Chaos in non-autonomous equations without any time periodicity

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List of Contents

Part I

Introduction

Unperturbed equation

Let $(x, y) \in \mathbb{R}^2$ be the phase variables,

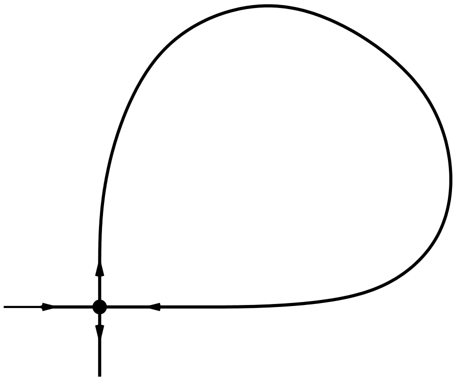
$$\frac{dx}{dt} = -\alpha x + f(x, y),$$
$$\frac{dy}{dt} = \beta y + g(x, y)$$

where f, g are higher order terms at $(0, 0)$.

- Dissipative saddle: $0 < \beta < \alpha$.

- Homoclinic solution: $(0, 0)$ is with a homoclinic solution, which we denote as

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$



Non-autonomously Perturbed equation

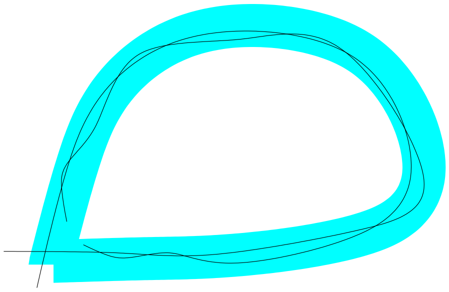
$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t)$$

where

$$|P(x,y,t)|_{C^2}, \quad |Q(x,y,t)|_{C^2} < K.$$

No periodicity conditions in t, not even quasi-periodic, nor almost periodicity, are imposed.

Fundamental object: $\Lambda =$ the maximum set of solutions staying around the homoclinic loop for all time.



A brief review of history

(I) Time-periodic equations

(1) Smale-Melnikov method (Smale, Melnikov, Marsden, Homles, Guckenheimer)

(2) Analytic shadowing (Palmer, Stoffer, Meyer, Sell, etc)

(3) Method of Lyapunov-Schmidt: (Hale, Chow, Mallet-Paret, etc)

(II) Quasi-periodic equations:

(1) Extension of Smale-Melnikov (Shilnikov, Wiggins)

(2) Extension of analytic shadowing (Palmer, Meyer, Sell)

(III) Without any periodicity in time

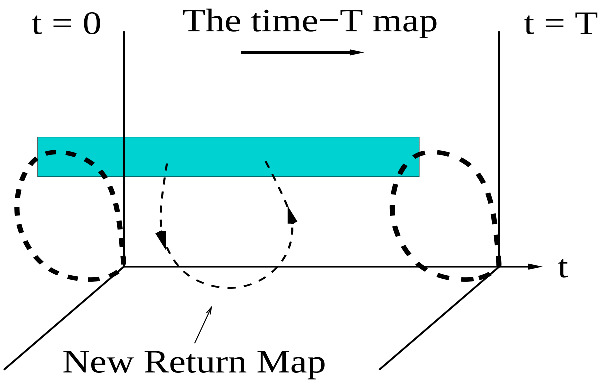
Isolated efforts: (Lerman and Shilnikov; Siegmund, Marsden, etc.)

Part II

Fundamental Objects

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t)$$

The separatrix map $\mathcal{F}$:



- Derivation is tedious, long, and technically difficult

- We need to assume that $P(x,y,t), Q(x,y,t)$ are both higher order terms in (x,y) at $(x,y) = (0,0)$.

Formula for $\mathcal{F}$

- Domain of $\mathcal{F}$: $\Sigma = \mathbb{R} \times I$.

- The Melnikov function:

$$M(t) = \int_{-\infty}^{\infty} (v(s)P(\ell(s), s+t) \\ - u(s)Q(\ell(s), s+t))e^{-\int_0^s E(\tau)d\tau} ds$$

- $(t_1, z_1) = \mathcal{F}(t, z)$: $t \in \mathbb{R}, \quad z \in I$

$$t_1 = t + \mathbf{a} - \beta^{-1} \ln \mathbb{F}(t, z) + \mathbb{E}_1$$
$$z_1 = \mathbf{b}[\mathbb{F}(t, z)]^{\frac{\alpha}{\beta}}$$

where

- $\mathbf{a} \approx \beta^{-1} \ln \mu^{-1}, \qquad \mathbf{b} \approx \mu^{\frac{\alpha}{\beta} - 1}$,

- $\mathbb{F}(t, z) = \mathbf{k}z + M(t) + \mathbb{E}_2$,

- $1 >> \mathbf{k} >> \mu$.

- $\mathbb{E}_1$, $\mathbb{E}_2$ are error terms.

$\mathcal{F}$ is a new kind of map

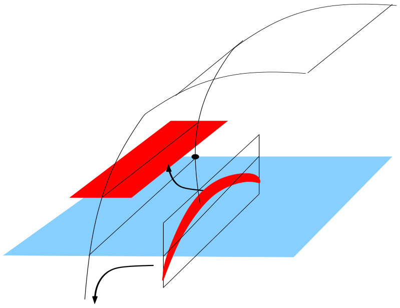
- It appears not suitable for chaos theory:

  (1) Phase space is not compact.

  (2) Orbits are not recurrent.

- It is, however, a global map that catches the dynamics of ALL solutions close to ℓ in the extended phase space.

- $\mathcal{F}$ is only partially defined on $\Sigma = \mathbb{R} \times I$ if $\min_{(t,z)\in\Sigma} \mathbb{F}(t,z) < 0$.



Dynamical Objects:

$$V = \{(t,z) \in \Sigma : \quad \mathbb{F}(t,z) > 0\}$$

$$\Omega = \{p \in V : \mathcal{F}^n(p) \in V, \quad \text{for all} \quad n \geq 1\}$$

$$\Lambda = \cap_{n\geq 1}\mathcal{F}^n(\Omega).$$

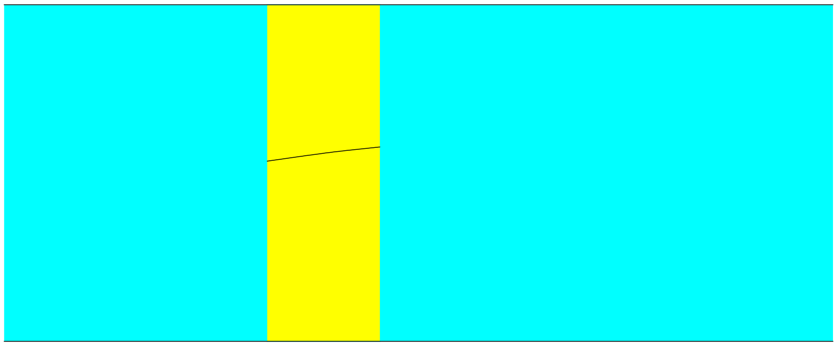
Ultimate Goal: To understand and to describe the geometric and dynamical structure of Λ in its entirity.

A Limited Goal: To construct complicated subsets of Λ through $\mathcal{F}$.

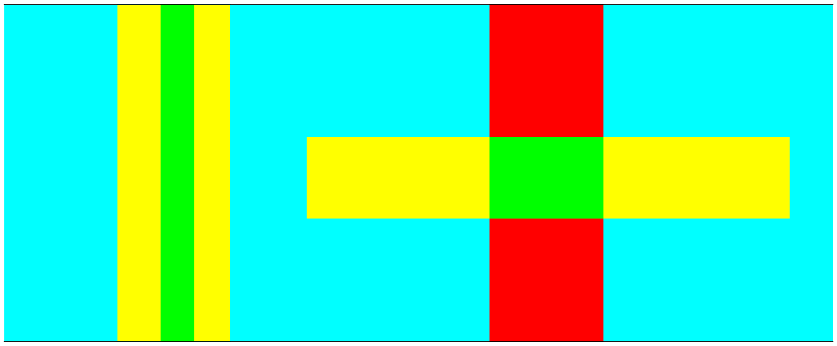
Aim of This Work: A Melnikov-like method applicable to a given equation.

Horseshoe for $\mathcal{F} : \Sigma \to \Sigma$

Geometric terms: Horizontal direction, vertical direction; vertical curves, vertical strips V; fully extended horizontal curve in V;



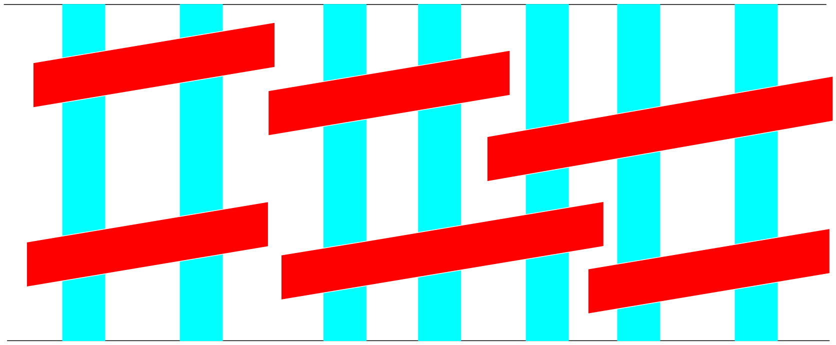
Horizontal crossing: $\mathcal{F}(V_1)$ horizontally cross V_2



Definition (Topological horseshoe) $\mathcal{F}$ admits a topological horseshoe of k-branches if there exists a bi-infinite sequence of non-intersecting vertical strips $\{V_n\}_{n=-\infty}^{\infty}$ lined up monotonically from $t = -\infty$ to $t = +\infty$ in Σ, such that

(1) For every m, there exists an $n > m$, such that $\mathcal{F}(V_m)$ crosses $V_n, V_{n+1}, \cdots V_{n+k}$ horizontally.

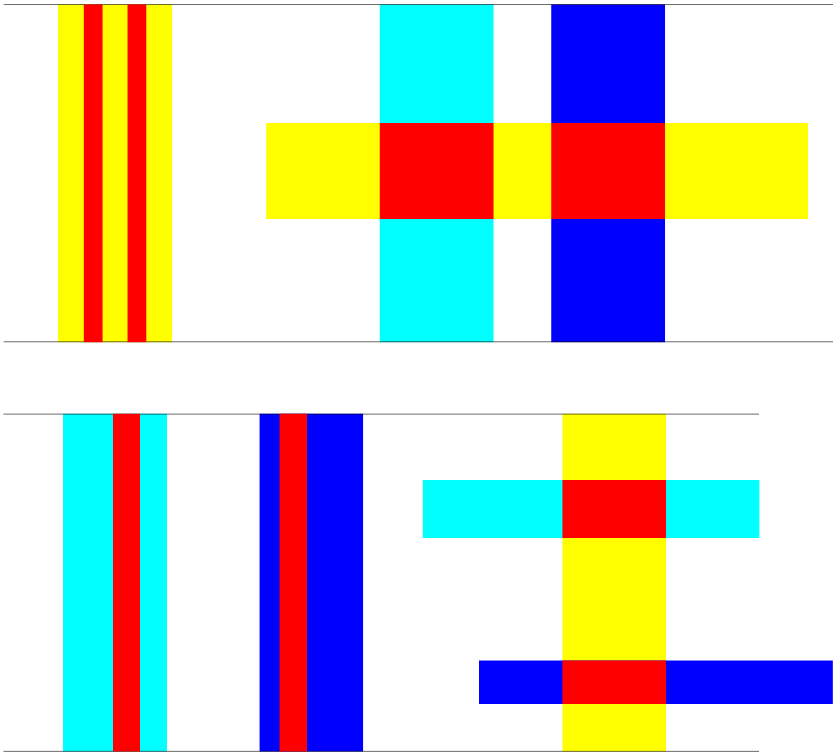
(2) For every m, there exists $\hat{n} < m$, such that $\mathcal{F}(V_{\hat{n}-k}), \cdots, \mathcal{F}(V_{\hat{n}})$ crosses V_m horizontally.



A horseshoe of two branches

Implication of a Horseshoe

Claim: In every vertical strip V_i, there is a Cantor set Λ_i of k branches, so that $\Lambda_i \subset \Lambda$.



For Periodic Equations:

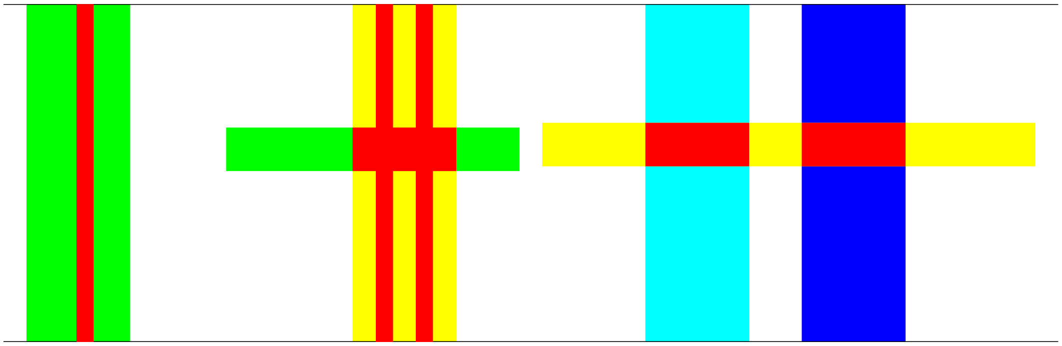
- $\{V_n\}_{n=-\infty}^{+\infty}$ are period in t-direction.
- Σ is reduced to an annulus, and this structure is reduced to a classical horseshoe.

Half-horseshoe

Definition (Half Horseshoe) $\mathcal{F}$ admits a Half Horseshoe of k-branches if there exists a bi-infinite sequence of non-intersecting vertical strips $\{V_n\}_{n=-\infty}^{\infty}$ lined up monotonically from $t=-\infty$ to $t=+\infty$ in Σ, such that

(1) For every $m>0$, there exists an $n>m$, such that $\mathcal{F}(V_m)$ crosses $V_n, V_{n+1}, \cdots V_{n+k}$ horizontally.

(2) For all m, there exists $\hat{n}<m$, such that $\mathcal{F}(V_{\hat{n}})$ crosses V_m horizontally.



Geometric structure: In every V_m, there is a 1D Cantor set Λ_m of k-branches, so that $\Lambda_m \subset \Lambda$.

Part III

Results

$$
\begin{aligned}
M(t) = \int_{-\infty}^{\infty} (v(s)P(\ell(s), s+t) \\
- u(s)Q(\ell(s), s+t))e^{-\int_0^s E(\tau)d\tau} ds
\end{aligned}
$$

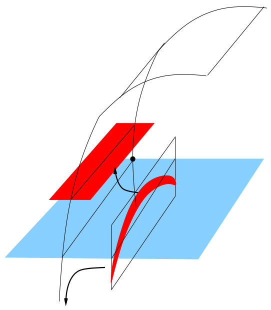
$$M = \sup_{t\in\mathbb{R}} M(t), \quad m = \inf_{t\in\mathbb{R}} M(t)$$

Proposition 1 (Homoclinic Intersections)

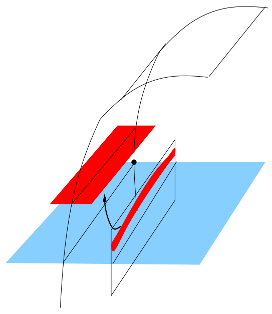
(a) $m < 0 < M$ $(W^s \cap W^u \neq \emptyset)$;

(b) $0 < m \leq M$ $(W^s \cap W^u = \emptyset)$;

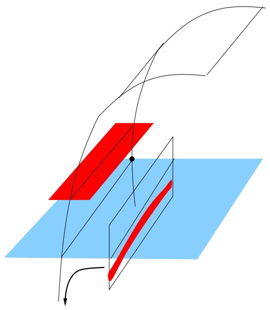
(c) $m \leq M < 0$ $(W^s \cap W^u = \emptyset)$.



(a)



(b)

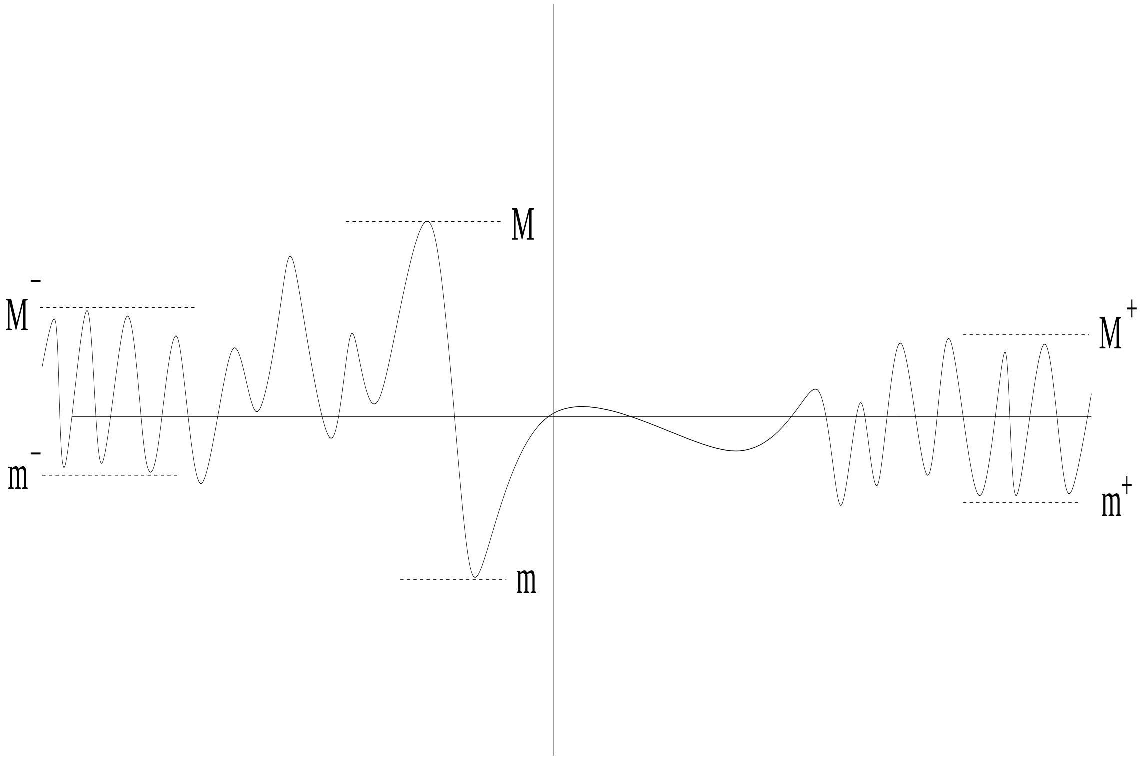


(c)

Limit Values for $M(t)$:

$$m^+ = \liminf_{t\to+\infty} M(t); \qquad M^+ = \limsup_{t\to+\infty} M(t).$$

$$m^- = \liminf_{t\to-\infty} M(t); \qquad M^- = \limsup_{t\to-\infty} M(t).$$



Equations with Periodicity:

$$m = m^+ = m^-; \quad M = M^+ = M^-$$

Scenario (a): With Homoclinic Solutions

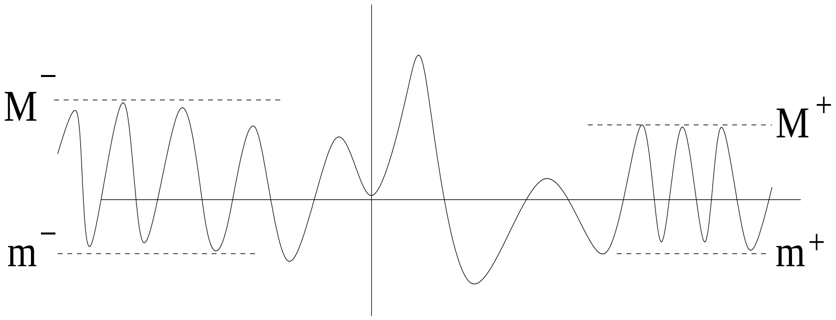
Theorem 1 (Chaos) Assume that

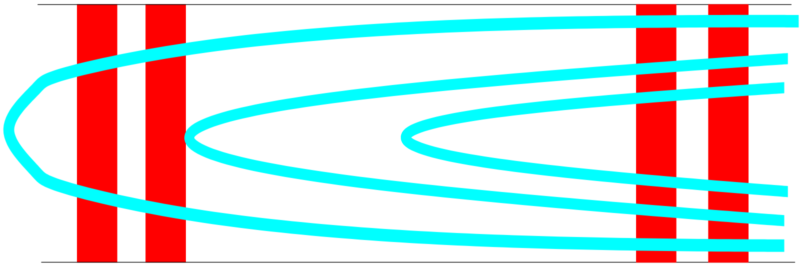
(i) $m^-, m^+ < 0 < M^-, M^+$.

(ii) There exists $t_0,\ c > 0$, s.t.

$$min\{|M'(t)| : M(t) = 0,\ |t| > t_0\} > c.$$

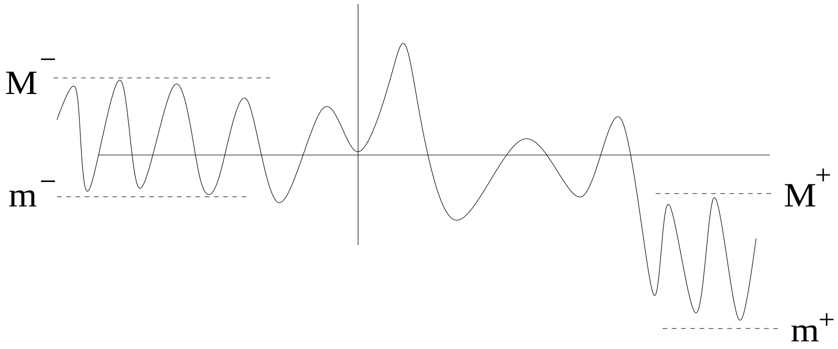
Then $\mathcal{F}$ admits a topological horseshoe of infinite branches.



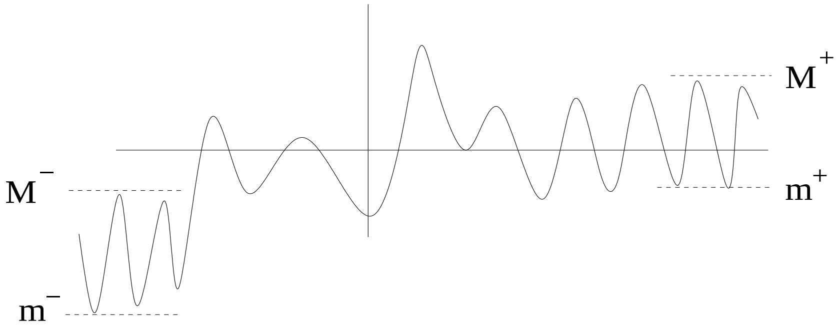


Homoclinic solutions: Even Infinitely many homoclinic solutions do not necessarily induce chaotic dynamics.

Example: $m^- < 0 < M^-; \quad m^+ < M^+ < 0$



Case for trivial dynamics: If either $M^- < 0$ or $M^+ < 0$, then Λ is empty.



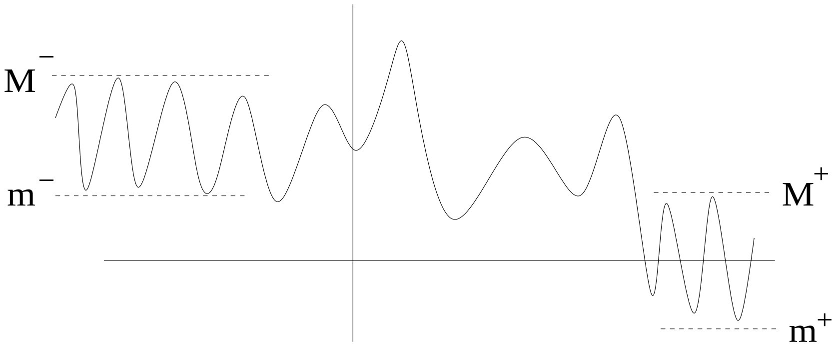
Theorem 2 (Half-horseshoe) Assume

(i) $m^-, M^- > 0, \quad m^+ < 0 < M^+$;

(ii) There exists $t_0,\ c > 0$, s.t.

$$min\{|M'(t)| : M(t) = 0,\ t > t_0\} > c.$$

Then Λ admits a half topological horseshoe.



Remarks: For almost periodic equations

$$M = M^{\pm}, \quad m = m^{\pm}.$$

So this theorem is exclusively for equations without any periodicity.

Scenario (b): No Homoclinic Solutions

Theorem 3 (Integral Manifold) Assume

$$m > 0 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \frac{1}{\beta} \left| \frac{M'(t)}{M(t)} \right| < 1.$$

Then

(a) Λ admits an invariant curve h defined by $h(t) : \mathbb{R} \to I$.

(b) h is globally attracting: $\forall p \in \Sigma$, $\exists p_0 \in h$, s.t.

$$\lim_{n \to +\infty} |\mathcal{F}^n(p) - \mathcal{F}^n(p_0)| \to 0.$$

Remark: (1) $m > 0$ implies $W^s \cap W^u = \emptyset$.

(2) This Theorem is an extension of classic results of Levinson, Krylov, Bogoliubov Mitropolsky, Hale, etc.

Theorem 4 (Horseshoe) Assume

(i) $m^+, M^+, M^- > 0$;

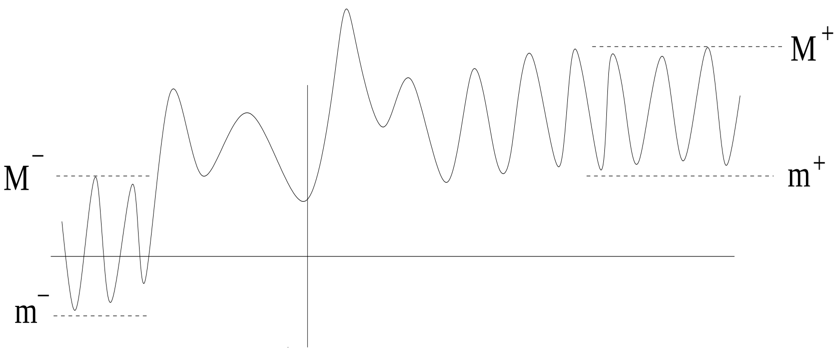
(ii) (L^+-dense sequences) $\exists L^+$, and sequences $a_k, b_k \to \infty$ satisfying $0 < a_{k+1} - a_k,\ b_{k+1} - b_k < L^+$ for all $k \geq 0$ so that

$$\lim_{k\to\infty} M(a_k) = m^+, \quad \lim_{k\to\infty} M(b_k) = M^+;$$

(iii) (Persistent Oscillations) $M^+ > m^+ e^{3\beta L^+}$.

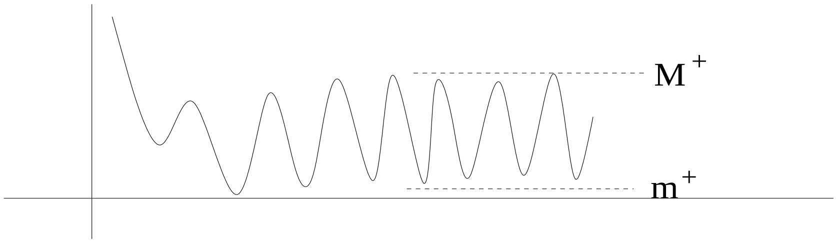
Then $\mathcal{F}$ admits a half horseshoe.

Assumptions (i) and (ii):

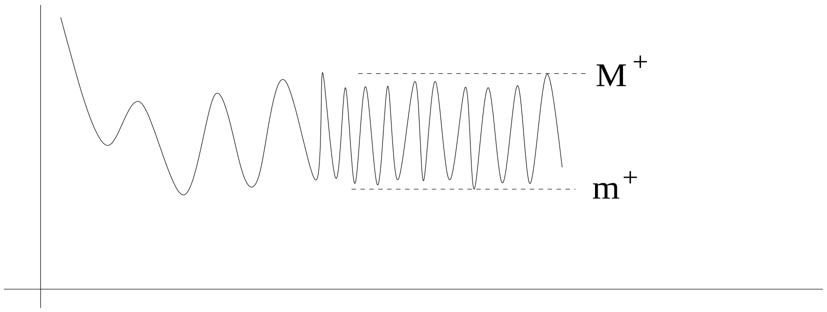


Assumption (iii): $M^+ > m^+ e^{3\beta L^+}$

- m^+ small (W^u and W^s persistently getting close as $t \to +\infty$)



- L^+ small (high frequency oscillation of $M(t)$ between m^+ and M^+ as $t \to +\infty$)



Part IV

An Example of Applications

Application to a given Equation:

(i) Computing $M(t)$;

(ii) Compute $m, M, m^{-}, m^{+}, M^{-}, M^{+}$.

Step 1: We start with the Duffing equation

$$\frac{d^2q}{dt^2} - q + q^3 = 0.$$

Step 2: Add non-linear dumping terms

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^3 = 0.$$

For $\lambda > 0$ small, there exists γ_λ so that it has a homoclinic loops to a dissipative saddle.

Step 3: Add a periodic perturbation

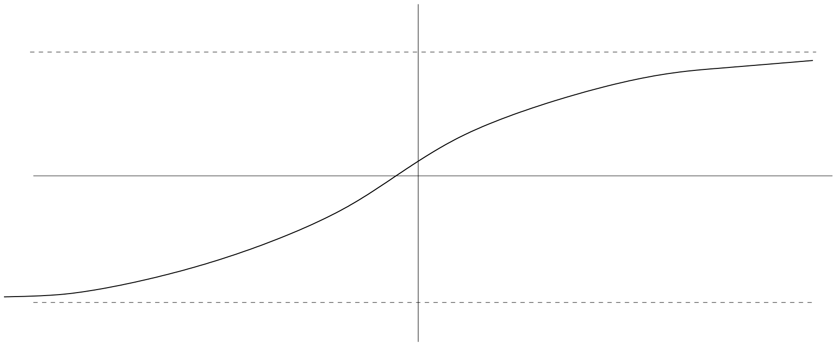
$$\frac{d^2q}{dt^2} + (\lambda - \gamma_\lambda q^2)\frac{dq}{dt} - q + q^3 = \mu \sin \omega t$$

Step 4: Introduce Non-periodicity

- Let $\Phi_{c_1,c_2} : \mathbb{R} \to [\min\{c_1, c_2\}, \max\{c_1, c_2\}]$ be a C^r function such that

$$\lim_{t\to-\infty} \Phi_{c_1,c_2}(t) = c_1, \qquad \lim_{t\to+\infty} \Phi_{c_1,c_2}(t) = c_2$$

where c_1, c_2 are two real numbers.



- (i) Multiplying a function $\Phi_{\eta^-,\eta^+}(t)$ onto the right hand side; (ii) replacing the damping constant γ by $\gamma_\lambda + \mu\Phi_{\tau^-,\tau^+}(t)$.

$$\frac{d^2q}{dt^2} + (\lambda - (\gamma_\lambda + \mu\Phi_{\tau^-,\tau^+}(t))q^2)\frac{dq}{dt} - q + q^3$$
$$= \mu q^2 \Phi_{\eta^-,\eta^+}(t) \sin \omega t$$

Denote

$$J = \int_{-\infty}^{+\infty} (v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s))e^{-\int_0^s E_\lambda(\tau)d\tau} ds$$

$$J_s = \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s)) \sin(\omega s)e^{-\int_0^s E_\lambda(\tau)d\tau} ds$$

$$J_c = \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s)) \cos(\omega s)e^{-\int_0^s E_\lambda(\tau)d\tau} ds.$$

Proposition 2 *We have for all $\lambda \in (0, \lambda_0)$ sufficiently small,*

(a) $J > 0$, $J_s^2 + J_c^2 \neq 0$;

(b) $m^\pm = J\tau^\pm - \sqrt{J_s^2 + J_c^2}\eta^\pm$, $M^\pm = J\tau^\pm + \sqrt{J_s^2 + J_c^2}\eta^\pm$; and

(c) $m^\pm$, $M^\pm$ are all approached by T-dense sequences where $T = \frac{4\pi}{\omega}$.

Conclusion: (1) All combinations of $m^\pm, M^\pm$ are realized for this example.

(2) Theorems 1-4 applies.

Conclusive Remarks:

(1) Derived the separatrix map (for a given non-autonomously perturbed equations without any periodicity in time assumed).

(2) A full extension (of the classical Smale-Melnikov method).

(3) More results (than Smale-Melnikov method even in periodic case).

(4) Method is fundamentally different (from Shilnikov-Wiggins, Meyer-Sell, and Shilnikov-Lerman).

(5) Extensions of other results? (based on the separatrix map, e.g. positive Liapunov exponents, etc)