+) = \mathcal{O}(\varepsilon^2)$. We finally obtain

$$s = L^+ + \mathcal{O}_{Z,t}(\mu)$$

by solving ξ . This completes the proof of Lemma B.1. \square

Lemma B.1 is not precise enough. We need the following refinement.

Lemma B.2. *We have on Σ^+ ,*

$$s - L^+ = -\frac{v(L^+) + \mathcal{O}(\varepsilon)u(L^+)}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)}z + \frac{\mu}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)}\mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu^2).$$

Proof: It suffices for us to drop all terms that are $\mathcal{O}_{Z,t}(\mu^2)$ in equation (B.6) to solve for ξ . From Lemma B.1 we conclude that all terms in ξ, z of degree higher than one are $\mathcal{O}_{Z,t}(\mu^2)$. With these terms all dropped, (B.6) becomes

$$(B.9) \quad (a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+))\xi + (v(L^+) + \mathcal{O}(\varepsilon)u(L^+))z = \mu\mathcal{O}_t(1),$$

from which the estimates of Lemma B.2 on Σ^+ follows. \square

Recall that $\mathbb{X} = \mu^{-1}X$, $\mathbb{Y} = \mu^{-1}Y$.

Lemma B.3. *On Σ^+ we have*

$$\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))Z + \mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu).$$

Proof: We have

$$\begin{aligned}
 (B.10) \quad Y &= (1 + \mathcal{O}(\varepsilon))(b'(L^+)\xi - u(L^+)z - \mu\tilde{Q}(\varepsilon, 0, t)) + \mathcal{O}_{Z,t}(\mu^2) \\
 &= (1 + \mathcal{O}(\varepsilon)) \left(- \left(u(L^+) + b'(L^+) \frac{v(L^+) + \mathcal{O}(\varepsilon)u(L^+)}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)} \right) z \right. \\
 &\quad \left. + \frac{\mu b'(L^+)}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)} \mathcal{O}_t(1) - \mu\tilde{Q}(\varepsilon, 0, t) \right) + \mathcal{O}_{Z,t}(\mu^2) \\
 &= (1 + \mathcal{O}(\varepsilon))z + \mu\mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu^2),
 \end{aligned}$$

where the first equality follows from using (B.5), (B.8) and Lemma B.1; the second equality from using Lemma B.2. To obtain the third equality we use $u(L^+) = -1 + \mathcal{O}(\varepsilon)$, $a'(L^+) \approx -\alpha\varepsilon$, $b'(L^+) = \mathcal{O}(\varepsilon^2)$. \square

Lemma B.1 is Proposition 4.2(a)(i) and Lemma B.3 is Proposition 4.2(a)(ii). Proposition 4.2(b) follows from parallel computations.

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