Dynamics of Periodically Perturbed Homoclinic Solutions

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This talk is based on:

[1] Q. Wang and W. Ott, preprint (2007)

[2] Q. Wang and A. Oksasoglu, preprint (2008)

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V. Afraimovich for

- Led us to Afraimovich-Shilnikov
- Many helpful feedbacks.

Kening Lu for

Help shaping my outlook on this project.

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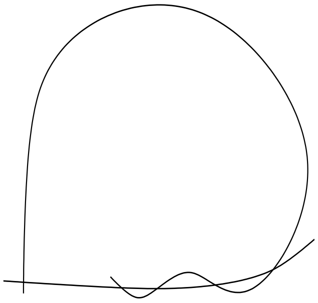
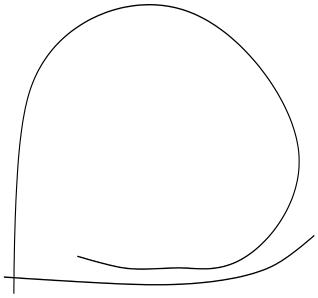
Part I

Introduction





Unperturbed equation

Let $(x, y) \in \mathbb{R}^2$ be the phase variables.

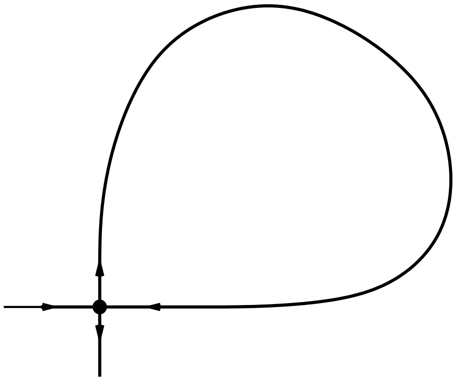
$$\frac{dx}{dt} = -\alpha x + f(x, y),$$
$$\frac{dy}{dt} = \beta y + g(x, y)$$

where f, g are high order terms at $(0, 0)$.

- **Dissipative saddle:** $0 < \beta < \alpha$.

- **Homoclinic solution:** $(0, 0)$ is with a homoclinic solution

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$



Periodically perturbed equation

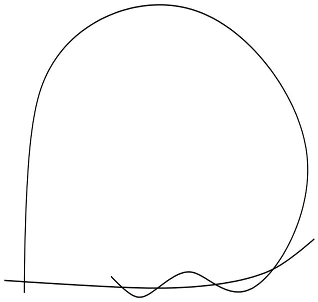
$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t)$$

where

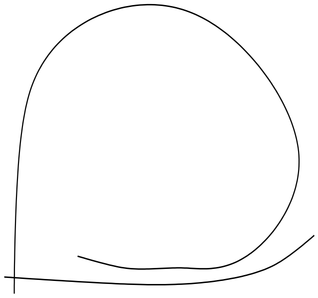
$$P(x,y,t+T) = P(x,y,t),$$
$$Q(x,y,t+T) = Q(x,y,t)$$

for some $T > 0$.

Two Scenarios:



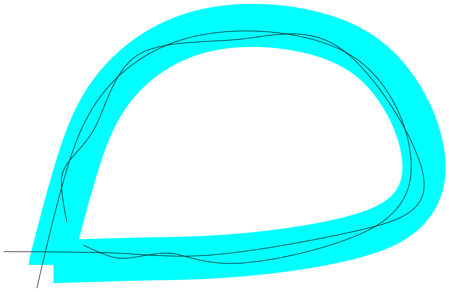
Scenario (a)



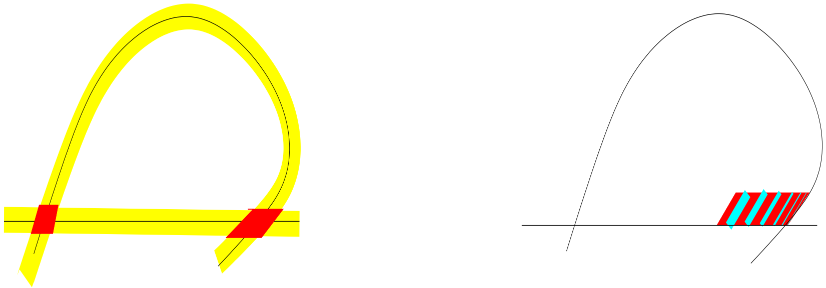
Scenario (b)

Scenario (a): Case of intersections

Fundamental object: $\Lambda =$ the maximum set of solutions staying around the homoclinic loop.



Existing results: (Complicated subsets of Λ)



Basic question: Can we describe the geometric and dynamical structure of Λ in its entirety?

Our Results: With assumptions that are generic and explicitly verifiable on

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t)$$

We have periodic occurrence of three types of tangles as $\mu \to 0$,

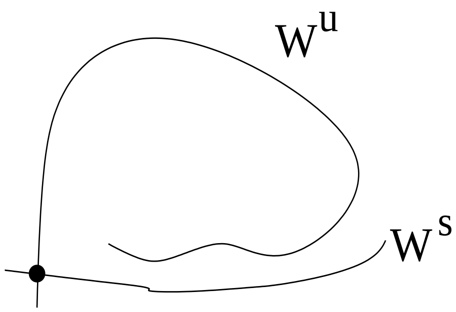
(i) Transient Tangle: Λ contains nothing else but one horseshoe of infinitely many branches.

(ii) Tangle Dominated by Sinks: Λ contains nothing else but some sinks and one horseshoe of infinitely many branches.

(iii) Tangle with Henon-like Attractors: Positive measure set of μ for which Λ admit strange attractors with SRB measures.

Newhouse Tangency: Λ is in and out of Newhouse domain infinitely many times as $\mu \to 0$.

Scenario (b): Case of non-intersections



Question: Is this a relevant case for chaos theory?

Answer: Yes. It exhibits rich dynamics.

— For small ω: Quasi-periodic torus;

— As ω increases: Transition to chaos;

— For large ω: Non-hyperbolic attractors:

- Horseshoes
- Newhouse sinks
- Rank one attractors

Part II

Structure of Homoclinic Tangles

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t)$$

A Characteristic function:

$$M(t) = \int_{-\infty}^{\infty} (v(s)P(\ell(s), s+t) \\ - u(s)Q(\ell(s), s+t))e^{-\int_0^s E(\tau)d\tau} ds$$

where

$$(u(t), v(t)) = \left|\frac{d}{dt}\ell(t)\right|^{-1} \frac{d}{dt}\ell(t);$$

$$E(t) = v^2(t)(-\alpha + \partial_x f(a(t), b(t))) \\ + u^2(t)(\beta + \partial_y g(a(t), b(t))) \\ - u(t)v(t)(\partial_y f(a(t), b(t)) \\ + \partial_x g(a(t), b(t))).$$

We call $M(t)$ the Melnikov function.

Standing Assumption: All intersections of $z = M(t)$ to the t-axis is non-tangential.

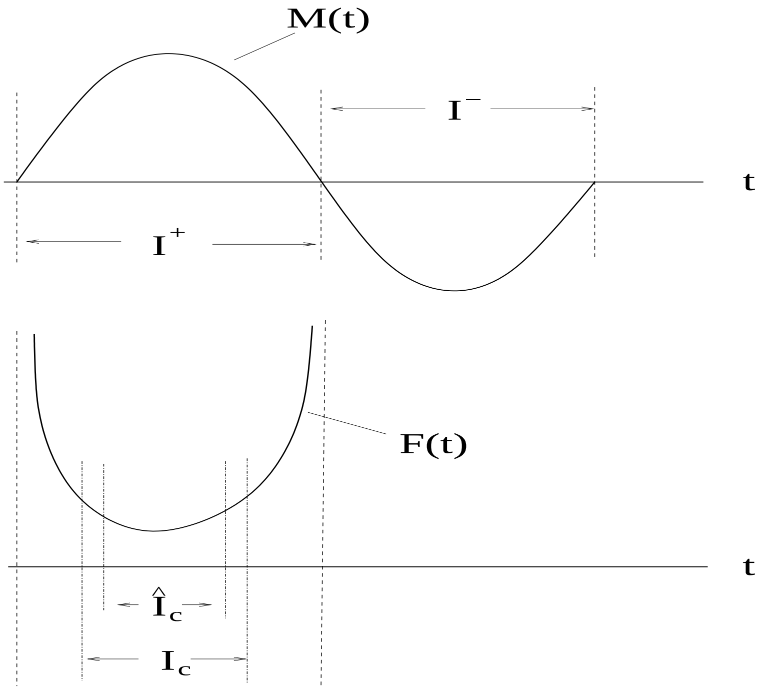
Structure of Homoclinc tangles

— We have $\min_t M(t) < 0 < \max_t M(t)$.

$$I^+ = \{M(t) > 0\}, \quad I^- = \{M(t) \le 0\}.$$

— Let $F(t) := t - \beta^{-1} \ln M(t)$,

$$I_c = \{|F'(t)| < 1^+\}, \quad \hat{I}_c = \{|F'(t)| < 1^-\}.$$

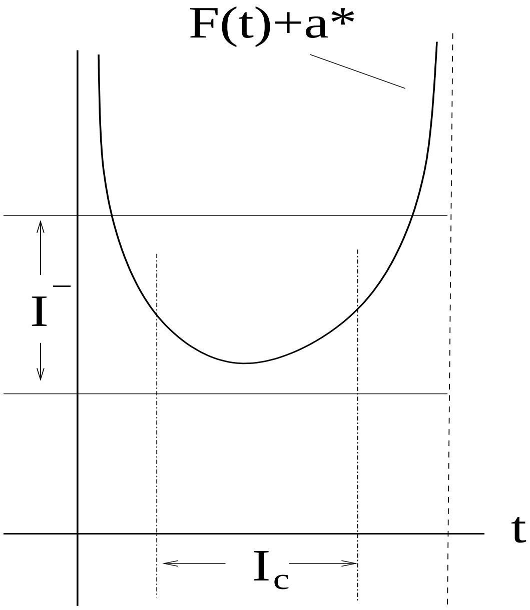


$$F_a(t) = F(t) + a.$$

Theorem 1 (Transient tangles) If $\exists a^*$ s.t.

$$F_{a^*}(I_c) \subset I^-.$$

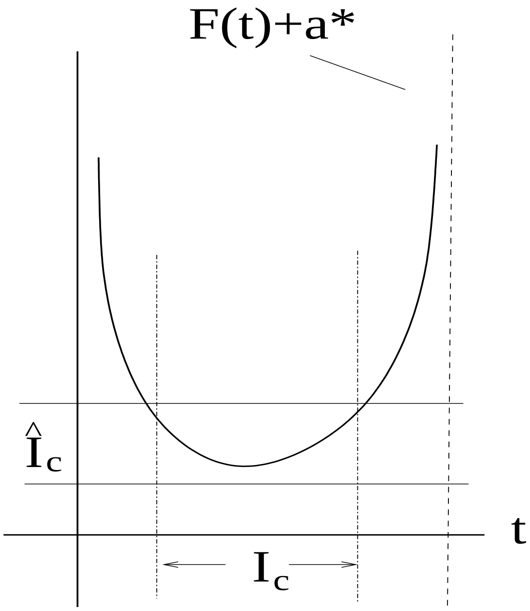
Then $\exists\{\Delta_n\} \to 0$ of μ (open intervals), s. t. Λ contains nothing else but solutions represented by one horseshoe of ∞-branches.



Theorem 2 (Tangles dominated by sinks) If $\exists a^*$ s.t.

$$F_{a^*}(I_c) \subset \hat{I}_c.$$

Then $\exists\{\Delta_n\} \to 0$ of μ (open intervals), s. t. Λ contains nothing else but periodic sinks and one horseshoe of ∞-branches.



Remarks:

- For tangles of Theorem 1: Λ possesses an attractive basin of zero Lesbegue measure.

  Λ does not have physical measures.

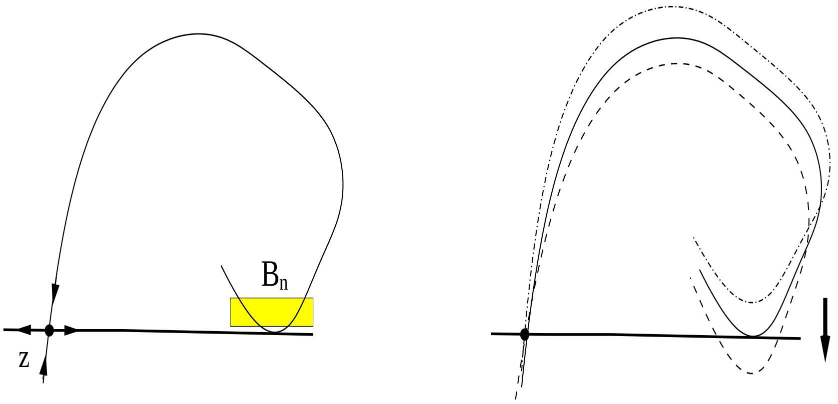
- For tangles of Theorem 2: sinks are the only physical measures admitted.

- Tangles of Theorems 1-2: We understand completely the geometric and dynamic structures.

- Sinks of Theorem 2: They are not Newhouse sinks.

Tangles of Theorems 1 and 2 are out of Newhouse domain. They do not admit homoclinic tangency.

Newhouse Theory



(a) (b)

- Persistency of tangency. (Newhouse, Palis-Takens)

- Infinitely many sinks. (Newhouse)

- Henon-like attractors. (Mora-Viana, Benedick-Young based on Benedick-Carleson)

A sufficient condition: $F(t) = t - \beta^{-1} \ln M(t)$ has a non-degenerate critical point on I^+.

Theorem 3 (Newhouse Tangency) Assume $F(t)$ admits a non-degenerate critical point in I^+. Then $\exists\{\mu_n\} \to 0$, s.t. for $\hat{\mu} = \mu_n$

(a) Newhouse sinks: $\exists \mu_k \to \hat{\mu}$, s.t. Λ admit periodic sinks;

(b) Henon-like attractors: $\exists \mu$ of positive Lesbegue measure close to $\hat{\mu}$, s.t. Λ admit strange attractors with SRB measures.

Remarks:

- Assumptions of Theorems 1-3 could all hold for a given equation.
- As $\mu \to 0$, Λ getting in and out of Newhouse domain infinitely many times.

Application to a Duffing Equation:

(i) Computing $M(t)$ and $F(t) = t - \beta^{-1} \ln M(t)$.

(ii) Verify the assumptions of Theorems 1-3.

Step 1: We start with the Duffing equation

$$\frac{d^2q}{dt^2} - q + q^3 = 0.$$

Step 2: Add non-linear dumping terms

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^3 = 0.$$

For $\lambda > 0$ small, there exists γ_λ so that it has a homoclinic loops to a dissipative saddle.

Step 3: Add periodic perturbations

$$\frac{d^2q}{dt^2} + (\lambda - \gamma_\lambda q^2)\frac{dq}{dt} - q + q^3 = \mu \sin \omega t$$

- $M(t)$ and $F(t)$:

$$M(t) = A(\omega)\sin\omega(t - t_0)$$
$$F(t) = t - \beta^{-1}\ln[A(\omega)\sin\omega(t - t_0)]$$

where $A(\omega) \neq 0$ for all $\omega \neq 0$.

- Standing Assumption:

$$M'(t) \neq 0 \quad at \quad t = t_0,\ t_0 + \omega^{-1}\pi$$

- Condition for Theorem 1:

$$|F(I_c)| \quad (\approx \tfrac{1}{2}\beta\omega^{-2}) < |I^+| \quad (= \omega^{-1}\pi)$$

- Condition for Theorem 2:

$$|F(I_c)| \quad (\approx \tfrac{1}{2}\beta\omega^{-2}) < |\hat{I}_c| \quad (\approx \beta\omega^{-2})$$

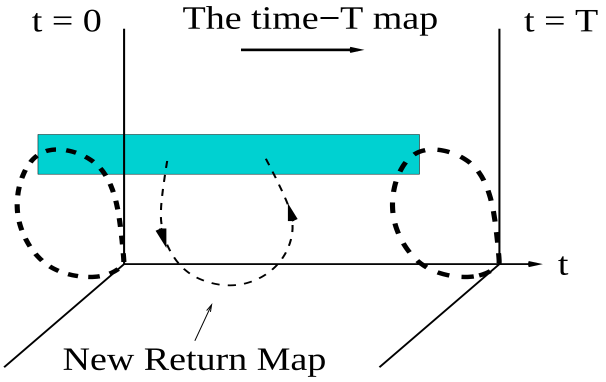
- Condition for Theorem 3:

$$F'(\hat{t}) = 0, \qquad F''(\hat{t}) \neq 0$$

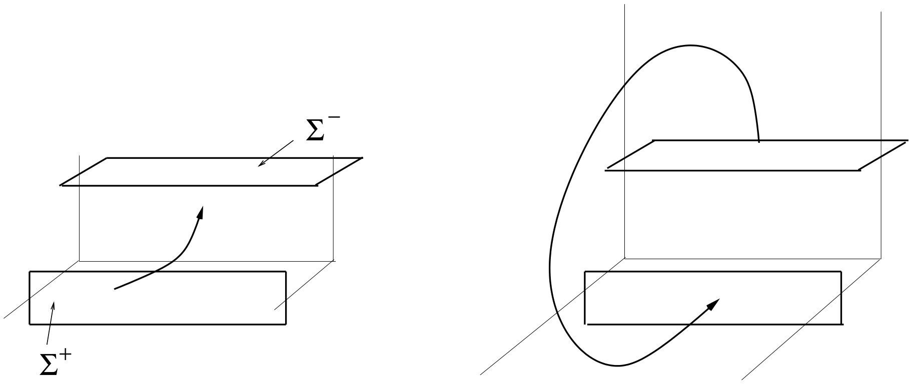
where $\hat{t} = tan^{-1}(\omega\beta^{-1})$,

Proof of Theorems 1-3

Looking into a different direction



- This separatix map is explicitly computable.



- The impact of different initial time are more directly represented.

The Separatrix Map $\mathcal{F} : \mathcal{A} \to \mathcal{A}$:

Result of a long, tedious, and somehow difficult computation.

Let $(t_1, z_1) = \mathcal{F}(t, z)$.

$$t_1 = t + \mathbf{a} - \beta^{-1} \ln \mathbb{F}(t, z) + \mathbb{E}_1$$
$$z_1 = \mathbf{b}[\mathbb{F}(t, z)]^{\frac{\alpha}{\beta}}$$

where

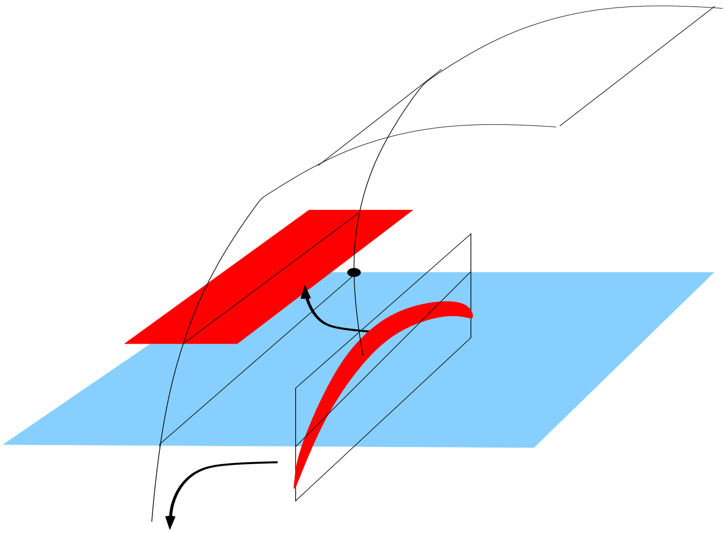
- $\mathbf{a} \approx \beta^{-1} \ln \mu^{-1}$,
- $\mathbf{b} \approx \mu^{\frac{\alpha}{\beta}-1}$,
- $\mathbb{F}(t, z) = \mathbf{k}z + M(t) + \mathbb{E}_2$,
- $1 >> \mathbf{k} >> \mu$.
- $\mathbb{E}_1$, $\mathbb{E}_2$ are error terms.

- $\mathcal{F}$ is only defined on

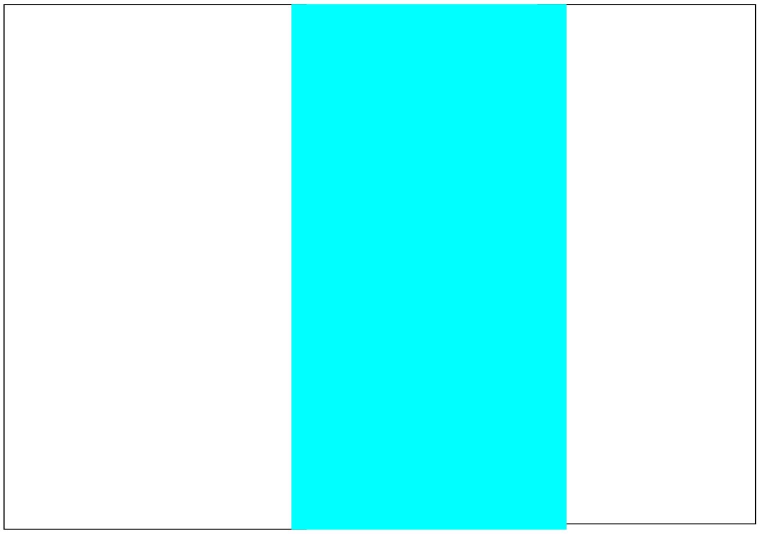
$$\mathbb{F}(t,z) > 0,$$

and recall that

$$\mathbb{F} \approx \mathbf{k}z + M(t).$$



A comprehensive description on Λ

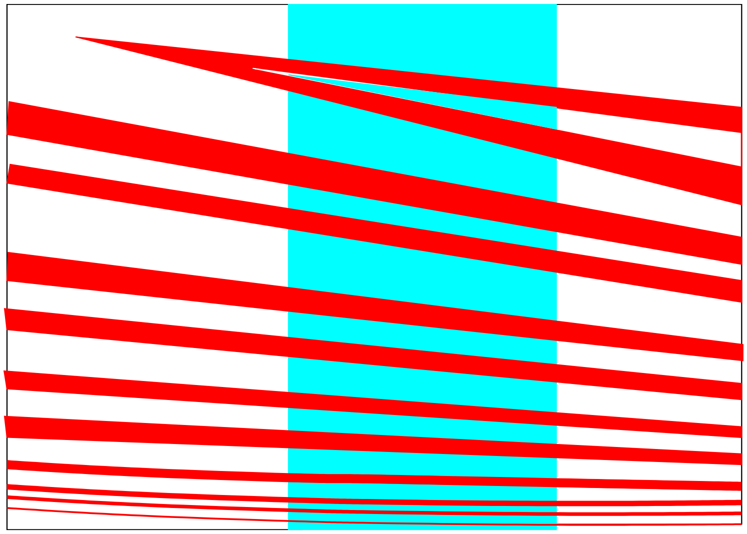


— $\mathcal{A} = U \cup V$, where U, V are two collections of vertical strips. $\mathcal{F}$ is defined on V, not on U.

— On $V = \cup V_i$, $\mathbb{F}(z,t) \approx \mathbf{k}z + M(t) > 0$

— On $U = \cup U_i$, $\mathbb{F}(z,t) \approx \mathbf{k}z + M(t) \leq 0$.

— Here we need the assumption that all intersections of $z = M(t)$ to the t-axis are non-tangential.



— $\mathcal{F}$ acts on V_i as follows:

(i) It compresses V_i in the vertical direction,

(ii) It stretch V_i in horizontal direction, making the image infinitely long towards both ends.

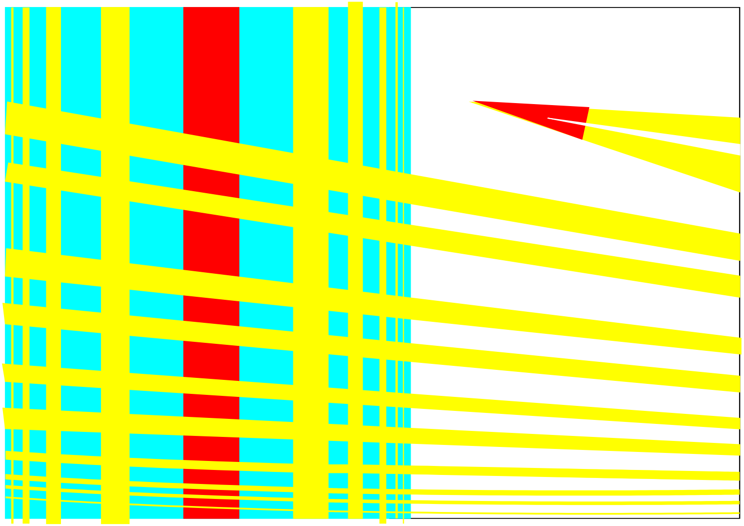
(iii) $\mathcal{F}$ then folds the image and wraps it around $\mathcal{A}$ infinitely many times.

— As $\mu \to 0$, the image of all V_i moves toward $t = +\infty$ with a speed $\approx$ 1 with respect to $\mathbf{a} \approx \ln \mu^{-1}$.

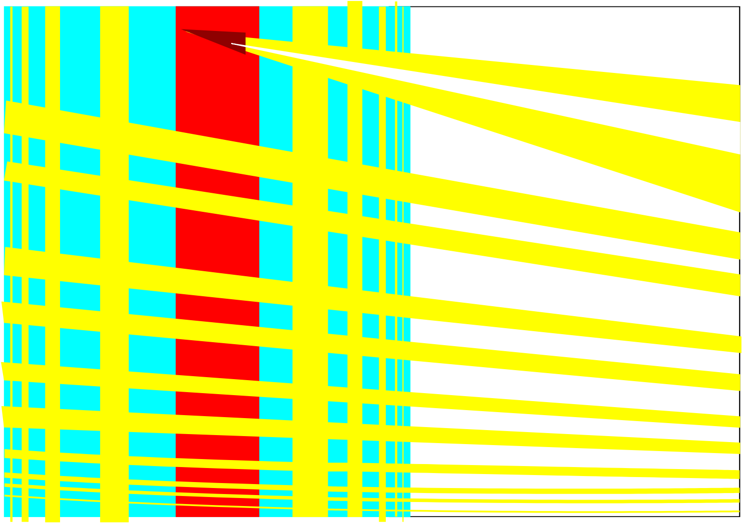
Proof of Theorem 1 The assumption of Theorem 1 implies that all folded part of the images of V are maps into U. In this case

$$\Lambda = \cap_{n\in\mathbb{Z}}\mathcal{F}^n(V)$$

is a uniformly hyperbolic horseshoe of infinitely many branches.



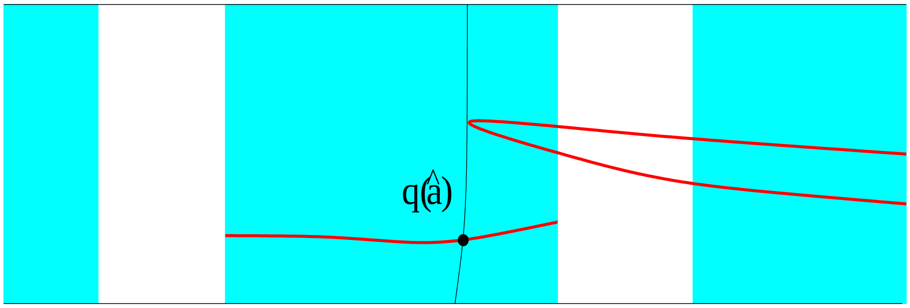
Proof of Theorem 2 The assumption of Theorem 2 implies that the image of the part of V that is contracting in horizontal direction is mapped into itself, shrinking eventually to periodic sinks. The rest of Λ is again a horseshoe of infinitely many branches.



Proof of Theorem 3 Observe that for all μ, there is a horseshoe of infinite many branches in Λ. This is the Smale horseshoe.

As $\mathbf{a} \to \infty$, the stable and unstable manifolds of this horseshoe will admit tangential intersections for infinitely many values of μ.

The assumption of Theorem 3 assures that these are quadratic tangency.



Remark on History

(1) A very long history, and a vast literature.

- Smale-Melnikov method: Smale, Melnikov, Marsden, Homles, Guckenheimer, Levi, Wiggins, etc,
- Analytic shadowing: Palmer, Stoffer, Meyer, Sell, etc,
- Method of Lyapunov-Schmidt: Hale, Chow, Mallet-Paret, Deng, etc,

(2) Our detailed computation of the separatrix map is motivated by a paper of Afraimovich and Shilnikov.

Similar maps were previously used in studies of

- Homoclinic bifurcations (Autonomous equations): Shilnokov, Afraimovich, Turaev, Gavrilov, etc,
- Arnold diffusions (Hamiltonian equations): Treschev, Piftankin, Zaslavsky, etc.

(3) Two things distinguish our results from all previous studies:

- Previous studies have been all on the existence of certain (complicated) subsets inside of Λ.

We offer a comprehensive description of the overall dynamical structure of the entire homoclinic tangles Λ.

- We offer a method of analyzing concrete equations, through $F(t) = t - \beta^{-1} \ln M(t)$, a function explicitly computable.

Results including applications:

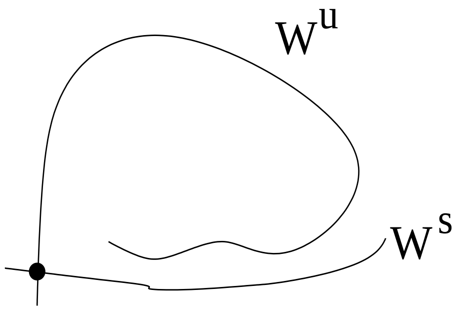
— The Newhouse theory,

— Theory of Henon-like attractors,

— Theory of SRB measures.

Part III

Case of Non-intersections



Theory of rank one maps (Wang and Young)

The theory of rank one maps:

[1] Com. Math. Phy. 218 (2001), 1-97

[2] Annals. Math. 167 (2008), 349-480

- An Abstract setting.
- A comprehensive dynamics theory.

Applications to ODE's:

[3] Com. Math. Phy. 225(2002), 275-304

[4] Com. Math. Phy. 240(2002), 509-529

- Applications of the theory of [1], [2] to periodically kicked stable limit cycles.

Scenario (b): The case of non-intersections

Space of forcing functions:

- $\mathcal{N}$ be a fixed open neighborhood of ℓ.

- H_T be the collection of all C^k functions on $\mathcal{N} \times \mathbb{R}$ that are periodic in t of period T.

- $H = \cup_{T>0} H_T \times H_T$ is the space for forcing functions (P, Q).

Space of Melnikov Functions

- $\omega = T^{-1}$ be the forcing frequency.

- $\theta = \omega t$.

- W be the collection of all C^k functions from S^1 to $\mathbb{R}$, the space for Melnikov functions $M(\theta)$.

$$\theta_1 = \theta + \mathbf{a} - \omega\beta^{-1}\ln(\mathbf{k}z + M(\theta) + \mathbb{E}_2) + \mathbb{E}_1$$
$$z_1 = \mathbf{b}[\mathbf{k}z + M(\theta) + \mathbb{E}_2]^{\frac{\alpha}{\beta}}$$

Theorem 4 (Attainable Melnikov Functions) Let $F(\theta) \in W$ be a given. Then for almost all $\omega \in \mathbb{R}$, there exists $(P, Q) \in H_{\omega^{-1}}$ such that the separatrix map is written as in the above, in which $F(\theta)$ is in the places of $M(\theta)$.

Remarks **(1)** We can first fix $M(\theta) \in W$, then treat ω as an independent parameter.

Assumption: $M(\theta) \in W$ is a Morse function.

(2) These maps are exactly the kind of 2D maps systematically studied by Wang and Young in [3] and [4].

Results

– For small ω: Quasi-periodic torus;

– As ω increases: Transition to chaos;

– For large ω: Non-hyperbolic attractors:

  - Horseshoes
  - Newhouse tangency
  - Rank one attractors

Historic remarks

- Horseshoe for large ω in scenario (b) was first observed by Afraimovich and Shilnikov.

- Hale and Chow also proved the existence of many periodic solutions in scenario (b).

Other Results Include:

(1) (With Kening Lu) A full extension of the Smale-Melnikov method for non-autonomously perturbed equations without any periodicity in time.

(2) Analysis of strange attractors in equations with two homoclinic loops to a dissipative saddle.

(3) (With Ali Oksasoglu) Numerical investigations based on the theory developed.