ON THE DYNAMICS OF CERTAIN HOMOCLINIC TANGLES

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Abstract. In this paper we study homoclinic tangles formed by transversal intersections of the stable and the unstable manifold of a non-resonant, dissipative homoclinic saddle point in periodically perturbed second order equations. We prove that the dynamics of these homoclinic tangles are those of infinitely wrapped horseshoe maps (See Section 1A). Using \( \mu \) as a parameter representing the magnitude of the perturbations, we prove that (a) there exist infinitely many disjoint open intervals of \( \mu \), accumulating at \( \mu = 0 \), such that the entire homoclinic tangle of the perturbed equation consists of one single horseshoe of infinitely many symbols, (b) there are parameters in between each of these parameter intervals, such that the homoclinic tangle contains attracting periodic solutions, and (c) there are also parameters in between where the homoclinic tangles admit non-degenerate transversal homoclinic tangency of certain dissipative hyperbolic periodic solutions.

1. Introduction

We start with an autonomous second-order ordinary differential equation that contains a non-resonant, dissipative saddle fixed point with a homoclinic solution. This autonomous equation is then subjected to time periodic perturbations. In this paper we study the dynamics of the homoclinic tangles formed by transversal intersections of the stable and the unstable manifold of the perturbed saddle point. These homoclinic tangles have been one of the major inspirations for the dynamical systems theory and a long standing puzzle in the studies of ordinary differential equations in modern times.

A. Description of results. In this paper we unravel this long standing puzzle. Instead of focusing on the picture of the globally induced time-T maps, by which H. Poincaré observed an exceedingly complicated mess [P] and S. Smale constructed an embedded horseshoe map [S], we compute the return maps induced by periodically perturbed equations around the homoclinic solution in the extended phase space. It has turned out that the return maps for these homoclinic tangles are infinitely wrapped horseshoe maps, the geometric structure of which is as follows. Take an annulus \( \mathcal{A} = S^1 \times I \). We represent points in \( S^1 \) and \( I \) by using variables \( \theta \) and \( z \) respectively. We call the direction of \( \theta \) the horizontal direction and the direction of \( z \) the vertical direction. To form an infinitely wrapped horseshoe map, which we denote as \( \mathcal{F} \), we first divide \( \mathcal{A} \) into two vertical strips, which we denote as \( V \) and \( U \). \( \mathcal{F} : V \to \mathcal{A} \) is defined on \( V \) but not on \( U \). We compress \( V \) in the vertical direction and stretch it in the horizontal direction, making the image infinitely long towards both ends. Then we fold it and wrap it around the annulus \( \mathcal{A} \) infinitely many times. See Fig. 1.
We use a small parameter $\mu$ to represent the magnitude of the time periodic perturbations. Denote the return maps obtained from the periodically perturbed equations as $F_\mu$ and let

$$\Omega_\mu = \{ (\theta, z) \in V : F_\mu^n(\theta, z) \in V \quad \forall n \geq 0 \}, \quad \Lambda_\mu = \bigcap_{n \geq 0} F_\mu^n(\Omega_\mu).$$

Then $\Omega_\mu$ represents all solutions that stay close to the unperturbed homoclinic loop in forward times; $\Lambda_\mu$ is the set $\Omega_\mu$ is attracted to, representing all solutions that stay close to the unperturbed homoclinic loop in both the forward and the backward times. The geometrical and dynamical structures of the homoclinic tangle represented by $F_\mu$ are manifested in those of $\Omega_\mu$ and $\Lambda_\mu$.

$\Omega_\mu$ and $\Lambda_\mu$ depend sensitively on the location of the folded part of $F_\mu(V)$. If this part is deep inside of $U$, then the entire homoclinic tangle is reduced to one horseshoe of infinitely many symbols. If it is located inside of $V$, then the homoclinic tangles are likely to have attracting periodic solutions or sinks and observable chaos associated with non-degenerate transversal homoclinic tangency. We prove that, as $\mu \to 0$, the folded part of $F_\mu(V)$ moves horizontally towards $\theta = +\infty$ with a roughly constant speed with respect to $p = \ln \mu$, crossing $V$ and $U$ infinitely many times along the way. It then follows, under mild assumptions, that there are infinitely many disjoint open intervals of $\mu$, accumulating at $\mu = 0$, such that the entire homoclinic tangle is reduced to one single horseshoe that conjugates to a full shift of infinitely many symbols. We also prove that there are other parameters in between these intervals, such that the homoclinic tangle contains attracting periodic solutions. In addition, there are also parameters in between where the homoclinic tangle admits non-degenerate transversal homoclinic tangency. See Theorems 1-3 in Sect. 4.2 for more details.

Fig. 1 Infinitely wrapped horseshoe maps.

B. Method of study. We use variables $(x, y)$ to represent the phase space of the unperturbed equation and let $(x, y) = (0, 0)$ be the saddle fixed point. Denote the homoclinic solution for $(x, y) = (0, 0)$ as $\ell$. We construct a small neighborhood of $\ell$.

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1We caution that $V$ and $U$ depend also on $\mu$. So to be completely rigorous we ought to write $V_\mu$ and $U_\mu$ instead of $V$ and $U$. However, for $F_\mu$ derived from the periodically perturbed equations, $V$ and $U$ vary only slightly as $\mu$ varies, and we could practically think them as being independent of $\mu$ at this stage.
by taking the union of a small neighborhood \( U_\varepsilon \) of \((0,0)\) and a small neighborhood \( D \) around \( \ell \) out of \( U_{1/4} \). See Fig. 2. Let \( \sigma^\pm \in U_\varepsilon \cap D \) be the two line segments depicted in Fig. 2, both perpendicular to the homoclinic solution. We use an angular variable \( \theta \in S^1 \) to represent the time.

In the extended phase space \((x, y, \theta)\) we denote

\[
U_\varepsilon = U_\varepsilon \times S^1, \quad D = D \times S^1
\]

and let

\[
\Sigma^\pm = \sigma^\pm \times S^1.
\]

Let \( \mathcal{N} : \Sigma^+ \to \Sigma^- \) be the maps induced by the solutions on \( U_\varepsilon \) and \( \mathcal{M} : \Sigma^- \to \Sigma^+ \) be the maps induced by the solutions on \( D \). See Fig. 3. We first compute \( \mathcal{M} \) and \( \mathcal{N} \) separately, then compose \( \mathcal{N} \) and \( \mathcal{M} \) to obtain an explicit formula for the return map \( \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^- \).

![Fig. 2 U, D and \( \sigma^\pm \).](image2)

![Fig. 3 \( \mathcal{N} \) and \( \mathcal{M} \).](image3)

We follow the steps of [WO] to derive the return maps. There are, however, two main differences between the classical scenario of homoclinic tangles we now consider and the ones studied in [WO]. First, the return maps of this paper are only partially defined on \( \Sigma^- \). After following the entire length of the homoclinic loop of
the unperturbed equation, part of $\Sigma^-$ (represented by $V$ in $A$) would hit $\Sigma^+$ on one side of the local stable manifold of the perturbed saddle where they return to $\Sigma^-$; and the rest (represented by $U$ in $A$) would hit on the other side where they sneak out. See Fig. 4. Second, analytic controls represented by the $C^3$ estimates in [WO] would deteriorate as we approach to the transversal intersections of the stable and the unstable manifold of the perturbed saddle, potentially devastating the usefulness of the formulas obtained for the return maps. Between the two, the second is essentially a technical issue we need to overcome. The first is an intrinsic character of these homoclinic tangles.

This paper is organized as follows. In Section 2 we introduce the equations of study and a set of changes of variables to transform the equations into certain canonical forms. In Section 3 we compute the return maps. Proposition 3.3 in Section 3 is the main result of this paper. Theorems about homoclinic tangles are then formulated and proved in Section 4 assuming the forcing function is in the form of $\sin \omega t$. Homoclinic tangles associated with general forcing functions are studied in Section 5.

**C. Remarks on history.** Homoclinic tangles formed by transversal intersections of the stable and the unstable manifold of a periodically perturbed homoclinic saddle in systems of ordinary differential equations were first observed by H. Poincaré [P] more than one hundred years ago. His observation is regarded in general as the event that gave birth to the modern theory of chaos and dynamical systems.

There exists a vast literature on periodically perturbed differential equations (see for instance, the reference list of [GH]). We could put the related studies roughly into two categories. The first category contains the ones that attempted to unravel the dynamics of the associated homoclinic tangles and the second contains the ones that attempted to verify the existence of these homoclinic tangles in concrete systems of differential equations. Among the most influential in the first category are the studies of Cartwright and Littlewood [CL] and Levinson [L] on van der Pol’s equation and the studies of Sitnikov [Sit] and Alekseev [A] on Sitnikov’s motions. These studies led eventually to Smale’s construction of his horseshoe map [S]. To the best of our knowledge, Smale’s horseshoe is essentially all that has been rigorously claimed for
these homoclinic tangles. In the second category, the most influential is the development of Melnikov’s method [M], purposed on verifying the existence of homoclinic tangles in concrete systems of differential equations.

Our idea of constructing return maps such as those derived in [WO] and in this paper is motivated by a work of Afraimovich and Shilnikov published about thirty years ago [AS], see also [AH]. With the return maps obtained in [WO] and in this paper, we now know much more. In particular, we know that these classical homoclinic tangles are only one of the chaos scenarios in the surroundings of periodically perturbed homoclinic solutions. By using the derived return maps, Wang and Ott proved the existence of rank one attractors in [WO] by applying a theory of rank one maps developed in recent years by Wang and Young [WY1]-[WY3] based on [BC]. See Sect. 4.4 for an overview on various dynamics scenarios newly found around periodically perturbed homoclinic solutions.

2. Equations and Canonical Forms

In this section we first introduce the equations. We then introduce coordinate changes to transform these equations into canonical forms, which we will use in Section 3 to compute the return maps.

2.1. Equations of study. Let \((x, y) \in \mathbb{R}^2\) be the phase variables and \(t\) be the time. We start with an autonomous system

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha x + f(x, y), \\
\frac{dy}{dt} &= \beta y + g(x, y)
\end{align*}
\]

where \(f(x, y), g(x, y)\) are real-analytic functions defined on an open domain \(V \subset \mathbb{R}^2\) satisfying \(f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0\). First we assume that \((0, 0)\) is a non-resonant, dissipative saddle point. To be more precise we assume

(H1) (i) there exists \(d_1, d_2 > 0\) so that for all \(n, m \in \mathbb{Z}^+\),

\[|n\alpha - m\beta| > d_1(|n| + |m|)^{-d_2};\]

(ii) \(0 < \beta < \alpha\).

(H1)(i) is a Diophantine non-resonance condition on \(\alpha\) and \(\beta\). (H1)(ii) claims that the saddle point \((0, 0)\) is dissipative. Let us also assume that the positive \(x\)-side of the local stable manifold and the positive \(y\)-side of the local unstable manifold of \((0, 0)\) are included as part of a homoclinic solution, which we denote as

\[
\ell = \{\ell(t) = (a(t), b(t)) \in V, \ t \in \mathbb{R}\}.
\]

Let \(Q(t) : \mathbb{R} \to \mathbb{R}\) be a real analytic function satisfying \(Q(t) = Q(t + 2\pi)\). To the right of equation (2.1) we add forcing terms to form a non-autonomous system

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha x + f(x, y) + \mu A(x, y)(\rho + Q(\omega t)), \\
\frac{dy}{dt} &= \beta y + g(x, y) + \mu B(x, y)(\rho + Q(\omega t))
\end{align*}
\]
where \( A(x, y), B(x, y) \) are also real analytic on \( V \) satisfying \( A(0, 0) = B(0, 0) = 0, \partial_x A(0, 0) = \partial_y A(0, 0) = \partial_x B(0, 0) = \partial_y B(0, 0) = 0. \) We regard \( \alpha, \beta, f(x, y), g(x, y), A(x, y), B(x, y) \) and \( Q(t) \) as been fixed. \( \omega, \rho, \mu \) are forcing parameters. The range \( P \) for \((\omega, \rho, \mu)\) is as follows: Let \( R_\omega >> 1 \) be an arbitrarily picked real number sufficiently large, \( R_\rho >> R_\omega \). Define
\[
P = \{ (\omega, \rho, \mu) \in \mathbb{R}^3, 0 < \omega < R_\omega, R_\rho^{-1} < \rho < R_\rho, 0 < \mu < R_\mu^{-1} \}. \]

This study is exclusively on equation (2.2) with parameters inside of \( P \). From \( R_\mu >> R_\rho >> R_\omega \) we have, for all \((\omega, \rho, \mu) \in P\),
\[
\mu \rho, \mu \omega << 1.
\]
We can make the magnitude of \(|\mu \rho|\) and \(|\mu \omega|\) as small as we desire by adjusting \( R_\mu \).

We also fix the values of \( \omega \) and \( \rho \), leaving \( \mu \) then as the only parameter to vary. We study the solutions of equation (2.2) in the surroundings of the homoclinic loop \( \ell \) in the original phase space \((x, y)\), which we divide into a small neighborhood \( U_\varepsilon \) of \((0, 0)\) and a small neighborhood \( D \) around \( \ell \) out of \( U_{1/2} \). See Fig. 2 in Section 1. In Sect. 2.2 we introduce a change of variables to linearize equation (2.2) on \( U_\varepsilon \). In Sect. 2.3 we introduce coordinate changes to transform (2.2) on \( D \) into a canonical form. In the rest of this paper \( r > 0 \) is reserved for an integer arbitrarily fixed, and we will control the \( C^r \)-norm of the derived return maps in phase variables and parameters.

**Two small scales**: \( \mu << \varepsilon << 1 \) represent two small scales of different magnitude. \( \varepsilon \) represents the size of a small neighborhood of \((x, y) = (0, 0)\), where the linearizations of Sect. 2.2 is valid. Define
\[
U_\varepsilon = \{ (x, y) : x^2 + y^2 < 4\varepsilon^2 \}
\]
and let \( L^+, -L^- \) be the respective times at which the homoclinic solution \( \ell(t) \) enters \( U_{1/2} \) in the positive and the negative directions. \( L^+ \) and \( L^- \) are related, both determined completely by \( \varepsilon \) and \( \ell(t) \). The parameter \( \mu << \varepsilon \) controls the magnitude of the time-periodic perturbation. We make
\[
R_\mu >> \varepsilon^{-1} >> R_\rho >> R_\omega.
\]

**Notation**: Quantities that are independent of the phase variables, time and \( \mu \) are regarded as constants and \( K \) is used to denote a generic constant, the precise value of which is allowed to change from line to line. On occasion, a specific constant is used in different places. We use subscripts to denote such constants as \( K_0, K_1, \ldots \). We will also make distinctions between constants depend on \( \varepsilon \) and those do not by making such dependencies explicit. A constant that depends on \( \varepsilon \) is written as \( K(\varepsilon) \). A constant written as \( K \) is independent of \( \varepsilon \).

2.2. **Linearization on \( U_\varepsilon \).** From this point on all functions are regarded as functions in phase variables, time \( t \) and the parameter \( \mu \). Let \( X, Y \) be such that
\[
x = X + P(X, Y) + \mu \tilde{P}(X, Y, \theta; \mu)
\]
\[
y = Y + Q(X, Y) + \mu \tilde{Q}(X, Y, \theta; \mu)
\]
(2.4)
where \( P, Q, \tilde{P}, \tilde{Q} \) as functions of \( X \) and \( Y \) are real-analytic on \( |(X,Y)| < 2\varepsilon \), and the values of these functions and their first derivatives with respect to \( X \) and \( Y \) at \((X,Y) = (0,0)\) are all zero. As is explicitly indicated in (2.4), \( P \) and \( Q \) are independent of \( \theta \) and \( \mu \). We also assume that
\[
\tilde{P}(X,Y,\theta + 2\pi; \mu) = \tilde{P}(X,Y,\theta; \mu), \quad \tilde{Q}(X,Y,\theta + 2\pi; \mu) = \tilde{Q}(X,Y,\theta; \mu)
\]
are periodic of period \( 2\pi \) in \( \theta \) and they are also real-analytic with respect to \( \theta \) and \( \mu \) for all \( \theta \in \mathbb{R} \) and \( |\mu| < R_\mu^{-1} \). Substituting \( \theta \) by \( \omega t \) in (2.4) defines a non-autonomous, near identity coordinate transformation from \( x, y \) to \( (X, Y) \), which we write explicitly as
\[
(2.5) \quad x = X + P(X,Y) + \mu \tilde{P}(X,Y,\omega t; \mu) \\
y = Y + Q(X,Y) + \mu \tilde{Q}(X,Y,\omega t; \mu).
\]

We have

**Proposition 2.1.** Assume that \( \alpha \) and \( \beta \) satisfy the Diophantine non-resonance condition \((H1)(i)\). Then there exists a small neighborhood \( U_\varepsilon \) of \((0,0)\), the size of which are completely determined by equation (2.1) and \( d_1, d_2 \) in \((H1)(i)\), such that there exists an analytic coordinate transformation in the form of (2.5) that transforms equation (2.2) into
\[
\frac{dX}{dt} = -\alpha X, \quad \frac{dY}{dt} = \beta Y.
\]

Moreover, the \( C^r \)-norms of \( P, Q, \tilde{P}, \tilde{Q} \) as functions of \( X, Y, \theta, \mu \) are all uniformly bounded from above by a constant \( K \) that is independent of both \( \varepsilon \) and \( \mu \) on \((X,Y) \in U_\varepsilon, \theta \in \mathbb{R} \) and \( \mu \in (-R_\mu^{-1}, R_\mu^{-1}) \).

**Proof:** This is a standard linearization result. See for instance [CLS] for a proof. □

2.3. A canonical form around homoclinic loop. In this subsection we derive a standard form for equation (2.2) around the homoclinic loop of equation (2.1) outside of \( U_{4\varepsilon} \). Let
\[
\ell = \{ \ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R} \}
\]
be the homoclinic solution of the unperturbed equation (2.1), and
\[
(u(t), v(t)) = \left| \frac{d}{dt} \ell(t) \right|^{-1} \frac{d}{dt} \ell(t)
\]
be the unit tangent vector of \( \ell \) at \( \ell(t) \). Let us regard \( t \) in \( \ell(t) = (a(t), b(t)) \) not as time, but as a parameter that parameterize the curve \( \ell \) in \((x,y)\)-space. We replace \( t \) by \( s \) to write this homoclinic loop as \( \ell(s) = (a(s), b(s)) \). We have
\[
(2.6) \quad \frac{da(s)}{ds} = -\alpha a(s) + f(a(s), b(s)), \quad \frac{db(s)}{ds} = \beta b(s) + g(a(s), b(s)).
\]
By definition,
\[ u(s) = \frac{-\alpha a(s) + f(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}, \]
(2.7)
\[ v(s) = \frac{\beta b(s) + g(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}. \]

Let
\[ e(s) = (v(s), -u(s)). \]

We now introduce new variables \((s, z)\) such that
\[ (x, y) = \ell(s) + z e(s). \]

This is to say that
\[ x = x(s, z) := a(s) + v(s)z, \quad y = y(s, z) := b(s) - u(s)z. \]  

We derive the equations for (2.2) in new variables \((s, z)\) defined through (2.8). Differentiating (2.8) we obtain
\[ \frac{dx}{dt} = (-\alpha a(s) + f(a(s), b(s))) + v'(s)z \frac{ds}{dt} + v(s) \frac{dz}{dt}, \]
(2.9)
\[ \frac{dy}{dt} = (\beta b(s) + g(a(s), b(s))) - u'(s)z \frac{ds}{dt} - u(s) \frac{dz}{dt}, \]
where \(u'(s) = \frac{du(s)}{ds}, \quad v'(s) = \frac{dv(s)}{ds}\). Let us denote
\[ F(s, z) = -\alpha (a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)), \]
\[ G(s, z) = \beta (b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)), \]
\[ A(s, z) = A(x(s, z), y(s, z)), \]
\[ B(s, z) = B(x(s, z), y(s, z)). \]

By using equation (2.2), we obtain from equation (2.9) the new equations for \(s, z\) as
\[ \frac{dz}{dt} = v(s)F(s, z) - u(s)G(s, z) + \mu (v(s)A(s, z) - u(s)B(s, z))(\rho + Q(\omega t)) \]
\[ \frac{ds}{dt} = \frac{v(s)G(s, z) + u(s)F(s, z) + \mu (v(s)B(s, z) + u(s)A(s, z))(\rho + Q(\omega t))}{\sqrt{F(s, 0)^2 + G(s, 0)^2 + z(u(s)v'(s) - v(s)u'(s))}}. \]

We re-write these equations as
\[ \frac{dz}{dt} = E(s)z + z^2 w_2(s, z) + \mu (v(s)A(s, z) - u(s)B(s, z))(\rho + Q(\omega t)) \]
(2.10)
\[ \frac{ds}{dt} = 1 + zw_1(s, z, \omega t; \mu) + \mu (v(s)B(s, z) + u(s)A(s, z))(\rho + Q(\omega t)) \]
\[ \sqrt{F(s, 0)^2 + G(s, 0)^2} \]

where
\[ E(s) = v^2(s)(-\alpha + \partial_x f(a(s), b(s))) + u^2(s)(\beta + \partial_y g(a(s), b(s))) \]
\[ - u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s))). \]  

(2.11)
and \( w_1(s, z, \theta + 2\pi; \mu) = w_1(s, z, \theta; \mu) \) is periodic in \( \theta \) of period \( 2\pi \). Also in the rest of this section we let \( K_1(\varepsilon) \) be a given constant independent of \( \mu \), and regard equation (2.10) as been defined on

\[
\{ s \in [-2L^-, 2L^+], \quad |z| < K_1(\varepsilon)\mu; \quad \mu \in (0, R^{-1}_\mu) \}.
\]

The \( C^r \)-norms of \( w_1(s, z, \theta; \mu) \) and \( w_2(s, z) \) are bounded above by a constant \( K(\varepsilon) \).

Finally we re-scale the variable \( z \) by letting

\[
(2.12) \quad Z = \mu^{-1}z.
\]

We arrive at the following equations

\[
\begin{align*}
\frac{dZ}{dt} &= E(s)Z + \mu \tilde{w}_2(s, Z, \omega t; \mu) + \mathbb{H}(s)(\rho + Q(\omega t)) \\
\frac{ds}{dt} &= 1 + \mu \tilde{w}_1(s, Z, \omega t; \mu)
\end{align*}
\]

where

\[
(2.14) \quad \mathbb{H}(s) = v(s)A(a(s), b(s)) - u(s)B(a(s), b(s));
\]

and \((s, Z; \mu)\) are defined on

\[
\mathbb{D} = \{ (s, Z; \mu) : s \in [-2L^-, 2L^+], \quad |Z| \leq K_1(\varepsilon), \quad \mu \in (0, R^{-1}_\mu) \}.
\]

Note that here we assume that \( R^{-1}_\mu \) is sufficiently small so that

\[
\mu << \min_{s \in [-2L^-, 2L^+]} (F(s, 0)^2 + G(s, 0)^2).
\]

Again, the \( C^r \) norms of \( \tilde{w}_1, \tilde{w}_2 \) with respect to \( s, Z; \mu \) and \( t \) are uniformly bounded by a constant \( K(\varepsilon) \) for \((s, Z; \mu) \in \mathbb{D} \) and \( t \in \mathbb{R} \). Equation (2.13) is the one we need.

2.4. **Technical estimates.** Estimates presented in this subsection are directly taken from [WO], which we include for completeness.

**Notation:** We are going to adopt the following convention in comparing the magnitude of two functions \( f(t) \) and \( g(t) \). We denote \( f(t) < g(t) \) if there exists \( K > 0 \) independent of \( t \) so that \( |f(t)| < K|g(t)| \) as \( t \to \infty \) (or \(-\infty\)). We denote \( f(t) \sim g(t) \) if in addition we have \( |f(t)| > K^{-1}|g(t)| \). We also denote \( f(t) \approx g(t) \) if

\[
\frac{f(t)}{g(t)} \to 1
\]
as \( t \to \infty \) (or \(-\infty\)).

Recall that \( \ell(t) = (a(t), b(t)) \) is the homoclinic solution for the hyperbolic fixed point \((0, 0)\) of equation (2.1). \((u(t), v(t))\) is the unit tangent vector of \( \ell \) at \( \ell(t) \) defined through (2.7).

**Lemma 2.1.** As \( t \to +\infty \),

\[
\begin{align*}
&\quad a(t) \sim e^{-\alpha t}, \quad b(t) \sim e^{-2\alpha t}, \quad u(t) \approx -1, \quad v(t) \sim e^{-\alpha t}; \\
&\quad a(-t) \sim e^{-2\beta t}, \quad b(-t) \sim e^{-\beta t}, \quad u(-t) \sim e^{-\beta t}, \quad v(-t) \approx 1.
\end{align*}
\]
Proof: We are simply re-stating the fact that $\ell(t) \to (0,0)$ with an exponential rate $-\alpha$ in the positive time direction along the $x$-axis, and an exponential rate $\beta$ in the negative time direction along the $y$-axis. $\square$

Let $E(s)$ be as in (2.11).

Lemma 2.2. As $L^\pm \to +\infty$,

(i) $\int_{-L^-}^0 (E(s) + \alpha)ds < 1$, $\int_{0}^{L^+} (E(s) - \beta)ds < 1$.

(ii) $\int_{-L^-}^{L^+} E(s)ds \approx -\alpha L^-$, $\int_{0}^{L^+} E(s)ds \approx \beta L^+$.

Proof: (i) claims that the integrals are convergent as $L^\pm \to \infty$. For the first integral, we observe that by adding $\alpha$ to $E(t)$, we obtain $E(t) + \alpha$ as a collection of terms, each of which decays exponentially as $t \to -\infty$ according to Lemma 2.1. Similarly, taking $\beta$ away from $E(t)$, we obtain $E(t) - \beta$ as a collection of terms, each of which decays exponentially as $t \to \infty$.

For (ii) we write

\[ \int_{-L^-}^{0} E(s)ds = -\alpha L^- + \int_{-L^-}^{0} (E(s) + \alpha)ds \]
\[ \int_{0}^{L^+} E(s)ds = \beta L^+ + \int_{0}^{L^+} (E(s) - \beta)ds. \]

(ii) now follows from (i). $\square$

We also have

Lemma 2.3. As $\varepsilon \to 0$, $\varepsilon \sim e^{-\alpha L^+} \sim e^{-\beta L^-}$.

Proof: This follows directly from the definition of $L^\pm$ and Lemma 2.1. $\square$

3. Derivation of Return Maps

Let $\theta \in S^1$ be an angular variable for time. We re-write equation (2.2) as

\[ \frac{dx}{dt} = -\alpha x + f(x, y) + \mu A(x, y)(\rho + Q(\theta)), \]
\[ \frac{dy}{dt} = \beta y + g(x, y) + \mu B(x, y)(\rho + Q(\theta)). \]
\[ \frac{d\theta}{dt} = \omega. \]

$\Sigma^\pm$ are formally defined in Sect. 3.1 (See Section 1B). In Sect. 3.2, we study coordinate conversions between $(X, Y, \theta)$ and $(s, Z, \theta)$ on $\Sigma^\pm$. $\mathcal{N} : \Sigma^+ \to \Sigma^-$, $\mathcal{M} : \Sigma^- \to \Sigma^+$ and the return map $\mathcal{F} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$ are computed in Sect. 3.3.

3.1. Poincaré sections $\Sigma^\pm$. We start with equation (3.1) on $\mathcal{U}_c$. We have obtained in Sect. 2.2 a change of variables on $\mathcal{U}_c$ in the form of (2.4), that is,

\[ x = X + P(X, Y) + \mu \tilde{P}(X, Y; \mu) \]
\[ y = Y + Q(X, Y) + \mu \tilde{Q}(X, Y; \mu) \]

(3.2)
that transforms equation (3.1) to the linear equation

\[
\frac{dX}{dt} = -\alpha X, \quad \frac{dY}{dt} = \beta Y, \quad \frac{d\theta}{dt} = \omega
\]

on \( U_\varepsilon \). We define \( \Sigma^\pm \) inside of \( U_\varepsilon \cap D \) by letting

\[
\Sigma^- = \{(X, Y, \theta) : Y = \varepsilon, \ |X| < \mu, \ \theta \in S^1\},
\]

and

\[
\Sigma^+ = \{(X, Y, \theta) : X = \varepsilon, \ |Y| < K_1(\varepsilon)\mu, \ \theta \in S^1\}.
\]

\( K_1(\varepsilon) \) will be precisely defined in Sect. 3.3. Observe that in [WO], \( \Sigma^\pm \) are defined in slightly different terms. The current definition is designed to avoid the long and deteriorating derivative estimates of [WO].

We turn to the canonical form for equation (3.1) on \( D \). Let

\[
x = a(s) + \mu v(s)Z, \quad y = b(s) - \mu u(s)Z.
\]

Then according to Sect. 2.3, equation (3.1) on \( D \) is written in \((s, Z, \theta)\) as

\[
\frac{dZ}{dt} = E(s)Z + \mu \tilde{w}_2(s, Z, \theta; \mu) + \mathbb{H}(s)(\rho + Q(\theta))
\]

\[
\frac{ds}{dt} = 1 + \mu \tilde{w}_1(s, Z, \theta; \mu)
\]

\[
\frac{d\theta}{dt} = \omega
\]

where

\[
E(s) = v^2(s)(-\alpha + \partial_x f(a(s), b(s))) + u^2(s)(\beta + \partial_y g(a(s), b(s))) - u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s)));
\]

\[
\mathbb{H}(s) = v(s)A(a(s), b(s)) - u(s)B(a(s), b(s));
\]

and the \( C^r \)-norms of \( \tilde{w}_1, \tilde{w}_2 \) are bounded from above by a \( K(\varepsilon) \) on \( D \times (0, R^{-1}_\mu) \) where

\[
D = \{(s, Z, \theta) : s \in [-2L^-, 2L^+], \ |Z| \leq K_1(\varepsilon), \ \theta \in S^1\}.
\]

Let \( q \in \Sigma^+ \) or \( \Sigma^- \). We represent \( q \) by using the \((X, Y, \theta)\)-coordinates, for which we have \( X = \varepsilon \) on \( \Sigma^+ \) and \( Y = \varepsilon \) on \( \Sigma^- \). We can also use \((s, Z, \theta)\)-coordinate to represent the same \( q \), for which the defining equations for \( \Sigma^\pm \) are not as direct. To compute the return maps, we need to first attend two issues that are technical in nature. First, we need to derive the defining equations for \( \Sigma^\pm \) for \((s, Z, \theta)\). Second, we need to be able to change coordinates from \((X, Y, \theta)\) to \((s, Z, \theta)\) and vice versa on \( \Sigma^\pm \). We start with some preparations in notation.

**Notation:** The intended formula for the return maps would inevitably contain terms that are explicit and terms that are implicit. Implicit terms are usually “error” terms, and the usefulness of a derived formula would depend completely on how well the error terms are controlled. In this paper we aim on \( C^r \)-control on all error terms. The derivations of the return maps involve a composition of maps and multiple coordinate changes. To facilitate our presentation, from this point on we adopt specific
conventions for indicating controls on magnitude. For a given constant, we write $O(1)$, $O(\varepsilon)$ or $O(\mu)$ to indicate that the magnitude of the constant is bounded by $K$, $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. For a function of a set $V$ of variables on a specific domain, we write $O_V(1)$, $O_V(\varepsilon)$ or $O_V(\mu)$ to indicate that the $C^r$-norm of the function on the specified domain is bounded by $K$, $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in the notation. For example, $O_{Z,\theta}(\mu)$ represents a function of $Z, \theta$, the $C^r$-norm of which is bounded above by $K\mu$.

**The new parameter $p$:** For the formulas obtained to be most useful, it is also desirable that we have control on the derivatives with respect to the forcing parameter $\mu$. Taking derivative with respect $\mu$, however, are problematic because such action takes $\mu$ out of the needed places. To resolve this potentially damaging problem we introduce a new parameter $p = \ln \mu$ and regard $p$, not $\mu$, as our bottom-line parameter. In another word, we regard $\mu$ as a shorthand for $e^p$, and all functions written in $\mu$ as functions in $p$. Observe that $\mu \in (0, \mu_0)$ corresponds to $p \in (-\infty, \ln \mu_0)$. This is a very important conceptual point because by regarding a function $F(\mu)$ of $\mu$ as a function of $p$, we have

$$\partial_p F(\mu) = \mu \partial_\mu F(\mu).$$

So regarding $F(\mu)$ as a function of $p$ would give back to us that much needed factor $\mu$ in derivative estimates.

### 3.2. Conversion of coordinates on $\Sigma^\pm$. We start with the defining equations for $\Sigma^+$ in $(s, Z, \theta)$. Results for $\Sigma^-$ are similar.

**Lemma 3.1.** We have for $(s, Z, \theta) \in \Sigma^+$

$$s = L^+ + O_{Z,\theta,p}(\mu).$$

**Proof:** From (3.2) and (3.4), we have on $\Sigma^+$

$$a(s) + v(s)z = \varepsilon + P(\varepsilon, Y) + \mu \tilde{P}(\varepsilon, Y, \theta; \mu)$$

$$b(s) - u(s)z = Y + Q(\varepsilon, Y) + \mu \tilde{Q}(\varepsilon, Y, \theta; \mu).$$

By definition

$$a(L^+) = \varepsilon + P(\varepsilon, 0)$$

$$b(L^+) = Q(\varepsilon, 0).$$

Let

$$W_1 = a(s) - a(L^+) + v(s)z - \mu \tilde{P}(\varepsilon, 0, \theta; \mu),$$

$$W_2 = b(s) - b(L^+) - u(s)z - \mu \tilde{Q}(\varepsilon, 0, \theta; \mu).$$

We have from (3.8) and (3.9),

$$W_1 = P(\varepsilon, Y) - P(\varepsilon, 0) + \mu(\tilde{P}(\varepsilon, Y, \theta; \mu) - \tilde{P}(\varepsilon, 0, \theta; \mu))$$

$$W_2 = Y + Q(\varepsilon, Y) - Q(\varepsilon, 0) + \mu((\tilde{Q}(\varepsilon, Y, \theta; \mu) - \tilde{Q}(\varepsilon, 0, \theta; \mu))$$
which we re-write as
\begin{align}
W_1 &= (\mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))Y + \mathcal{O}_{Y,\theta,p}(1)Y^2 \\
W_2 &= (1 + \mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))Y + \mathcal{O}_{Y,\theta,p}(1)Y^2.
\end{align}

(3.11)

We first obtain
\begin{equation}
Y = (1 + \mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))W_2 + \mathcal{O}_{W_2,\theta,p}(1)W_2^2
\end{equation}

(3.12)

by inverting the second line in (3.11). We then substitute into the first line in (3.11) to obtain
\begin{align*}
W_1 &= (\mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))((1 + \mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))W_2 + \mathcal{O}_{W_2,\theta,p}(1)W_2^2) \\
&+ \mathcal{O}_{Y,\theta,p}(1)((1 + \mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))W_2 + \mathcal{O}_{W_2,\theta,p}(1)W_2^2)^2 \\
&= (\mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))W_2 + \mathcal{O}_{W_2,\theta,p}(1)W_2^2.
\end{align*}

Consequently,
\begin{equation}
F(s, Z, \theta, \mu) := W_1 - (\mathcal{O}(\varepsilon) + \mu \mathcal{O}_{\theta,p}(1))W_2 + \mathcal{O}_{W_2,\theta,p}(1)W_2^2 = 0,
\end{equation}

(3.13)

where \(W_1, W_2\) as function of \(s, Z, \theta\) and \(\mu\) are defined by (3.10). To re-write \(W_1, W_2\) we let
\begin{equation}
\xi = s - L^+
\end{equation}

(3.14)

and expand \(a(s)\) in \(\xi\) as
\begin{equation}
a(s) = a(L^+) + a'(L^+)\xi + \sum_{i=2}^{\infty} a_i(L^+)\xi^i.
\end{equation}

Expansions for \(b(s), u(s)\) and \(v(s)\) are similar. We have
\begin{align}
W_1 &= a'(L^+)\xi + \sum_{i=2}^{\infty} a_i(L^+)\xi^i + v(L^+)z + (v'(L^+)\xi + \sum_{i=2}^{\infty} v_i(L^+)\xi^i)z
-
\mu \tilde{P}(\varepsilon, 0, \theta; \mu) \\
W_2 &= b'(L^+)\xi + \sum_{i=2}^{\infty} b_i(L^+)\xi^i - u(L^+)z - (u'(L^+)\xi + \sum_{i=2}^{\infty} u_i(L^+)\xi^i)z
-
\mu \tilde{Q}(\varepsilon, 0, \theta; \mu).
\end{align}

(3.15)

We now put (3.15) for \(W_1, W_2\) back into equation (3.13) and replace \(z\) by \(\mu Z\). We obtain
\begin{equation}
(a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+) + h(\theta, p, \xi)\xi)\xi = \mathcal{O}_{Z,\theta,p}(\mu)
\end{equation}

where the \(C^\alpha\) norm of \(h(\theta, p, \xi)\) is bounded from above by \(K(\varepsilon)\). From Lemma 2.1, \(a'(L^+) \approx -\alpha \varepsilon, \ b'(L^+) = \mathcal{O}(\varepsilon^2)\). We finally obtain
\begin{equation}
s = L^+ + \mathcal{O}_{Z,\theta,p}(\mu)
\end{equation}

by solving \(\xi\). This proves Lemma 3.1.

Lemma 3.1 is not precise enough. We need the following refinement.
Lemma 3.2. We have on $\Sigma^+$,
\[ s - L^+ = -\frac{v(L^+)}{a'(L^+)} + O(\varepsilon)u(L^+)z + \frac{\mu}{a'(L^+)}O_{\varepsilon}(1) + O_{Z,\varepsilon,p}(\mu^2). \]

Proof: It suffices for us to drop all terms that is $O_{Z,\varepsilon,p}(\mu^2)$ in equation (3.13) to solve for $\xi$. From Lemma 3.1 we conclude that all terms in $\xi, z$ of degree higher than one are $O_{Z,\varepsilon,p}(\mu^2)$. With these terms all dropped, (3.13) becomes
\[ (a'(L^+))\xi + (v(L^+) + O(\varepsilon)u(L^+))z = \mu O_{\varepsilon}(1), \]
from which the estimates of Lemma 3.2 on $\Sigma^+$ follows. \qed

From this point on we let
\[ X = \mu^{-1}X, \quad Y = \mu^{-1}Y. \]

Lemma 3.3. On $\Sigma^+$ we have
\[ Y = (1 + O(\varepsilon))Z + O_{\varepsilon,p}(1) + O_{Z,\varepsilon,p}(\mu). \]

Proof: We have
\[ Y = (1 + O(\varepsilon))(b'(L^+)\xi - u(L^+)z - \mu\tilde{Q}(\varepsilon, 0, \theta; \mu)) + O_{Z,\varepsilon,p}(\mu^2) \]
\[ = (1 + O(\varepsilon))\left( -\left( u(L^+) + b'(L^+)\frac{v(L^+) + O(\varepsilon)u(L^+)}{a'(L^+)} \right)z \right. \]
\[ + \left. \frac{\mu b'(L^+)}{a'(L^+)}O_{\varepsilon,p}(1) - \mu\tilde{Q}(\varepsilon, 0, \theta; \mu) \right) + O_{Z,\varepsilon,p}(\mu^2) \]
\[ = (1 + O(\varepsilon))z + \mu O_{\varepsilon,p}(1) + O_{Z,\varepsilon,p}(\mu^2), \]
where the first equality follows from using (3.12), (3.15) and Lemma 3.1; the second equality from using Lemma 3.2. To obtain the third equality we use $u(L^+) = -1 + O(\varepsilon)$, $a'(L^+) \approx \varepsilon$, $b'(L^+) = O(\varepsilon^2)$. \qed

Along similar lines we can also prove

Lemma 3.4. On $\Sigma^-$, we have
(i) \[ s = -L^- + O_{Z,\varepsilon,p}(\mu); \]
(ii) \[ Z = (1 + O(\varepsilon))X + O_{\varepsilon,p}(1) + O_{Z,\varepsilon,p}(\mu). \]

Proof: Left to the reader as an exercise. \qed

3.3. The return map $F = N \circ M$. First we compute $N : \Sigma^+ \to \Sigma^-$ and $M : \Sigma^- \to \Sigma^+$ separately. We then compose $N$ and $M$ by using Lemmas 3.3 and 3.4.

A. The induced map $N : \Sigma^+ \to \Sigma^-$. For $(X, Y, \theta) \in \Sigma^+$ we have $X = \varepsilon \mu^{-1}$ by definition. Similarly, for $(X, Y, \theta) \in \Sigma^-$ we have $Y = \varepsilon \mu^{-1}$. Denote a point on $\Sigma^+$ by using $(Y, \theta)$ and a point on $\Sigma^-$ by using $(X, \theta)$, and let
\[ (X_1, \theta_1) = N(Y, \theta) \]
for $(Y, \theta) \in \Sigma^+$. 


Proposition 3.1. We have for $(Y, \theta) \in \Sigma^+$,
\[ X_1 = (\mu \varepsilon^{-1})^{\frac{\beta}{\beta} - 1} Y^{\frac{\beta}{\beta}} \]
\[ \theta_1 = \theta + \frac{\omega}{\beta} \ln(\varepsilon^{-1}) - \frac{\omega}{\beta} \ln Y. \]
(3.18)

Proof: Let $T$ be the time it takes for the solution of (3.3) from $(\varepsilon, Y, \theta) \in \Sigma^+$ to get to $(X_1, \varepsilon, \theta_1) \in \Sigma^-$. We have
\[ X_1 = \varepsilon e^{-\alpha T}, \quad \varepsilon = Y e^{\beta T}, \quad \theta_1 = \theta + \omega T, \]
from which (3.18) follows. \qed

B. The induced map $\mathcal{M}: \Sigma^- \to \Sigma^+$. Let $H(s)$ be as in (3.7). In what follows, we write
\[ A_L = \int_{-L^-}^{L^+} \mathbb{H}(s) e^{-\int_0^s E(\tau) d\tau} ds \]
\[ \phi_L(\theta) = \int_{-L^-}^{L^+} \mathbb{H}(s) Q(\theta + \omega s + \omega L^-) e^{-\int_0^s E(\tau) d\tau} ds \]
We also write
\[ P_L = e^{\int_{-L^-}^{L^+} E(s) ds}, \quad P_L^+ = e^{\int_0^{L^+} E(s) ds}. \]
Note that for $P_L$ we integrate from $s = -L^-$ to $s = L^+$, while for $P_L^+$ the integration starts from $s = 0$. First we have

Lemma 3.5.
\[ P_L \sim \varepsilon^{\frac{\beta}{\beta} - \frac{\omega}{\beta}} \ll 1, \quad P_L^+ \sim \varepsilon^{-\frac{\omega}{\beta}} >> 1. \]

Proof: Both estimates follow directly from Lemmas 2.2 and 2.3. \qed

For $q = (s^-, Z, \theta) \in \Sigma^-$, the value of $s^-$ is uniquely determined by that of $(Z, \theta)$ through Lemma 3.4(i). So it is allowed for us to use $(Z, \theta)$ to represent $q$. Let $(s(t), Z(t), \theta(t))$ be the solution of equation (3.5) initiated at $(s^-, Z, \theta)$, and $\tilde{t}$ be the time $(s(\tilde{t}), Z(\tilde{t}), \theta(\tilde{t}))$ hit $\Sigma^+$. By definition $\mathcal{M}(q) = (s(\tilde{t}), Z(\tilde{t}), \theta(\tilde{t}))$. In what follows we write
\[ s^+ = s(\tilde{t}), \quad \tilde{Z} = Z(\tilde{t}), \quad \tilde{\theta} = \theta(\tilde{t}). \]

Proposition 3.2. Denote $(\tilde{Z}, \tilde{\theta}) = \mathcal{M}(Z, \theta)$. We have
\[ \dot{\tilde{Z}} = P_L^+(\rho A_L + \phi_L(\theta)) + P_L Z + O_{Z, \theta, \rho}(\mu) \]
\[ \dot{\tilde{\theta}} = \theta + \omega(L^+ + L^-) + O_{Z, \theta, \rho}(\mu). \]
(3.21)

Proof: Let us re-write equation (3.5) as
\[ \frac{dZ}{ds} = E(s) Z + \mathbb{H}(s)(\rho + Q(\theta)) + O_{s, Z, \theta, \rho}(\mu) \]
\[ \frac{d\theta}{ds} = \omega + O_{s, Z, \theta, \rho}(\mu) \]
(3.22)
on $D \times (0, R^{-1}_{\mu})$ where
\[ D = \{(s, Z, \theta) : s \in [-2L^-, 2L^+], |Z| < K_1(\varepsilon), \theta \in S^1\}. \]

Dropping all error terms in (3.22) we have
\[
\frac{dZ}{ds} = E(s)Z + H(s)(\rho + Q(\theta)) \quad \frac{d\theta}{ds} = \omega.
\]

(3.23)

We estimate the solution of equation (3.22) initiated at $(Z, \theta)$ from $s = s^-$ to $s = s^+$ by the solution of equation (3.23) initiated at the same $(Z, \theta)$ from $s = -L^-$ to $s = L^+$. By the smooth dependencies of solutions with respect to equations and initial conditions, the error of such estimates, according to Lemma 3.1 and Lemma 3.4(i), is
\[ O_{Z,\theta,p}(\mu) + O_{\hat{Z},\hat{\theta},p}(\mu) \]
provided that both solutions stay inside of $D$. By solving (3.23), we obtain
\[
\hat{Z} = P_L(Z + \Phi_L(\theta)) + O_{Z,\theta,p}(\mu) + O_{\hat{Z},\hat{\theta},p}(\mu)
\]
where $P_L$ is as in (3.20) and
\[
\Phi_L(\theta) = \int_{L^-}^{L^+} H(s)(\rho + Q(\theta + \omega L^- + \omega \tau)) \cdot e^{-\int_{L}^{\tau} E(\tau)d\tau} d\tau.
\]

(3.25)

From (3.24) we have
\[
\hat{Z} = P_L(Z + \Phi_L(\theta)) + O_{Z,\theta,p}(\mu)
\]
\[
\hat{\theta} = \theta + \omega(L^+ + L^-) + O_{Z,\theta,p}(\mu).
\]

(3.26)

Let
\[
K_1(\varepsilon) = \max_{\theta \in S^1, s \in [-2L^-, 2L^+], \mu \in (-R^{-1}_{\mu}, R^{-1}_{\mu})} P_s(2 + |\Phi_s(\theta)|)
\]
where $P_s$ and $\Phi_s$ are obtained by replacing $L^+$ with $s$ in $P_L$ and $\Phi_L$. $K_1(\varepsilon)$ is the one we use for $D$ and $\Sigma^+$. Solutions of (3.22) initiated on $\Sigma^-$ will stay inside of $D$ before hitting $\Sigma^+$. To finish, we observe that
\[
P_L\Phi_L(\theta) = P_L^+ \cdot \int_{-L^-}^{L^+} H(s)(\rho + Q(\theta + \omega L^- + \omega s)) e^{-\int_0^s E(\tau)d\tau} ds
\]
\[= P_L^+(\rho A_L + \phi_L(\theta)).\]

This finishes the proof of Proposition 3.2. 

\[\square\]

C. The return map $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$ We are now ready to compute the return map
\[\mathcal{F} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^- \]
We use $(\tilde{X}, \tilde{\theta})$ to represent a point on $\Sigma^-$ and denote
\[(\tilde{X}, \tilde{\theta}) = \mathcal{F}(X, \theta).\]
Proposition 3.3. The map $\mathcal{F} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$ is given by

\begin{align}
\tilde{X} &= (\mu \varepsilon^{-1})^{\frac{1}{\beta}} \cdot \left[ (1 + \mathcal{O}(\varepsilon)) P_L^+ \mathcal{F}(X, \theta) \right]^{\frac{1}{\beta}} \\
\tilde{\theta} &= \theta + \omega(L^+ + L^-) + \frac{\omega}{\beta} \ln \mu^{-1} \varepsilon (1 + \mathcal{O}(\varepsilon)) P_L^+ - \frac{\omega}{\beta} \ln \mathcal{F}(X, \theta) + \mathcal{O}_{X, \theta, p}(\mu)
\end{align}

where

\begin{align}
\mathcal{F}(X, \theta) &= (\rho A_L + \phi_L(\theta)) + P_L(P_L^+)^{-1}(1 + \mathcal{O}(\varepsilon))X \\
&\quad + (P_L^+)^{-1}(1 + P_L)\mathcal{O}_{\theta, p}(1) + \mathcal{O}_{X, \theta, p}(\mu),
\end{align}

and $P_L, P_L^+$ and $\phi_L(\theta)$ are as in (3.19) and (3.20).

Proof: By using Proposition 3.2 and Lemma 3.4, we have

\begin{align}
\dot{Z} &= P_L(1 + \mathcal{O}(\varepsilon))X + P_L^+(\rho A_L + \phi_L(\theta)) + P_L\mathcal{O}_{\theta, p}(1) + \mathcal{O}_{X, \theta, p}(\mu) \\
\dot{\theta} &= \theta + \omega(L^+ + L^-) + \mathcal{O}_{X, \theta, \rho}(\mu).
\end{align}

Let $\tilde{Y}$ be the $Y$-coordinate for $(\tilde{Z}, \tilde{\theta})$, we have from Lemma 3.3,

\begin{align}
\tilde{Y} &= (1 + \mathcal{O}(\varepsilon)) P_L^+ \mathcal{F}(X, \theta)
\end{align}

where $\mathcal{F}(X, \theta)$ is as in (3.29). We then obtain (3.28) by using (3.18). \qed

We remark that $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$ is only defined on the part of $\Sigma^-$ where

$\mathcal{F}(X, \theta) > 0$,

and the set in $\Sigma^-$ defined by $\mathcal{F} = 0$ is on the stable manifold of the saddle $(x, y) = (0, 0)$. Proposition 3.3 is the main result of this paper.

4. Dynamics of Homoclinic Tangles: $Q(t) = \sin t$

In this section we let $Q(t) = \sin t$ in equation (2.2). In Sect. 4.1 we derive the return maps. In Sect. 4.2, we prove that these return maps are infinitely wrapped horseshoe maps (See Section 1A). In particular, we prove that there exist infinitely many disjoint open intervals of $\mu$, accumulating at $\mu = 0$, such that the entire homoclinic tangle is one single horseshoe represented by a full shift of infinitely many symbols (Theorem 1). We also prove that there are parameters in between each of these intervals, such that the homoclinic tangle contains attracting periodic solutions (Theorem 2). In addition, there are also parameters in between where the homoclinic tangle admits non-degenerate transversal homoclinic tangency (Theorem 3). In Sect. 4.3 we study the associated homoclinic tangles by numerically iterating the derived return maps. Finally in Sect. 4.4, we summarize various dynamics scenarios in the surroundings of periodically perturbed homoclinic solutions newly found through the return maps of Proposition 3.3.
4.1. **The return maps for homoclinic tangle.** Let $Q(t) = \sin t$ in equation (2.2). Let $\mathbb{F}(\mathbb{X}, \theta)$ be as in Proposition 3.3. The stable and the unstable manifold of $(x, y) = (0, 0)$ of equation (2.2) intersect if and only if there exists $\theta$ such that $\mathbb{F}(0, \theta) = 0$. In [WO], the authors excluded the possibility of these intersections by restricting to a specific range of forcing parameters. We now allow $\mathbb{F}(0, \theta) = 0$.

Let
\[
A = \int_{-\infty}^{\infty} \mathbb{H}(s)e^{-\int_{0}^{s} E(\tau) d\tau} ds
\]
\[
C(\omega) = \int_{-\infty}^{\infty} \mathbb{H}(s) \cos(\omega s)e^{-\int_{0}^{s} E(\tau) d\tau} ds
\]
\[
S(\omega) = \int_{-\infty}^{\infty} \mathbb{H}(s) \sin(\omega s)e^{-\int_{0}^{s} E(\tau) d\tau} ds.
\]

Recall that $\ell(s) = (a(s), b(s)), s \in \mathbb{R}$ is the homoclinic solution of equation (2.1) and $(u(s), v(s))$ is the unit tangent vector of $\ell(s)$. Also recall that $\mathbb{H}(s)$ is as in (2.14) and $E(s)$ is as in (2.11). Using the conclusions of Sect. 3.2, it is easy to verify that $A, C$ and $S$ are all well-defined. In the rest of this section we assume that

**H2** (i) $A \neq 0$; and (ii) $C^2(\omega) + S^2(\omega) \neq 0$.

For a given equation (2.1) satisfying (H1), (H2)(i) holds for majority of $A(x, y)$ and $B(x, y)$. (H2)(ii) requires that, as a function of $s$, the Fourier spectrum of the function
\[
R(s) = \mathbb{H}(s)e^{-\int_{0}^{s} E(\tau) d\tau}
\]
is not identically zero. We know that $R(s)$ decays exponentially as a function of $s$, and it follows that the Fourier transform $\hat{R}(\omega)$ is analytic in a strip contain the real $\omega$-axis. Consequently, $\hat{R}(\omega) = 0$ for at most a discrete set of values of $\omega$ because $R(s)$ is not identically zero. Note that $\hat{R}(\omega) = C(\omega) + iS(\omega)$.

**Specifications on parameters:** The parameters $\omega, \rho, \varepsilon, \mu$ are specified as follows. First we fix (arbitrarily) an $\omega$ such that (H2)(ii) holds. Then we fix a value of $\rho$ such that\(^2\)
\[
3 < \frac{1}{\rho A} \sqrt{C^2(\omega) + S^2(\omega)} < 9.
\]
3 and 9 here has no specific meaning and can be replaced by any other two numbers larger than one. We let $\varepsilon$ be small enough for a variety of reasons: one is to validate the derivations of the previous sections and another is to make
\[
2 < \frac{1}{\rho A_L} \sqrt{C_L^2(\omega) + S_L^2(\omega)} < 10
\]
where $A_L, C_L, S_L$ are obtained by replacing the integral bounds $\pm \infty$ with $\pm L^\pm$ respectively in $A, C, S$. $\mu (<< \varepsilon)$ is the only parameter we allow to vary.

\(^2\)Let us assume $A > 0$ here to maintain a positive range for $\rho$. If $A < 0$ we need to switch $\rho$ to $-\rho$. 18
The return maps: In the rest of this section we use $z$ for $X$, $A$ for $\Sigma$. So we write

$$A = \{(\theta, z) : \theta \in \mathbb{R}/(2\pi \mathbb{Z}), \ |z| < 1\}.$$ 

We regard $\omega, \rho, \varepsilon$ as been fixed. Let $(\theta_1, z_1) = \mathcal{F}(\theta, z)$ for $(\theta, z) \in A$ where $\mathcal{F}$ is from Proposition 3.3. We have

$$\begin{align*}
\theta_1 &= \theta + a - \frac{\omega}{\beta} \ln \mathcal{F}(\theta, z, \mu) \\
z_1 &= b[\mathcal{F}(\theta, z, \mu)]^{\frac{\beta}{2}}
\end{align*}$$

where

$$\begin{align*}
a &= \frac{\omega}{\beta} \ln \mu^{-1} + \omega(L^++L^-) + \frac{\omega}{\beta} \ln(\varepsilon(1 + \mathcal{O}(\varepsilon))P_L^+A_L\rho) \\
b &= (\mu\varepsilon^{-1})^{\frac{\beta}{2}}[(1 + \mathcal{O}(\varepsilon))P_L^+A_L\rho]^{\frac{\beta}{2}}
\end{align*}$$

and

$$\mathcal{F}(\theta, z, \mu) = 1 + c \sin \theta + kz + \mathcal{E}(\theta, \mu) + \mathcal{O}(\theta, z, \mu).$$

in which

$$\begin{align*}
c &= (A_L\rho)^{-1}\sqrt{C_L^2 + S_L^2} \\
k &= (A_L\rho)^{-1}P_L(P_L^+)^{-1}(1 + \mathcal{O}(\varepsilon))
\end{align*}$$

and

$$\mathcal{E}(\theta, \mu) = (A_L\rho)^{-1}(P_L^+)^{-1}(1 + P_L)\mathcal{O}(\theta, \mu)(1).$$

Note that in getting (4.3) we have changed $\theta + \omega L^- + c_0$ to $\theta$ where $c_0$ is such that $\tan c_0 = C_L^2S_L$. $a, b, c, k$ and $\mathcal{E}(\theta, \mu)$ are as follows:

(i) $b \rightarrow 0$ as $\mu \rightarrow 0$. We can think $\mathcal{F}$ as an unfolding of the 1D maps

$$f(\theta) = \theta + a - \frac{\omega}{\beta} \ln (1 + c \sin \theta + \mathcal{E}(\theta, 0)).$$

(ii) $a \rightarrow +\infty$ as $\mu \rightarrow 0$. $a$ is a large number. But since it appears in the angular component we can module it by $2\pi$. With $\omega, \rho$ and $\varepsilon$ been fixed, $a$ is essentially $\omega\beta^{-1}\ln \mu^{-1}$. Varying $\mu$ from a small $\mu_0 > 0$ to zero is to run $a$ over $(a_0, +\infty)$ for some $a_0 \sim \omega\beta^{-1}\ln \mu_0^{-1}$.

(iii) By (4.2), $c \in [2, 10]$ is a constant independent of $\mu$. Consequently, there exists an interval for $\theta$ so that $1 + c \sin \theta \leq 0$, and the stable and the unstable manifold of the perturbed saddle of equation (2.2) do intersect. Also observe that from Lemma 3.5 we have

$$\mathcal{E}(\theta, \mu) \sim \varepsilon^{\beta\alpha-1}\mathcal{O}(\theta, \mu)(1).$$

When $\varepsilon$ is sufficiently small, $\mathcal{E}(\theta, \mu)$ is a $C^r$-small perturbation to $1 + c \sin \theta$.

(iv) $k$ is a small number independent of $\mu$. In fact, $k \sim \varepsilon^{\alpha\beta^{-1}}$ from Lemma 3.5. $k$ is, however, much larger than $\mu$ and it follows that the first derivative of $\mathcal{F}(\theta, z, \mu)$ with respect to $z$ is $\approx k$. This implies that the unfolding from $f(\theta)$ in (i) to $\mathcal{F}$ is non-degenerate in $z$-direction, and is controlled completely by the linear term $kz$. 

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(v) It is important that $\mathbb{E}(\theta, \mu)$ is independent of $z$. Otherwise we would have trouble in controlling what happens in $z$-direction. See (iv) above.

**New notation on parameter:** In the rest of this section we put $a$ in the place of $p$, regarding it as the bottom line parameter. Both $\mu$ and $p$ are regarded as functions of $a$. Since we have fixed $\omega$, $\rho$ and $\varepsilon$, $c$ and $k$ are fixed constants independent of $a$. We denote the return maps as $F_a$ to emphasize that $a$ is the parameter. $b$ is a function of $a$. Because $a$ and $p$ are linearly related, we have $O_{\theta, p}(1) = O_{\theta, a}(1)$ in (4.7) and $O_{\theta, z, p}(\mu) = O_{\theta, z, a}(\mu)$ in (4.5).

4.2. **Homoclinic tangles as an infinitely wrapped horseshoe map.** For $q = (\theta, z) \in A$, let $v = (u, v)$ be a tangent vector of $A$ at $q$ and let $s(v) = vu^{-1}$. $s(v)$ is the slope of $v$. We say that $v$ is horizontal if $|s(v)| < \frac{1}{100}$ and $v$ is vertical if $|s(v)| > 100$. A curve in $A$ is a horizontal curve if all its tangent vectors are horizontal and it is a vertical curve if all its tangent vectors are vertical. A vertical curve is fully extended if it reaches both boundaries of $A$ in $z$-direction. A region in $A$ that is bounded by two non-intersecting, fully extended vertical curves is a vertical strip. For a given vertical strip $V$, a horizontal strip in $V$ is a region bounded by two non-intersecting horizontal curves traversing $V$ in $\theta$-direction.

Observe that

\begin{equation}
F(\theta, z, \mu) = kz + 1 + c \sin \theta + \mathbb{E}(\theta, \mu) + O_{\theta, z, a}(\mu) = 0
\end{equation}

defines two fully extended vertical curves that divide $A$ into two vertical strips, which we denote as $V$ and $U$. Let $F > 0$ on $V$ and $F < 0$ on $U$. $F_a$ is well-defined on $V$ but not on $U$. $U$ is the window through which the solutions of equation (2.2) sneak out.

Let

\begin{equation}
\Omega_a = \{ (\theta, z) \in V : F_a^n(\theta, z) \in V, \forall n \geq 0 \}, \quad \Lambda_a = \bigcap_{n \geq 0} F_a^n(\Omega_a).
\end{equation}

$\Omega_a$ represents all solutions of equation (2.2) that stay close to $\ell$ in forward times; $\Lambda_a$ is the set $\Omega_a$ is attracted to, representing all solutions that stay close to $\ell$ in both the forward and the backward times. $\Omega_a$ and $\Lambda_a$ together represent the homoclinic tangles, the structure of which we now unravel through $F_a$.

For a fixed $z \in [-1, 1]$, let

\[ I_z = \{ \theta \in (-\frac{1}{2} \pi, \frac{3}{2} \pi) : (\theta, z) \in V \}. \]

$I_z$ is an interval in $(-\frac{1}{2} \pi, \frac{3}{2} \pi)$, which we denote as $(\theta_1(z), \theta_1(z))$. Let $h_z = \{ (\theta, z) : \theta \in I_z \}$. $F_a(h_z)$ is a 1D curve in $A$ parameterized in $\theta$, which we denote as $(z_1(\theta), \theta_1(\theta))$.

By definition

\begin{equation}
\theta_1(\theta) = \theta + a - \frac{\omega}{\beta} \ln F(\theta, z, \mu).
\end{equation}

We have

**Lemma 4.1.** Assume that $\omega/\beta^{-1} > 100$.

(a) $\lim_{\theta \to \theta_1(z)} (\theta_1, z_1) = \lim_{\theta \to \theta_1(z)} (\theta_1, z_1) = (+\infty, 0)$. 


(b) For every fixed \( z \in [-1, 1] \), there exists a unique value of \( \theta \), which we denote as \( \theta_c(z) \), such that
\[
\frac{d\theta_1}{d\theta}(\theta_c(z)) = 0.
\]

(c) Let
\[
(4.11) \quad V_f = \bigcup_{z \in [-1, 1]} \{(\theta, z) \in V : \left| \frac{d\theta_1}{d\theta} \right| < 2\}.
\]

Then \( V_f \) is a vertical strip, the horizontal size of which is \(< 10^{-1/\beta}\).

**Proof:** Observe, from (4.8), that \( \theta_l \in (-\frac{1}{2}\pi, 0) \), where \( \cos \theta > 0 \), and \( \theta_r \in (\pi, \frac{3}{2}\pi) \), where \( \cos \theta < 0 \). (a) follows directly from the fact that, as \( \theta \to \theta_l^+ \), \( \theta_r^+ \), \( F \to 0 \). To prove (b) we first observe that, because \( F \to 0 \) as \( \theta \to \theta_r^- \),
\[
|1 + c \sin \theta_r^-| < K \varepsilon^{\beta^{-1}} \lessapprox 1,
\]
and it follows that
\[
\frac{\partial F}{\partial \theta} (\theta_r(z)^-, z) \approx c \cos \theta_r < -1.
\]

Consequently,
\[
\lim_{\theta \to \theta_r^-} \frac{d\theta_1}{d\theta} = \lim_{\theta \to \theta_r^-} \left( 1 - \omega \beta^{-1} \frac{1}{F} \frac{\partial F}{\partial \theta} \right) = +\infty.
\]

Similarly, we have
\[
\lim_{\theta \to \theta_l^+} \frac{d\theta_1}{d\theta} = \lim_{\theta \to \theta_l^+} \left( 1 - \omega \beta^{-1} \frac{1}{F} \frac{\partial F}{\partial \theta} \right) = -\infty.
\]

Therefore there exists at least one \( \theta_c(z) \) satisfying \( \frac{d\theta_1}{d\theta} = 0 \). For the uniqueness we observe that
\[
\frac{d^2 \theta_1}{d\theta^2} = -\frac{\omega \beta^{-1}}{F^2} \left( \frac{\partial^2 F}{\partial \theta^2} - \left( \frac{\partial F}{\partial \theta} \right)^2 \right) \approx \frac{\omega \beta^{-1}}{F^2} (c^2 + c \sin \theta) > 0
\]
for all \( \theta \). Recall that \( c > 2 \).

To prove (c) we observe that the boundary of \( V_f \) is defined by
\[
\left| 1 - \omega \beta^{-1} \frac{1}{F} \frac{\partial F}{\partial \theta} \right| = 2,
\]
from which we obtain
\[
|\cos \theta| \leq \frac{9}{2} \omega^{-1/\beta} + K \varepsilon^{\beta^{-1}}.
\]
(c) follows directly from this estimate. \(\square\)

We are now ready to formally state and prove the first of our theorems.

**Theorem 1 (Horseshoe of infinitely many symbols).** Let \( Q(t) = \sin t \) and assume (H1) and (H2) for equation (2.2). Let the parameters \( \omega, \rho, \varepsilon \) been specified as in Sect. 4.1. If in addition \( \omega \beta^{-1} > 100 \), then there exists a sequence of \( \mu \), which we denote as
\[
1 \gg \mu_1^{(r)} > \mu_1^{(l)} > \cdots > \mu_n^{(r)} > \mu_n^{(l)} > \cdots > 0
\]
such that for all $\mu \in [\mu_n^{(l)}, \mu_n^{(r)}]$, $F_a$ on
\[ \Lambda = \{(\theta, z) \in V : F_a^i(\theta, z) \in V, \forall i \in \mathbb{Z}\} \]
conjugates to a full shift of countably many symbols.

**Proof:** For different values of $\mu$, the corresponding vertical curves in $A$ defined by (4.8) are $O(\mu)$ close. So $V$ and $U$ are almost stationary as $a$ varies from $a_0$ to $+\infty$. On the other hand, it follows from (4.3) that, by varying $a$ from $a_0$ to $+\infty$, we move $F_a(V)$ horizontally towards $\theta = +\infty$. Denote $F = F_a$ and let $V_f$ be the vertical strip defined through (4.11). The horizontal size of $F(V_f)$ is smaller than $20\beta\omega^{-1}$ from Lemma 4.1 assuming $\omega\beta^{-1} > 100$, which is in turn smaller than the horizontal size of $U$. Therefore $F(V_f)$ traverses $A$ infinitely many times in horizontal direction as we vary $a$ from $a_0$ to $+\infty$ and there are infinitely many sub-intervals of $a$, such that $F(V_f) \subset U$. For these parameter values $F(V) \cap V$ consists of countably many horizontal strips in $V$ (see Fig. 1 in Section 1A), to each of which we assign a positive integer according naturally to the order in which these strips are stacked in the downward $z$-direction.

For $q \in A$, let $v$ be a tangent vector at $q$. Let $C_h(q)$ be the collection of all $v$ satisfying $|s(v)| < \frac{1}{100}$, and $C_v(q)$ be the collection of all $v$ satisfying $|s(v)| > 100$. To prove that $\Lambda$ conjugates to a full shift of all positive integers, it suffices to verify that we have, assuming $F(V_f) \subset U$,

(i) $DF(C_h(q)) \subset C_h(F(q))$ on $F^{-1}(F(V) \cap V)$, and
(ii) $DF^{-1}(C_v(q)) \subset C_v(F(q))$ on $F(V) \cap V$.

To prove (i) we first compute $DF$ by using (4.3). Let $(\theta_1, z_1) = F(\theta, z)$, we have

\begin{equation}
DF = \left( \frac{\partial \theta_1}{\partial \theta} \frac{\partial \theta_1}{\partial z_1} \frac{\partial z_1}{\partial z} \right) = \left( 1 - \omega\beta^{-1} \frac{\partial F}{\partial \theta} \frac{\partial F}{\partial z} \omega \beta^{-1} \frac{\partial F}{\partial \theta} \frac{\partial F}{\partial z} \right)
\end{equation}

where $F = F(\theta, z, \mu)$ is as in (4.5) and

\[ \frac{\partial F}{\partial \theta} = c \cos \theta + \varepsilon \beta \alpha^{-1} O_{\theta, a}(1) + O_{\theta, z, a}(\mu) \]
\[ \frac{\partial F}{\partial z} = k + O_{\theta, z, a}(\mu). \]

Let $v$ be such that $|s(v)| < \frac{1}{100}$, we have from (4.12)

\begin{equation}
|s(DF(v))| = \left| \frac{\alpha \beta^{-1} b \partial F}{\partial \theta} \partial F + \beta^{-1} \partial F}{\partial z} s(v) \right|
\end{equation}

We have two cases to consider.

**Case 1:** $F \geq \sqrt{k}$. In this case we have
\[ \omega \beta^{-1} \frac{1}{F} \frac{\partial F}{\partial z} < \omega \beta^{-1} \sqrt{k} << 1. \]
From \((\theta, z) \in \mathcal{F}^{-1}(\mathcal{F}(V) \cap V)\) and \(\mathcal{F}(V_f) \subset U\), it follows that \((\theta, z) \notin V_f\) therefore
\[
\left| \frac{\partial \theta_1}{\partial \theta} \right| = \left| 1 - \omega \beta^{-1} \frac{1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial \theta} \right| > 2.
\]
These two estimates together implies that the denominator for \(|s(D\mathcal{F}(v))|\) in (4.13) is \(> 1\), and it follows that \(|s(D\mathcal{F}(v))| < \frac{1}{100}\).

**Case 2:** \(\mathbb{F} < \sqrt{k}\). In this case
\[
|1 + c \sin \theta| < K \epsilon \frac{\beta}{\pi} + \sqrt{k},
\]
from which we have
\[
(4.14) \quad |c \cos \theta| > 1.
\]
It then follows that the denominator for \(|s(D\mathcal{F}(v))|\) in (4.13) is \(> \frac{1}{2} \sqrt{k}\), which implies \(|s(D\mathcal{F}(v))| < \frac{1}{100}\). This finishes our proof for (i).

To prove (ii) we let \(v\) be such that \(|s(v)| > 100\). From (4.12),
\[
(4.15) \quad D\mathcal{F}^{-1} = \frac{1}{\alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial z}} \left( \begin{array}{cc}
\alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial \theta} - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial z} & 1 - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial \theta} \\
- \alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial \theta} & - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial z} \end{array} \right),
\]
and we have
\[
|s(D\mathcal{F}^{-1}(v))| = \left| \frac{-\alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial \theta} s^{-1}(v) + (1 - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial \theta})}{\alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial z} s^{-1}(v) - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial \theta}} \right|.
\]
We again divide into the cases of \(\mathbb{F} > \sqrt{k}\) and \(\mathbb{F} < \sqrt{k}\). If \(\mathbb{F} > \sqrt{k}\), the magnitude of the denominator \(< 1\) and that of the numerator is \(> 1\) again because
\[
\left| 1 - \omega \beta^{-1} \frac{1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial \theta} \right| > 2
\]
from the assumption that \((\theta, z) \notin V_f\). For the case of \(\mathbb{F} < \sqrt{k}\), we re-write \(|s(D\mathcal{F}^{-1}(v))|\) as
\[
|s(D\mathcal{F}^{-1}(v))| = \left| \frac{-\alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial \theta} s^{-1}(v) + (\mathbb{F} - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial \theta})}{\alpha \beta^{-1} b \mathcal{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathcal{F}}{\partial z} s^{-1}(v) - \omega \beta^{-1} \frac{\partial \mathcal{F}}{\partial \theta}} \right|.
\]
The denominator is again \(< 1\) and the dominating term in the numerator is
\[
\left| \omega \beta^{-1} \frac{1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial \theta} \right| > 1.
\]
The last estimate is from
\[
\left| \frac{\partial \mathcal{F}}{\partial \theta} \right| \approx |c \cos \theta| > 1
\]
again by (4.14). This proves (ii). \(\square\)

We refer the reader to Chapter III.1 of [Mo] for a detailed discussion on horseshoes of infinitely many symbols.

**Remarks:**

1. For the parameters of Theorem 1, the entire homoclinic tangle consists of a single horseshoe of infinitely many symbols.
2. For all $|\mu| < \mu_0$, $\mathcal{F}_a$ induces a horseshoe of infinitely many symbols on $V \setminus V_f$. This horseshoe covers Smale’s horseshoe and all its variations. It is the one that hides inside all homoclinic tangles.

3. $\Lambda$ is much more complicated when $\mathcal{F}_a(V_f)$ intersects $V$. As $\mathcal{F}_a(V_f)$ traverses $V$, we encounter complicated dynamical patterns caused by our allowing the images of the unstable manifold of the horseshoe in $V \setminus V_f$ to come back to traverse the stable manifold of the same horseshoe. We will prove, momentarily, that there are parameters that admit periodic sinks and there are also others that admit non-degenerate transversal homoclinic tangency. We unfortunately do not have a bifurcation diagram for $\mathcal{F}_a$. However, we know from (4.3) that the same diagram are repeated infinitely many times as $\mu \to 0$.

4. We caution that, though the horseshoe of Theorem 1 represents all solutions of the perturbed equation that stay forever inside of a small neighborhood of the homoclinic loop $\ell$, solutions sneaked out through $U$ might find a way to come back to $\mathcal{A}$, creating more complicated sneaked structures. One particular mechanism for such coming back is for the unperturbed equation to have two homoclinic solutions. See Fig. 5(a). In this case, part of $U$ would come back to $\mathcal{A}$ following the other homoclinic loop. On the other hand, it is easy to obtain examples for which the solutions sneaked out of $U$ would never come back. In this case the entire homoclinic tangle for the perturbed equation is in fact reduced to the horseshoe of Theorem 1: all it takes for this to happen is for us to send the other branch of the local unstable manifold of $(0,0)$ to a sink. See Fig. 5(b).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{(a) $U$ gets back to $\mathcal{A}$, and (b) all points in $U$ approach a sink.}
\end{figure}

Our next Theorem is about the existence of periodic sinks. We remark that these periodic sinks are not Newhouse sinks associated with homoclinic tangency.

**Theorem 2 (Periodic sinks).** Let the assumptions be identical to that of Theorem 1. Then there exists an open set of $\mu$ inside each of the intervals $[\mu_n^{(r)}, \mu_n^{(l)}]$, such that the corresponding homoclinic tangle admits a periodic sink.

**Proof:** Let $\theta_c(z)$ be as in Lemma 4.1(b). To make the dependency on $\mu$ explicit we write it as $\theta_c(z, \mu)$. Let $a_n$ be the value of $a$ at $\mu = \mu_n^{(r)}$ and $[a_n] = a_n - a_n \mod(2\pi)$. 

\[24\]
Observe that there exists a $\hat{\mu} \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]$ so that $\theta_1(\theta_c) = \theta_c + [a_n]$ where $\theta_c = \theta_c(0, \hat{\mu})$. This is because when $\mu$ traverses $[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, $\theta_1(\theta_c)$ traverses the interval $(\theta_l + [a_n], \theta_r + [a_n])$. Let $\hat{a}$ be the value of $a$ for $\hat{\mu}$. To solve for a fixed point we let

$$
\theta + [a_n] = \theta + \hat{a} - \omega\beta^{-1} \ln F
$$

(4.16)

$$
z = bF^{\alpha \beta^{-1}}
$$

to obtain

$$
F = e^{\omega^{-1} \beta(\hat{a} - [a_n])},
$$

(4.17)

$$
z = be^{\omega^{-1} \alpha(\hat{a} - [a_n])}.
$$

From the first line we have

(4.18) \[ 1 + c \sin \theta + E(\theta, \hat{\mu}) + O_{\theta, z, a}(\hat{\mu}) = e^{\omega^{-1} \beta(\hat{a} - [a_n])}. \]

To solve (4.18) for $\theta$, first we observe that $\theta_c = \theta_c(0, \hat{\mu})$ is a solution of (4.18) for $z = 0$. We then observe that

$$
| \cos \theta_c | > K^{-1}.
$$

This estimate follows from the fact that $\theta_c$ is defined by

\[ 1 - \omega \beta^{-1} \frac{\partial F}{\partial \theta} = 0 \]

and $F = e^{\omega^{-1} \beta(\hat{a} - [a_n])}$ from (4.17). Applying the inverse value theorem to (4.18) we obtain a solution $\hat{\theta}$ satisfying

$$
| \hat{\theta} - \theta_c | < K \hat{\mu}.
$$

In summary we have obtained a fixed point $(\hat{\theta}, \hat{z})$ satisfying

$$
\hat{\theta} \approx \theta_c; \quad \hat{z} = b e^{\omega^{-1} \alpha(\hat{a} - [a_n])}.
$$

To prove that $(\hat{\theta}, \hat{z})$ is an attracting fixed point, we compute the eigenvalues. The eigen-equation for $DF$ is

$$
\lambda^2 - Tr(DF) \lambda + det(DF) = 0.
$$

From (4.12) we have

(4.19) \[ Tr(DF) = \frac{\partial \theta_1}{\partial \theta} + \alpha \beta^{-1} b F^{\alpha \beta^{-1} - 1} \frac{\partial F}{\partial z} \ll 1 \]

$$
det(DF) = \alpha \beta^{-1} b F^{\alpha \beta^{-1} - 1} \frac{\partial F}{\partial z} \ll 1
$$

where for the first inequality we use

$$
\frac{\partial \theta_1}{\partial \theta} < K(\| \hat{\theta} - \theta_c \| + |\hat{z}|)
$$

at $(\hat{\theta}, \hat{z})$ with

$$
K = \max_{\theta \in (\theta_c, \hat{\theta}), z \in [0, \hat{z}]} \left( \frac{|\partial^2 \theta_1|}{\partial \theta^2} + \frac{|\partial^2 \theta_1|}{\partial \theta \partial z} \right).
$$
Note that, on the domain the maximum is taken, $F > \frac{1}{2}$. The rest of (4.19) are obvious. It follows from (4.19) that both eigenvalues of $D\hat{F}$ are close to 0. □

Our next Theorem is about the existence of non-degenerate transversal homoclinic tangency.

**Theorem 3 (Homoclinic tangency).** Let the assumptions be identical to that of Theorem 1. Then for every $n > 0$ given, there exists $\hat{\mu} \in [\mu^{(r)}_n, \mu^{(l)}_{n+1}]$, the corresponding value for $\hat{a}$ we denote as $\hat{a}$, such that

(i) $F_{\hat{a}}$ has a saddle fixed point, which we denote as $q(\hat{a})$, so that $W^u(q(\hat{a})) \cap W^s(q(\hat{a}))$ contains a point of non-degenerate tangency.

(ii) Let $q(a)$ be the continuous extension of $q(\hat{a})$ for $a$ sufficiently close to $\hat{a}$. Then as $a$ passes through $\hat{a}$, $W^u(q(a))$ crosses $W^s(q(a))$ at the tangential intersection point of (i) with a speed $> \frac{1}{2}$ with respect to $a$.

**Proof:** Our plan of proof is as follows. We know that $F_a$ induces a horseshoe of infinitely many symbols on $V \setminus V_f$, creating many saddle fixed points. Pick one and denote it as $q$. We prove that $q$ is continuously extended over the $\mu$ interval $[\mu^{(r)}_n, \mu^{(l)}_{n+1}]$, which we denote as $q(a)$. Let $W^u(q(a))$ be the unstable and $W^s(q(a))$ be the stable manifold of $q(a)$. We prove that $W^u(q(a)) \cap V_f$ has a horizontal segment traversing $V_f$, which we denote as $\ell^u(a)$. We also prove that $W^s(q(a))$ has a vertical segment fully extended in $V$, which we denote as $\ell^s(a)$. Observe that $F_a(\ell^u(a))$ has a sharp quadratic turn, and as $\mu$ varies from $\mu^{(r)}_n$ to $\mu^{(l)}_{n+1}$, it moves from one side of $V$ to the other, transversally crossing $\ell^s(a)$. See Fig. 6.

![Fig. 6 Transversal homoclinic tangency.](image)

Detailed proof for Theorem 3 is long and include some tedious computations. A complete proof is included in the Appendices. □

**Remark:** We caution that, Theorem 3 does not directly imply the existence of Newhouse sinks of [N] and Hénon-like attractors of [MV] for $F_a$. In [N] and [MV], it is assumed that the maps are linearized around a dissipative saddle fixed point on an interval of parameters. For $F_a$, the eigenvalues of $q(a)$ are functions of $a$, implicating resonant eigenvalues that prevent the maps to be smoothly linearized.
around \( q(a) \). Nonetheless, Theorem 3 is a strong indication that Newhouse sinks and strange attractors with SRB measures are plausible dynamical scenarios for \( \mathcal{F}_a \).

4.3. Homoclinic tangles and observable chaos. Let \( \mathcal{F}_a \) be as in (4.3) and \( \Omega, \Lambda \) be as in (4.9). \( \Omega \) represents all solutions that stay close to \( \ell \) in forward times, and \( \Lambda \) represents all solutions that stay close to \( \ell \) in both the forward and the backward times. In this subsection we study numerically the structures of \( \Omega \) and \( \Lambda \).

We start with a concept of observability in numerical simulations. We say that a homoclinic tangle is observable in phase space if \( \Omega \) has positive Lebesgue measure. Otherwise we say that this homoclinic tangle is not observable. We only expect observable homoclinic tangles to show up in numerical simulations. For maps with parameters, there is also an issue of observability in parameter space: a sub-collection of maps is observable only if it is from a parameter set of positive Lebesgue measure. See [WOk] for more detailed discussions on observable dynamical scenarios in numerical simulations.

To numerically study homoclinic tangles through \( \mathcal{F}_a \), we drop the error terms in (4.3) and re-write \( k_z \) as \( z \). We obtain from (4.3)-(4.6) a family of 2D maps in the form of

\[
\begin{align*}
\theta_1 &= \theta + a - d \ln(1 + c \sin \theta + z) \\
\gamma z_1 &= b[1 + c \sin \theta + z]^\gamma.
\end{align*}
\]

(4.20)

where \( a, b, c, d, \gamma \) are parameters. \( a \in S^1, b \ll 1, c > 1, d \in \mathbb{R} \) and \( \gamma > 1 \). From Theorems 1-3, we would expect at least three dynamical scenarios that are observable in parameter space. They are as follows.

(i) For parameters of Theorem 1, \( \Lambda \) for \( \mathcal{F}_a \) is a uniformly hyperbolic invariant set, and \( \Omega \) is the stable manifold for \( \Lambda \) inside of \( \Sigma^- \). Both \( \Omega \) and \( \Lambda \) are Lebesgue measure zero sets. The corresponding homoclinic tangle for these parameters is therefore not observable in phase space.

(ii) For parameters of Theorem 2, \( \Omega \) contains an open neighborhood of a periodic sink so the associated homoclinic tangle is observable. Plots of individual orbits from \( \Omega \) would lead us to periodic sinks in \( \Lambda \).

(iii) From Theorem 3 and the existing theory related to the existences of non-degenerate transversal homoclinic tangency, we expect strange attractors with SRB measures to show up as an observable phenomenon in numerical simulations.

In Figs 7-9, we plot \( \Omega \) and \( \Lambda \) for the maps defined in (4.20) with various choices of parameters that reflect the scenarios (i)-(iii) above respectively. Fig. 7 is for scenario (i), with \( a = 0.2, b = 0.005, c = 3, d = 2 \) and \( \gamma = \sqrt{2} \). Fig. 7(a) is a plot of all points in \( V \), the orbits of which remain inside of \( V \) after 3 iterations, Fig. 7(b) is for after 6 iterations. Nothing is left in \( V \) after 15 iterations.
Fig. 7 Homoclinic tangles with no sinks nor observable chaos.
\((a = 0.2, b = 0.005, c = 3, d = 2 \text{ and } \gamma = \sqrt{2})\)

Fig. 8 is for scenario (ii) with \(a = 2\). The values for \(b, c, d, \gamma\) are kept the same as in Fig. 7. In this case \(\Lambda\) contains a periodic sink with a relatively large basin. Fig. 8(a) is for \(\Omega\). All orbits initiated from \(\Omega\) quickly converge to an attracting periodic orbit. In Fig. 8(b) we depict \(\theta_k\) v.s. \(k\) for one orbit from \(\Omega\). Picture for \(z_k\) v.s. \(k\) is similar. Only one periodic sink shows up for \(\Lambda\) in numerical simulations. The horseshoe associated with the singularity of the logarithmal function, though exists inside of \(\Lambda\), does not show up because the set it attracts is a set of zero Lebesgue measure.

Fig. 8 Homoclinic tangles with an attracting periodic solution.
\((a = 2, b = 0.005, c = 3, d = 2 \text{ and } \gamma = \sqrt{2})\)

Fig. 9 is for scenario (iii) with \(a = 1.5\). The values for \(b, c, d\) and \(\gamma\) are kept the same as before. \(\Omega\) is depicted in Fig. 9(a). In Fig. 9(b) we depicted again \(\theta_k\) v.s. \(k\) for one orbit. As \(k\) moves forward, \(\theta_k\) jumps randomly in a fixed range. These pictures represent a strange attractor with an SRB measure created by transversal
homoclinic tangency of a saddle periodic orbit of relatively high period. All orbits from \( \Omega \) in fact offer the same picture.

We also performed systematic search over all combinations of parameters with \( b \) reasonably small. We persistently run into one of the three scenarios above. In the case of Fig. 7, however, sometimes it takes much longer for all points to be completely iterated out of \( V \). This is particularly the case when \( d \) is small, and is more or less expected: as the overall strength of expansions around the horseshoe of Theorem 1 gets weaker, the points in \( V \) tends to linger longer inside of \( V \) before been pushed out into \( U \).

4.4. Dynamical scenarios for periodically perturbed homoclinic solutions.

In this paragraph we summarize all that have been obtain so far for equation (2.2) in [WO], [LW] and in this paper through the return map of Proposition 3.3. Again, we let \( Q(t) = \sin t \) and assume (H1) and (H2). The forcing parameters are inside of

\[
\mathbb{P} = \left\{ (\omega, \rho, \mu) : \omega \in (0, R_\omega), \rho \in (R_\mu^{-1}, R_\rho), \mu \in (0, R_\mu^{-1}) \right\}
\]

where \( R_\mu >> R_\rho >> R_\omega >> 1 \). Let \( W^s \) be the stable and \( W^u \) be the unstable manifold of \((x, y) = (0, 0)\) in the extended phase space. Various dynamics scenarios for different parts of \( \mathbb{P} \) are illustrated in Fig. 10. Since our purpose is to provide an overview, only descriptive statements are presented. Rigorous formulations and their proofs are either directly included in [WO], [LW] and in this paper, or obtained by reasonable modifications of existing text.

1. There is a surface \( S^* \) in \( \mathbb{P} \) (See Fig. 10), such that for all parameters under \( S^* \), \( W^u \cap W^s \neq \emptyset \) and we have homoclinic tangles for equation (2.2). The dynamics of these homoclinic tangles are studied in Sect. 4.2 of this paper. In particular, there are open sets of parameters, such that the entire homoclinic tangle is one uniformly hyperbolic horseshoe. There are also parameters for periodic sinks, and parameters for non-degenerate, transversal homoclinic tangency.
2. For parameters over $S^*$, $W^s \cap W^u = \emptyset$. The return maps are again defined through (4.3)-(4.7), but for these parameters $F > 0$ on $\Sigma^-$ so $F_a$ are well-defined on $\Sigma^-$. These maps have been studied systematically in [WY4]. We know that

(a) There is a surface $S$ above $S^*$ for which the following holds. For all parameters in between $S$ and $S^*$, $F_a$ admit global attractors in $\Sigma^-$ that are chaotic in the sense that they all contain a horseshoe (See [LW]). If the forcing frequency $\omega$ is reasonably large, then there is a positive measure set of forcing parameters such that the strange attractors are rank one attractors of [WY1] and [WY2] with SRB measures (See [WO]).

(b) There is a surface $Q$ (See Fig. 10) for which the following holds. For any give set of parameters on the left of $Q$, $\Sigma^-$ is attracted globally to a simple closed curve, on which the map induced conjugates to a circle diffeomorphisms. If the rotation number of this circle diffeomorphism is rational, then there are saddles and attracting periodic solutions. If the rotation number is irrational, then the solutions are quasi-periodic. We also know as a fact that there are positive measure sets of parameters such that the corresponding rotation numbers are irrational and the corresponding solutions are quasi-periodic.

(c) What happens between $Q$ and $S$ are as follows: as we move from the left to the right in the $\omega$-direction, larger forcing frequency first deforms, then breaks the attracting invariant curve, inducing sinks and saddles. The unstable manifolds of these induced saddles would eventually fold in $\theta$-direction, intersecting the stable manifolds to create strange attractors and rank one chaos. As we go down in the $\rho$-direction, $W^u$ and $W^s$ are pulled gradually together. Reflected in the return maps of Proposition 3.3 is the growing relevance of the expansions associated with the singularity of the logarithmical function.
5. Homoclinic tangles for general forcing functions

In this section we let \( Q(t) \) be an arbitrary periodic function of period \( 2\pi \). We explain how the different choices of the forcing function \( Q(t) \) affect the dynamics of the associated homoclinic tangles.

Let \( A, S(\omega) \) and \( C(\omega) \) be the same as before (See (4.1)). \( A, S(\omega) \) and \( C(\omega) \) are independent of \( Q(t) \). We assume (H1) and (H2)(i) for equation (2.2) and replace (H2)(ii) by (H3) below.

(H3) There exists a constant \( \xi > 0 \) so that
\[
\sqrt{S^2(\omega) + C^2(\omega)} \sim e^{-\xi|\omega|}
\]
as \( |\omega| \to +\infty \).

(H3) is stronger than (H2)(ii). It requires that the magnitude of the Fourier transformation \( \hat{R}(\omega) \) of the function
\[
R(s) = \mathbb{H}(s)e^{-\int_0^s E(\tau)d\tau}
\]
decays exponentially as \( |\omega| \to \infty \). We note that there is no lack of known systems satisfying (H3).

Expanding \( Q(t) \) in Fourier series we write
\[
Q(t) = \sum_{n=1}^{\infty} (c_n \cos nt + s_n \sin nt).
\]

If the mean value of a \( Q(t) \) is not zero, we give it to \( \rho \). So there is no loss of generality in assuming (5.1). Let us assume in addition that
\[
c_1^2 + s_1^2 \neq 0.
\]

Let \( Q(t) \) be as in (5.1) satisfying (5.2) and assume (H1), (H2)(i) and (H3) for equation (2.2). Parameters \( \omega, \rho, \varepsilon, \mu \) are specified as follows. First we fix (arbitrarily) an \( \omega \in \left[\frac{1}{100}R_\omega, R_\omega\right] \). We then fix a value of \( \rho \) such that
\[
3 < \frac{1}{\rho A} \cdot \sqrt{c_1^2 + s_1^2} \cdot \sqrt{C^2(\omega) + S^2(\omega)} < 9.
\]

After that we fix \( \varepsilon \) sufficiently small so that,
\[
|\phi(\theta) - \phi_L(\theta)| < \rho A
\]
where \( \phi_L(\theta) \) is from (3.19) and \( \phi(\theta) \) is obtained by replacing \(-L^-, L^+ \) with \(-\infty, +\infty \) respectively in \( \phi_L(\theta) \). We also make \( \varepsilon \) sufficiently small so that
\[
2 < \frac{1}{\rho A} \cdot \sqrt{c_1^2 + s_1^2} \cdot \sqrt{C_L^2(\omega) + S_L^2(\omega)} < 10.
\]

\( \mu (< \varepsilon) \) is the only parameter we allow to vary. In what follows \( F_p : \Sigma^- \to \Sigma^- \) is the return maps of Proposition 3.3 induced by equation (2.2). Recall that \( p = \ln \mu \).
Theorem 4. Let \( Q(t) \) be as in (5.1) satisfying (5.2) and assume (H1), (H2)(i) and (H3) for equation (2.2). Let the parameters \( \omega, \rho, \varepsilon \) be specified as in the above. Then

(a) there are infinitely many disjoint open intervals of \( \mu \), accumulating at \( \mu = 0 \), so that the corresponding homoclinic tangles of equation (2.2) are reduced to one single horseshoe of infinitely many symbols;

(b) in between each of these parameter intervals, there are values of \( \mu \) so that the homoclinic tangles of equation (2.2) contain stable periodic solutions; and

(c) there are also parameters in between where the homoclinic tangles admit non-degenerate transversal homoclinic tangency.

Proof: We argue that our previous proofs for Theorems 1-3 remain valid for the current setups. By assuming (H3), (5.2) and \( \omega \in [\frac{1}{1000} R_{\omega}, R_{\omega}] \), which is \( >> 1 \), we make the first order term for \( Q(t) \) dominate in \( \phi_L \). Let us recall that

\[
\phi_L(\theta) = \int_{-L^-}^{L^+} \mathbb{H}(s) Q(\theta + \omega s + \omega L^-) e^{-\int_0^s E(r) dr} ds
\]

is a critical element of the return map \( F_p \) in Proposition 3.3. By definition

\[
\phi(\theta) = \int_{-\infty}^{\infty} \mathbb{H}(s) Q(\theta + \omega s + \omega L^-) e^{-\int_0^s E(r) dr} ds
\]

\[
= \sum_{n=1}^{\infty} \sqrt{c_n^2 + s_n^2} \cdot \sqrt{C^2(n\omega) + S^2(n\omega)} \cdot \sin(n\theta - n\omega L^- - \theta_n)
\]

where \( \theta_n \) are constants completely determined by \( c_n, s_n, C(n\omega), S(n\omega) \). We re-write \( F_p \) of Proposition 3.3 following the steps of Sect. 4.1, using \( z \) for \( X \) and denoting \((\theta_1, z_1) = F_p(\theta, z)\) for \((\theta, z) \in \Sigma^-\). From Proposition 3.3 we have

\[
\theta_1 = \theta + a - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, z, \mu)
\]

\[
z_1 = b[\mathbb{F}(\theta, z, \mu)]^{\frac{\alpha}{\beta}}
\]

where

\[
a = \frac{\omega}{\beta} \ln \mu^{-1} + \omega(L^+ + L^-) + \frac{\omega}{\beta} \ln(1 + O(\varepsilon)) P_L^+ A_L \rho
\]

\[
b = (\mu e^{-1})^{\frac{\beta}{\alpha}} (1 + O(\varepsilon)) P_L^+ A_L \rho;\]

\[
\mathbb{F}(\theta, z, \mu) = 1 + c(\sin \theta + \Phi(\theta)) + k z + E(\theta, \mu) + O_{\theta, z, \rho}(\mu)
\]

with

\[
c = (A_L \rho)^{-1} \cdot \sqrt{c_1^2 + s_1^2} \cdot \sqrt{C^2(\omega) + S^2(\omega)}
\]

\[
k = (A_L \rho)^{-1} (P_L P_L^+)^{-1} (1 + O(\varepsilon))
\]
and
\[
\mathbb{E}(\theta, \mu) = (A_L\rho)^{-1}(P_L^+)^{-1}(1 + P_L)\mathcal{O}_{\theta,p}(1) + (A_L\rho)^{-1}(\phi(\theta) - \phi_L(\theta))
\]
(5.7)
\[
\Phi(\theta) = \sum_{n=2}^{\infty} \sqrt{c_n^2 + s_n^2} \cdot \sqrt{\frac{C^2(n\omega) + S^2(n\omega)}{C^2(\omega) + S^2(\omega)}} \cdot \sin(n\theta - n\omega L^- - \theta_n).
\]

By (H3) and the assumption that \(\omega > \frac{1}{100} R_{\omega} >> 1\), \(\Phi(\theta)\) is an added error term, toward which our previous proofs of Theorems 1-3 are indifferent. \(\square\)

From (5.3)-(5.7) for \(\mathcal{F}_p\) we see that (4.20) is a prototype of return maps for all \(\mathcal{Q}(\omega t)\) provided that \(\omega >> 1\). If the forcing frequency is lower, then \(\Phi(\theta)\) in (5.7) remains an important part of \(\phi_L(\theta)\) in \(\mathcal{F}_p\). It is then possible to have a number of disjoint vertical strips for \(V\), and an equal number of vertical strips for \(U\). The images of each of the \(V\)-components again wrap around \(\Sigma^-\) infinitely many times in \(\theta\) direction. We might, however, have more turns for \(\mathcal{F}(V)\), as shown in Fig. 11.

![Fig. 11](image)

Fig. 11 Infinitely wrapped horseshoe maps for \(\mathcal{Q}(t)\) in general.

We finish by presenting two more numerical pictures for \(\Omega\) and \(\Lambda\). These are for the maps assuming the form of (4.20), but with \(\sin \theta\) replaced by \(\sin \theta + \sin 3\theta\). Fig. 12 is for the case of an attracting periodic sink, with \(a = 1, b = 0.005, c = 1, d = 2\). \(\Omega\) is depicted in Fig. 11(a). \(\theta_k\) v.s. \(k\) for one orbit from \(\Omega\) is depicted in Fig. 12(b). All orbits in \(\Omega\) is attracted to a periodic sink.

Fig. 13 is for a strange attractor with an SRB measure. The values for \(b, c, d\) and \(\gamma\) are kept the same as in Fig. 12, but \(a\) is changed to 0.5. \(\Omega\) is depicted in Fig. 13(a), and \(\theta_k\) v.s. \(k\) for one orbit from \(\Omega\) is depicted in Fig 13(b). This orbit is attracted to an SRB measure.
Fig. 12 Homoclinic tangle with a periodic sink. 
\(a = 1, b = 0.005, c = 1, d = 2\) and \(\gamma = \sqrt{2}\)

Fig. 13 Homoclinic tangles with observable chaos. 
\(a = 0.5, b = 0.005, c = 1, d = 2\) and \(\gamma = \sqrt{2}\)

APPENDIX A. PROOF OF THEOREM 3

In this appendix we prove Theorem 3.\(^3\) Let \(a_n\) be the value of \(a\) at \(\mu = \mu_n^{(r)}\) and \([a_n] = a_n - a_n \mod(2\pi)\). Let \(a(\mu)\) be the value of \(a\) at \(\mu \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]\). We divide the proof of this theorem into the following steps.

**Step 1. Solving for hyperbolic fixed points** For \(\mu \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]\), let \(m\) be an integer \(\geq 3\omega / \beta^{-1}\) and \(q_m(a) = (\theta_m, z_m)\) be the solution of the equations

\[
\begin{align*}
\theta + [a_n] + 2\pi m &= \theta + a(\mu) - \frac{\omega}{\beta} \ln \mathcal{F}(\theta, z, \mu) \\
z &= b[\mathcal{F}(\theta, z, \mu)]^{\frac{\alpha}{\beta}}.
\end{align*}
\]

\(^3\)Minus Claim A.6(b), which we prove in Appendix B.
\( \theta_m \) is determined by

(A.2) \[ F(\theta_m, z_m, \mu) = e^{\omega^{-1} \beta(a(\mu) - [a_n] - 2\pi m)}, \]

and

(A.3) \[ z_m = be^{\omega^{-1} \alpha(a(\mu) - [a_n] - 2\pi m)}. \]

Claim A.1. \( q_m(a) = (\theta_m, z_m) \) is saddle fixed point.

Proof of Claim A.1: Recall that

\[
\text{Tr}(DF) = \frac{\partial \theta_1}{\partial \theta} + \alpha \beta^{-1} b F^{\alpha \beta^{-1} - 1} \frac{\partial F}{\partial z},
\]

\[
\text{det}(DF) = \alpha \beta^{-1} b F^{\alpha \beta^{-1} - 1} \frac{\partial F}{\partial z}.
\]

Observe that, from (A.2) and the assumption that \( m \geq 3\omega \beta^{-1} \), \( F(\theta_m, z_m, \mu) < \frac{1}{100} \).

It follows that \( |c \cos \theta_m| > 1 \), and

\[ \left| \frac{\partial \theta_1}{\partial \theta} \right| > 101. \]

This implies

\[ |\text{Tr}(DF)| > 100. \]

Observe that we also have \( \text{det}(DF) << 1 \). Therefore we have two eigenvalues, one is close to 0 and the other is with magnitude > 1. \( \diamond \)

We also have

Claim A.2. For \( m \geq 3\omega \beta^{-1} \),

\[
\left| \frac{d \theta_m}{da} \right| < \frac{1}{100}, \quad \left| \frac{dz_m}{da} \right| < Kb.
\]

Proof of Claim A.2: Estimate for \( \frac{dz_m}{da} \) follows directly from (A.3). To estimate \( \frac{d \theta_m}{da} \) we take derivative with respect to \( a \) on both-side of (A.2) and use \( F < \frac{1}{100} \) to obtain \( |c \cos \theta_m| > 1 \). \( \diamond \)

Step 2. The stable and the unstable manifold for \( q_m \) In the rest of this proof we let \( m \) be the smallest integer \( > 3\omega \beta^{-1} \). Denote \( q(a) = q_m(a) \). Let

\[ \hat{V} = \{ (\theta, z) \in V, \ F > F(q_{100m}(a), \mu) \} \]

We obtain \( \hat{V} \) from \( V \) by taking away two thin vertical strips at the vertical boundaries of \( V \). Observe that, by definition, \( q(a) \in \hat{V} \). We make \( \varepsilon \) sufficiently small so that

(1) the distance from \( q(a) \) to the vertical boundary of \( \hat{V} \) is \( >> k \); and

(2) \( K_m := F(q_{100m}) >> k \).

Denote the stable and the unstable manifold of \( q = q(a) \) as \( W^s(q) \) and \( W^u(q) \) respectively. The local stable and the local unstable manifold are denoted as \( W^s_{loc}(q) \) and \( W^u_{loc}(q) \). Let \( \ell^u(q) \) be the connected branch of \( W^u(q) \) in \( \hat{V} \setminus V_T \) that contains \( W^u_{loc}(q) \), and \( \ell^u(q) = F_{\alpha}(\ell^u(q)) \). \( \ell^u(q) \) is a horizontal curve traversing \( V_f \) in \( \theta \)-direction: it is straight forward to verify that

(1) \( \ell^u(q) \) is a horizontal curve, (2) the image of
\(\ell^u(q)\) is also a horizontal curve, and (3) the length of that image at least doubles the length of \(\ell^u(q)\) so it traverses \(V_f\). Let \(z = w^u(\theta)\) be such that \((\theta, w^u(\theta)) \in \ell^u_1\).

**Claim A.3.** We have on \(\ell^u_1\),

(a) \(|\frac{d w^u}{d \theta}| < b^\frac{1}{2}\);

(b) \(|\frac{d^2 w^u}{d \theta^2}| < b^\frac{3}{2}\).

**Proof of Claim A.3:** Denote \(\mathcal{F} = \mathcal{F}_a\). For \((\theta, z) \in \ell^u\), let \((\theta_1, z_1) = \mathcal{F}(\theta, z)\). We have from (4.12)

\[
\frac{d w^u(\theta_1)}{d \theta_1} = \left| \frac{\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial \theta} + \alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z} d w^u(\theta)}{(1 - \omega \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial \theta}) + \omega \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z} d w^u(\theta)} \right|.
\]

(a) holds because the magnitude of the denominator in (A.4) is \(> 1\) for \((\theta, z) \in \hat{V} \setminus V_f\). Remember that since \(\ell^u(q)\) is horizontal we have \(|\frac{d w^u}{d \theta}| < \frac{1}{100}\). To prove (b) we take derivative one more time to obtain

\[
\frac{d^2 w^u}{d \theta^2} = \frac{d}{d \theta} \left( \frac{d w^u}{d \theta} \right).
\]

Observe that \(\frac{\partial}{\partial \theta}\) is the denominator in (A.4), the magnitude of which is \(> 1\). Let

\[
M = \max_{(\theta, z) \in \ell^u_1} \left| \frac{d^2 w^u(\theta)}{d \theta^2} \right|.
\]

We have from (A.4) and (A.5)

\[
\left| \frac{d^2 z_1}{d \theta_1^2} \right| < K_1 b + K_2 b M.
\]

So

\[
M < K_1 b + K_2 b M,
\]

and \(M < K b < b^\frac{5}{4}\).

Let \(\ell^u_{-1}\) be the segment of \(W^u(q)\) in \(\hat{V}\) that contains \(W^u_{loc}(q)\), and \(\ell^u(q) = \mathcal{F}(\ell^u_{-1}(q))\). \(\ell^u_{-1}\) is a fully extended vertical curve in \(\hat{V}\), which we represent by a function \(\theta = w^u(z)\).

**Claim A.4.** We have on \(\ell^u_{-1}\),

(a) \(|\frac{d w^u}{d z}| < k^\frac{1}{2}\); and

(b) \(|\frac{d^2 w^u}{d z^2}| < b^\frac{4}{2}\).

**Proof of Claim A.4:** Let \((\theta, z) \in \ell^u\) and denote \((\theta_{-1}, z_{-1}) = \mathcal{F}^{-1}(\theta, z)\). We have from (4.16)

\[
\left( \frac{d^2 \theta_1}{d z_{-1}^2} \right) = \frac{1}{\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z}} \left( \begin{array}{c} \alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial \theta} \\ -\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z} \end{array} \right) \left( \begin{array}{c} \frac{d w^u}{d z} \\ 1 \end{array} \right),
\]

\[
\left( \frac{d \theta_1}{d z_{-1}} \right) = \frac{1}{\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z}} \left( \begin{array}{c} \alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial \theta} \\ -\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z} \end{array} \right) \left( \begin{array}{c} \frac{d w^u}{d z} \\ 1 \end{array} \right),
\]

\[
\left( \frac{d^2 \theta_1}{d z_{-1}^2} \right) = \frac{1}{\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z}} \left( \begin{array}{c} \alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial \theta} \\ -\alpha \beta^{-1} b \Re^{\alpha \beta^{-1} - 1} \frac{\partial \Re}{\partial z} \end{array} \right) \left( \begin{array}{c} \frac{d w^u}{d z} \\ 1 \end{array} \right).
\]
and it follows that
\[
(A.7) \quad \left| \frac{dw^s(z_{-1})}{dz_{-1}} \right| = \left| \frac{\alpha \beta^{-1} b^\alpha \beta^{-1-1} \frac{\partial w^s(z)}{\partial z}}{\alpha \beta^{-1} b^\alpha \beta^{-1-1} \frac{\partial y}{\partial z} - \omega \beta^{-1} \frac{\partial y}{\partial z}} \right|.
\]

Let
\[
M_1 = \max_{(\theta, z) \in \ell_{-1}(q)} \left| \frac{dw^s(z)}{dz} \right|.
\]

We have from (A.7),
\[
\left| \frac{dw^s(z_{-1})}{dz_{-1}} \right| < K_1 k + K_2 b M_1
\]
because $F > K_m$ and
\[
(1 - \omega \beta^{-1} \left[ \frac{\partial F}{\partial \theta} \right]) > 2.
\]

It then follows that $M_1 < K k < k^\frac{1}{2}$.

To prove (b) we write
\[
(A.8) \quad \frac{d^2 w^s(z_{-1})}{dz_{-1}^2} = \frac{d}{dz} \left( \frac{\frac{d w^s(z_{-1})}{d z_{-1}}}{\frac{dz_{-1}}{dz}} \right),
\]
where $\frac{dz_{-1}}{dz}$ is as in (A.7) and
\[
(A.9) \quad \frac{dz_{-1}}{dz} = \frac{1}{\alpha \beta^{-1} b^\alpha \beta^{-1-1} \frac{\partial y}{\partial z} \left( -\alpha \beta^{-1} b^\alpha \beta^{-1-1} \frac{\partial F}{\partial \theta} \frac{d w^s(z)}{dz} + 1 - \omega \beta^{-1} \left[ \frac{\partial F}{\partial \theta} \right] \right)},
\]

Let
\[
M_2 = \max_{(\theta, z) \in \ell_{-1}(q)} \left| \frac{d^2 w^s(z)}{dz^2} \right|.
\]

We have from (A.7), (A.8) and (A.9) that
\[
\left| \frac{d^2 w^s(z_{-1})}{dz_{-1}^2} \right| < K b (K_1 b M_2 + K_2),
\]
from which we obtain $M_2 < K b < b^\frac{1}{2}$.

\[\diamondsuit\]

**Step 3. Non-degenerate, transversal tangency** Let $\ell^u(a)$ be a connected segment of $\ell^u_1(q) \cap V_f$, and $\ell^s(a)$ be the vertical curve $\ell^s_{-1}(q)$ where $\ell^s_1(q)$, $\ell^s_{-1}(q)$ are as in Step 2.

We use $z = w^s(\theta)$ to represent $\ell^u(a)$ and $\theta = w^s(z)$ to represent $\ell^s(a)$. $F_\alpha(\ell^u(a))$ traverses $V$ in horizontal direction as $\mu$ runs through $[\mu_0^{(r)}, \mu_n^{(l)}]$. Consequently there exists $\hat{\mu} \in [\mu_0^{(r)}, \mu_n^{(l)}]$, the corresponding value for $a$ we denote as $\hat{a}$, so that $\ell^s(a)$ and $F_\alpha(\ell^u(a))$ intersect tangentially at a point we denote as $\hat{q} = (\hat{\theta}, \hat{z})$. Let $(\theta_0, z_0) \in \ell^u(a)$ be such that $(\hat{\theta}, \hat{z}) = F_\alpha(\theta_0, z_0)$. Our next claim implies that the tangential intersection of $\ell^s(a)$ and $F_\alpha(\ell^u(a))$ at $\hat{q}$ is not degenerate.

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Claim A.5. For $(\theta, z) \in \ell^u_a$, let $(\theta_1, z_1) = F_\ast(\theta, z)$. Then at $(\theta, z) = (\theta_0, z_0)$, we have
\[
\left| \frac{d^2 \theta_1}{dz_1^2} \right| > 1.
\]

Proof of Claim A.5: From (4.12) we have
\[
(A.10) \quad \frac{d\theta_1}{dz_1} = \frac{(1 - \omega \beta^{-1} \frac{\partial F}{\partial \theta}) + \omega \beta^{-1} \frac{\partial F}{\partial z} \frac{dw^u(\theta)}{d\theta} + \omega \beta^{-1} b \beta^{-1} \frac{\partial F}{\partial \theta} \frac{dw^u(\theta)}{d\theta}}{\alpha \beta b \beta^{-1} \frac{\partial F}{\partial \theta} + \alpha \beta b \beta^{-1} \frac{\partial F}{\partial z} \frac{dw^u(\theta)}{d\theta}}.
\]

At the point of tangential intersection, we have \( \left| \frac{d\theta_1}{dz_1} \right| < k^{\frac{1}{2}} \), which is not possible unless
\[
(A.11) \quad \left| 1 - \omega \beta^{-1} \frac{\partial F}{\partial \theta} \right| < b^{\frac{1}{2}}
\]
from (A.10). This is because \( \frac{dw^u(\theta)}{d\theta} < b^{\frac{1}{2}} \) from Claim A.3(a). The effect of \( b \) in the denominator cannot be possibly balanced if (A.11) is false.

For the estimate on second derivative we start from
\[
(A.12) \quad \frac{d^2 \theta_1}{dz_1^2} = \frac{\frac{d}{d\theta} \left( \frac{d\theta_1}{dz_1} \right)}{\frac{dz_1}{d\theta}}
\]
where \( \frac{dz_1}{d\theta} \) is the denominator in (A.10). To compute \( \frac{d}{d\theta} \left( \frac{d\theta_1}{dz_1} \right) \), we take derivative of the function on the right hand side of (A.10) with respect to \( \theta \). Applying the quotient rule we obtain a fraction, the bottom of which has a factor \( b^2 \). On the top, we have a collection of finitely many terms, each of which is \( < K b^{1+\frac{1}{2}} \) in magnitude except one in the form of
\[
(A.13) \quad \left( \alpha \beta^{-1} b \beta^{-1} \frac{\partial F}{\partial \theta} \right) \frac{d}{d\theta} \left( 1 - \omega \beta^{-1} \frac{\partial F}{\partial \theta} \right).
\]

Remember that we have \( F > K_m \) on \( \hat{V} \), and \( \frac{\partial F}{\partial \theta} > K^{-1} \) from (A.11). We also have
\[
\left| \frac{d}{d\theta} \left( 1 - \omega \beta^{-1} \frac{\partial F}{\partial \theta} \right) \right| \approx \frac{\omega \beta^{-1}}{F^2} (c^2 + c \sin \theta) > 1.
\]

Therefore, (A.13) is the dominating term on top and we obtain
\[
\left| \frac{d^2 \theta_1}{dz_1^2} \right| > K b^{-2}
\]
at \( \tilde{q} \).

To finish our proof of Theorem 3, we also need to prove that, as \( a \) varies, \( \ell^u_a \) and \( F_\ast(\ell^u_a) \) move with different speed at the point of tangency. To make the dependency on parameter \( a \) explicit, we write \( w^u = w^u(\theta, a) \), \( w^s = w^s(z, a) \). Claim A.3 applies to \( w^u(\theta, a) \) and Claim A.4 applies to \( w^s(z, a) \).
Claim A.6. Let \((\theta_1(\theta_0, a), z_1(\theta_0, a)) = \mathcal{F}_a(\theta_0, w^u(\theta_0, a))\). Then at \(a = \hat{a}\) we have

(a) \(|\partial_{\theta_1}(\theta_0, a)| > \frac{2}{3}\); and
(b) \(|\partial_{w^u}(\hat{a}, a)| \leq \frac{1}{2}\).

Recall that \(\tilde{q} = (\hat{\theta}, \hat{z})\) is the point of tangential intersection and \((\theta_0, z_0)\) is such that \(\mathcal{F}_a(\theta_0, z_0) = \tilde{q}\).

Proof of Claim A.6: In this prove we use \(\partial_z, \partial_{\theta}\) and \(\partial_a\) to denote partial derivative with respect to \(z, \theta\) and \(a\) respectively.

To prove (a) we let \(q = (\theta_m, z_m)\) be the saddle fixed point and \(\ell^u(q)\) and \(\ell^u(a) = \mathcal{F}_a(\ell^u(q))\) be as in Claim A.3. For \((\theta, z) \in \ell^u(q)\) and \((\theta_0, z_0) = \mathcal{F}_a(\theta, z)\). We have from (4.3),

\[
\theta_0 = f(\theta, z, a) = \theta + a - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, z, \mu)
\]

(A.14)

\[
z_0 = g(\theta, z, a) = b[\mathbb{F}(\theta, z, \mu)]^{\frac{1}{b}},
\]

and in (A.14), \(z_0 = w^u(\theta_0, a), z = w^u(\theta, a)\) because both \((\theta_0, z_0)\) and \((\theta, z)\) are on \(\ell^u(q)\). We first invert the first equality in (A.14), obtaining \(\theta = \theta(\theta_0, a);\) then we put it into the second equality in (A.14) to obtain \(z_0 = w^u(\theta_0, a)\). To estimate \(\partial_a w^u(\theta_0, a)\), we first let

\[
M_a = \max_{(\theta, z) \in \ell^u(q)} |\partial_a w^u(\theta, a)|
\]

and obtain from the first equality in (A.14),

\[
|\partial_a \theta(\theta_0, a)| = \frac{|\partial_a f + \partial_z f \cdot \partial_a w^u|}{|\partial_{\theta} f + \partial_z f \cdot \partial_{\theta} w^u|} < K_1 + K_2 M_a
\]

because \(|\partial_a f| > 1\) for \((\theta, z) \in \hat{V} \setminus V_f\) and \(|\partial_{\theta} w^u| < b \frac{1}{2}\) from Claim A.3(a). From the second equality in (A.14) we have

\[
|\partial_a w^u(\theta_0, a)| = |\partial g \cdot \partial_a \theta + \partial_z g \cdot (\partial g \cdot \partial_\theta \theta + \partial_\theta g) + \partial a g|
\]

\[
\leq K_3 b(K_1 + K_2 M_a) + K_4 b,
\]

from which it follows that

(A.15)

\[
M_a < b^{-\frac{1}{2}}.
\]

(a) now follows by taking \(\partial_a\) on

\[
\theta_1(\theta_0, a) = \theta_0 + a - \omega \beta^{-1} \ln \mathbb{F}(\theta_0, w^u(\theta_0, a), \mu)
\]

using (A.15).

Proof of (b) is more sophisticated than that of (a). We need to study the stable manifold through the field of most contracted directions, a method originally introduced in [BC] and fully developed in [WY1] and [WY2]. A detailed proof is included in Appendix B.

With Claim A.6 we know that, as \(a\) varies passing \(\hat{a}\), \(\mathcal{F}_a(\ell^u_a)\) crosses \(\ell^u_a\) transversally. This finishes our proof of Theorem 3 owing that of Claim A.6(b).
In order to produce the desired estimates in Claim A.6(b), we need more precise controls on the stable manifold of the saddle fixed point $q_m$. The main idea of our proof, that is, to approximate the stable manifold by using the integral curves of vector field defined by the most contracted directions of the Jacobi matrix, was originated from [BC], and was fully developed in [WY1] and [WY2]. Here we only need a specific version of the contents developed in the beginning part of Section 3 in [WY2].

**B.1. Most contracted directions.** In what follows $u_1 \wedge u_2$ is the wedge product and $\langle u_1, u_2 \rangle$ is the inner product for $u_1, u_2 \in \mathbb{R}^2$.

Let $M$ be a $2 \times 2$ matrix and assume $M \neq cO$ where $O$ is orthogonal and $c \in \mathbb{R}$. Then there is a unit vector $e$, uniquely defined up to a sign, that represents the most contracted direction of $M$, i.e. $|Me| \leq |Mu|$ for all unit vectors $u$. From standard linear algebra, we know $f = e^\perp$ is the most expanded direction, meaning $|Me^\perp| \geq |Mu|$ for all unit vectors $u$, and $Me \perp Me^\perp$. The numbers $|Me|$ and $|Me^\perp|$ are the singular values of $M$.

Let $u \perp v$ be two unit vectors in $\mathbb{R}^2$. The following formulas are results of elementary computations. First, we write down the squares of the singular values of $M$:

\begin{align}
|Me|^2 &= \frac{1}{2}(B - \sqrt{B^2 - 4C}) := \lambda, \quad |Mf|^2 = \frac{1}{2}(B + \sqrt{B^2 - 4C})
\end{align}

where

\begin{align}
B &= |Mu|^2 + |Mv|^2, \quad C = |Mu \wedge Mv|^2.
\end{align}

We write $e = \alpha_0 u + \beta_0 v$, and solve for $|Me| = \sqrt{\lambda}$ subject to $\alpha_0^2 + \beta_0^2 = 1$. There are two solutions (a vector and its negative): either $e = \pm v$, or the solution with a positive $u$-component is given by

\begin{align}
e &= \frac{1}{Z}(\alpha u + \beta v)
\end{align}

with

\begin{align}
\alpha &= |Mv|^2 - \lambda, \quad \beta = -\langle Mv, Mu \rangle
\end{align}

and

\begin{align}
Z &= \sqrt{\alpha^2 + \beta^2}.
\end{align}

From this we deduce that a solution for $f$ is

\begin{align}
f &= \frac{1}{Z}(-\beta u + \alpha v).
\end{align}
B.2. Stability of most contracted directions. In what follows we let \( q_i = \mathcal{F}_a^{i}(q_0), \quad M_i = D\mathcal{F}_a(q_{i-1}) \); 
\[
M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} = \begin{pmatrix} 1 - \omega \beta^{-1}1 \frac{\partial \xi}{\partial \theta} & \omega \beta^{-1} \frac{\partial \xi}{\partial z} \\ \alpha \beta^{-1} \frac{\partial \xi}{\partial \theta} & \alpha \beta^{-1} \frac{\partial \xi}{\partial z} \end{pmatrix}.
\]
We have for \( q_{i-1} \in \hat{V} \setminus V_f \),
\[
(B.7) \quad 2 < |A_i| < K, \quad |B_i| < K, \quad |C_i|, |D_i| < Kb. \tag{B.7}
\]
Let \( M^{(n)} = D\mathcal{F}_a^{n}(q_0) \). \( M^{(n)} = M_n \cdot M_{n-1} \cdots M_1 \). Let the most contracted direction for \( M^{(n)} \) be \( e_n \) and the most expanded direction be \( f_n \). Denote the values of \( \alpha, \beta \) and \( Z \) in (B.4) and (B.5) for \( M^{(n)} \) as \( \alpha_n, \beta_n \) and \( Z_n \). Observe that, assuming \( q_i \in \hat{V} \setminus V_f, \ i < n \),
\[
(B.8) \quad |M^{(n)} f_n| > 1.
\]
We have

**Lemma B.1.** Let \( q_0 \) be such that \( q_0, \ldots, q_n \in \hat{V} \setminus V_f \). Then for all \( 1 \leq i \leq n \),
\[
\begin{align*}
(a) \quad |e_{i+1} - e_i| < (Kb)^i, & \quad |M^{(i)} e_n| < (Kb)^i; \\
(b) \quad |\partial_a(e_{i+1} - e_i)| < (Kb)^i, & \quad |\partial_a M^{(i)} e_n| < (Kb)^i.
\end{align*}
\]

**Proof:** Let \( \Delta_i := |M^{(i)} u \wedge M^{(i)} v| \). We have
\[
(B.9) \quad \Delta_i = |\det(M^{(i)})| < (Kb)^i. \tag{B.9}
\]
It then follows from \( |M^{(i)} e_i| |M^{(i)} f_i| = \Delta_i \) and (B.8),
\[
(B.10) \quad |M^{(i)} e_i| < (Kb)^i. \tag{B.10}
\]
We substitute \( u = e_i, v = f_i \) and \( M = M^{(i+1)} \) into (B.3) for \( e_{i+1} \) and (B.5) for \( f_{i+1} \). By using (B.1) for \( M^{(i+1)} f_{i+1} \), we have
\[
(B.11) \quad |M^{(i+1)} f_i| = |M^{(i+1)} f_{i+1}| \pm O((Kb)^i).
\]
from (B.2), (B.9) and (B.10). We also have
\[
(B.12) \quad Z_{i+1} \approx |\alpha_{i+1}| \approx |M^{(i+1)} f_i|^2.
\]
We now prove Lemma B.1(a). Using \( u = e_i \) and \( v = f_i \), we have, from (B.3),
\[
(B.13) \quad e_{i+1} - e_i = \frac{1}{Z_{i+1}} \left( \frac{-\beta_{i+1}^2}{\alpha_{i+1} + Z_{i+1}} e_i + \beta_{i+1} f_i \right).
\]
To estimate \( |e_{i+1} - e_i| \), we need to obtain a suitable upper bound for \( |\beta_{i+1}| \) and lower bounds for \( |\alpha_{i+1}| \) and \( Z_{i+1} \). We have from (B.4), (B.10) and (B.12),
\[
(B.14) \quad |\beta_{i+1}| \leq |M^{(i+1)} e_i| |M^{(i+1)} f_i| < Kb^i \sqrt{Z_{i+1}}
\]
and \( |\alpha_{i+1}| \approx Z_{i+1} \). These estimates together with \( Z_{i+1} > 1 \) tell us
\[
|e_{i+1} - e_i| \approx \frac{|\beta_{i+1}|}{Z_{i+1}} < (Kb)^i.
\]
The second assertion follows easily from

\[ |M^{(i)}e_n| \leq |M^{(i)}(e_n - e_{n-1})| + \cdots + |M^{(i)}(e_{i+1} - e_i)| + |M^{(i)}e_i| < (Kb)^i. \]

This finished our proof for Lemma B.1(a).

To prove Lemma B.1(b) we start with

**Sublemma B.1.** \(|\partial_a e_1|, |\partial_a f_1| < K_1\) for some \(K_1\).

**Proof:** Let \(u = (0,1)^T\), \(v = (1,0)^T\) and use (B.3) for \(e_1\) and (B.6) for \(f_1\). We have \(Z_1 > \alpha \geq |M_1v|^2 - Kb > 1\). Differentiating (B.3) and (B.6) gives the desired result. \(\Box\)

In the rest of this proof, \(\partial = \partial_a\). Our plan of proof for Lemma B.1(b) is as follows: For \(k = 1, 2, \ldots\), we assume for all \(i \leq k\)

(*) \(|\partial e_i|, |\partial f_i| < 2K_1\) where \(K_1\) is as in Sublemma B.1,

and prove for all \(i \leq k\):

(A) \(|\partial(M^{(i)}f_i)| < K^i\), \(|\partial(M^{(i)}e_i)| < (Kb)^i\);

(B) \(|\partial(e_{i+1} - e_i)|, |\partial(f_{i+1} - f_i)| < (Kb)^i\).

Observe that for \(i = 1\), (*) is given by Sublemma B.1. It is easy to see that (B) above implies (*) with \(i = k + 1\), namely \(|\partial f_{k+1}| \leq |\partial(f_{k+1} - f_k)| + \cdots + |\partial(f_2 - f_1)| + |\partial f_1|\). From (B), we have \(|\partial(f_{i+1} - f_i)| < (Kb)^i\), and from Sublemma B.1, we have \(|\partial f_1| < K_1\). Hence \(|\partial f_{k+1}| < Kb + K_1\), which, for \(b\) sufficiently small, is < \(2K_1\). The computation for \(e_{k+1}\) is identical.

**Proof that (*) \(\Rightarrow\) (A):** First we prove the estimate for \(\partial(M^{(i)}f_i)\). Writing

\[
\partial(M^{(i)}f_i) = \sum_{j=1}^i M_i \cdots (\partial M_j) \cdots M_1 f_i + M^{(i)} \partial f_i,
\]

we obtain easily

\[
|\partial(M^{(i)}f_i)| \leq \sum_{j=1}^i |M_i \cdots (\partial M_j) \cdots M_1 f_i| + \|M^{(i)}\| |\partial f_i| \leq iK^i + K^i(2K_1).
\]

This estimate is used to estimate \(\partial(M^{(i)}e_i)\). Write \(\partial(M^{(i)}e_i) = (I) + (II)\) where (I) is its component in the direction of \(M^{(i)}f_i\) and (II) is its component orthogonal to \(M^{(i)}f_i\). Recall that \(\partial\langle M^{(i)}e_i, M^{(i)}f_i \rangle = 0\). We have

\[
|\langle I \rangle| = \left| \langle \partial(M^{(i)}e_i), \frac{M^{(i)}f_i}{|M^{(i)}f_i|} \rangle \right| = \frac{1}{|M^{(i)}f_i|} |\langle M^{(i)}e_i, \partial(M^{(i)}f_i) \rangle| < (Kb)^iK^i;
\]

\[
|\langle II \rangle| \|M^{(i)}f_i\| = |\partial(M^{(i)}e_i) \wedge M^{(i)}f_i| \leq |\partial(M^{(i)}e_i \wedge M^{(i)}f_i)| + |M^{(i)}e_i \wedge \partial(M^{(i)}f_i)|.
\]

The first term in the last line is < \((Kb)^i\), noting that we have established \(|\partial e_i|, |\partial f_i| < 2K_1\); the second term is < \((Kb)^i \cdot K^i\). This completes the proof of (A). \(\Box\)
To prove (B), we first compute some quantities associated with the next iterate. Substitute \( u = e_i, v = f_i, M = M^{(i+1)} \) in (B.1)-(B.6). The following is a straightforward computation.

**Sublemma B.2.** Assume (*) and (A). Then for all \( i \leq k \):

(a) \( |\partial \lambda_{i+1}| < (Kb)^{2(i+1)} \);
(b) \( |\partial \beta_{i+1}| < (Kb)^{i}\sqrt{Z_{i+1}} \);
(c) \( |\partial \alpha_{i+1}|, |\partial Z_{i+1}| < K^i\sqrt{Z_{i+1}} \).

**Proof that (A) \( \Rightarrow \) (B):** We work with \( e_i \); the computation for \( f_i \) is similar. From (23) we have

\[
\partial (e_{i+1} - e_i) = (III) + (IV) + (V)
\]

where

\[
| (III) | = \left| \frac{1}{Z_{i+1}} (e_{i+1} - e_i) \partial Z_{i+1} \right| < \frac{K^i\sqrt{Z_{i+1}}}{Z_{i+1}} \cdot (Kb)^i < (Kb)^i;
\]

\[
| (IV) | = \left| \frac{1}{Z_{i+1}} \partial (\beta_{i+1} f_i) \right| < \frac{1}{Z_{i+1}} (|\partial \beta_{i+1}| + |\beta_{i+1}| |\partial f_i|) < (Kb)^i;
\]

\[
| (V) | = \left| \frac{1}{Z_{i+1}} \partial \left( \frac{\beta_{i+1}^2}{\alpha_{i+1} + Z_{i+1}} e_i \right) \right| << (Kb)^i.
\]

To estimate (III), we have used Sublemma B.2(c) and part (a) of Lemma B.1. To estimate (IV), we have used Sublemma B.2(b), (*) and \( |\beta_{i+1}| < \frac{(Kb)^i}{K} \). The estimate for (V) is easy.

This completes the proof of Lemma B.1(b).

We also need to control the speed of change for the most contracted directions in \( \hat{V} \setminus V_f \). Let \( q_0(s,a) \) be a curve in \( \hat{V} \setminus V_f \) parameterized by a parameter \( s \) and assume that

\[
\| q_0(s,a) \|_{C^2} < K.
\]

Let \( M^{(n)}(s) = DF^n_a(q_0(s,a)) \), and \( e_n(s) \) be the most contracted direction for \( M^{(n)}(s) \).

**Lemma B.2.** Let \( q_0 \) be such that \( q_0, \ldots, q_n \in \hat{V} \setminus V_f \). Then for all \( 1 \leq i \leq n \),

(a) \( |\partial s(e_{i+1}(s) - e_i(s))| < (Kb)^i \), \( |\partial s M^{(i)}(s) e_n(s)| < (Kb)^i \); and
(b) \( |\partial s \partial \alpha(e_{i+1}(s) - e_i(s))| < (Kb)^i \), \( |\partial s \partial \alpha M^{(i)}(s) e_n(s)| < (Kb)^i \).

**Proof:** The proof for Lemma B.2(a) is identical to that of Lemma B.1(b). It suffices to regard all \( \partial \) as \( \partial s \) instead of \( \partial a \). The estimate for the second derivatives is proved by a similar argument. Here we skip the details.

**B.3. Temporary stable curves and the stable manifold.** In the rest of this proof we let \( \eta = b^\frac{1}{4\kappa} \) and denote \( A_\eta = \{ (\theta, z) \in A : |z| < \eta \} \). We view \( e_n \) as a vector field, defined where it makes sense, and let \( \gamma_n(s) \) be the integral curve to \( e_n \) with \( \gamma_n(0) = q_0 \).

**Lemma B.3.** Let \( q_0 = q_m \) be the saddle fixed point of Theorem 3, and \( \gamma_n(s) \) be the integral curve to \( e_n \) satisfying \( \gamma_n(0) = q_0 \) in \( A_\eta \). Then, for all \( n > 0 \),

(a) \( |F^s_a(q) - F^s_a(q_0)| < (Kb)^i |s| \) for all \( q = \gamma_n(s) \) and all \( i \leq n \);
(b) \( \gamma_n(s) \) is a fully extended vertical curve in \( (\hat{V} \setminus V_f) \cap A_\eta \).
(c) \(|\gamma_{n+1}(s) - \gamma_n(s)|, |\nabla_a \gamma_{n+1}(s) - \nabla_a \gamma_n(s)| < b^{\frac{1}{4}}\).

**Proof:** Lemma B.3(a) follows directly from \(|M^{(i)} e_i| < (Kb)^i\) for all \(i \leq n\) (Lemma B.1(a)). Denote \(\mathcal{F} = \mathcal{F}_a\). Let \(B_0\) be the ball of radius \(2\eta\) centered at \(q_0\). \(e_1\) is well defined on \(B_0\), and substituting \(u = (0, 1)^T, v = (1, 0)^T\) into (B.3) we obtain \(s(e_1) > Kb^{-1}\). Let \(\gamma_1 = \gamma_1(s)\) be the integral curve to \(e_1\) defined for \(s \in (-2\eta, 2\eta)\) with \(\gamma_1(0) = q_0\).

To construct \(\gamma_2\), let \(B_1\) be the \(\eta^2\)-neighborhood of \(\gamma_1\). For \(\xi \in B_1\), let \(\xi'\) be a point in \(\gamma_1\) with \(|\xi - \xi'| < \eta^2\). Then \(|\mathcal{F}(\xi) - \mathcal{F}(q_0)| \leq |\mathcal{F}(\xi) - \mathcal{F}(\xi')| + |\mathcal{F}(\xi') - \mathcal{F}(q_0)| \leq K\eta^2 + Kb\eta < K\eta^2\). This ensures that \(e_2\) is defined on all of \(B_1\). Let \(\gamma_2\) be the integral curve to \(e_2\) with \(\gamma_2(0) = q_0\). We verify that \(\gamma_2\) is defined on \((-2\eta, 2\eta)\) and runs alongside \(\gamma_1\). More precisely, let \(t \in [0, 1]\) and

\[q(t, s) = \gamma_1(s) + t(\gamma_2(s) - \gamma_1(s)).\]

We have

\[
\left| \frac{d}{ds} (\gamma_2(s) - \gamma_1(s)) \right| \leq |e_2(\gamma_2(s)) - e_1(\gamma_2(s))| + |e_1(\gamma_2(s)) - e_1(\gamma_1(s))| \\
\leq |e_2 - e_1| + |\nabla e_1| |\gamma_2(s) - \gamma_1(s)| \\
\leq Kb + K|\gamma_2(s) - \gamma_1(s)|.
\]

Here we use \(|\nabla e_1| < K\). By Gronwall’s inequality, \(|\gamma_2(s) - \gamma_1(s)| \leq Kb|s|e^{K|s|}\), which is \(< \eta^2\) for \(|s| < 2\eta\). This ensures that \(\gamma_2\) remains in \(B_1\) and hence is well defined for all \(s \in (-2\eta, 2\eta)\).

In general, we inductively construct \(\gamma_i\) by letting \(B_{i-1}\) be the \(\eta^i\)-neighborhood of \(\gamma_{i-1}\) in \(S\). Then for all \(\xi \in B_{i-1}\), \(|\mathcal{F}^j(\xi) - \mathcal{F}^j(q_0)| = |\mathcal{F}^j(\xi) - q_0| < K\eta^j\) for \(k < i\). Thus \(e_i\) is well defined. Integrating and arguing as above, we obtain \(\gamma_i\) with \(|\gamma_i(s) - \gamma_{i-1}(s)| < (Kb)^{i-1}|s| < \eta^i\) for all \(s\) with \(|s| < 2\eta\).

To estimate the derivative with respect to \(a\), we let

\[q(t, s) = \gamma_n(s) + t(\gamma_{n+1}(s) - \gamma_n(s)).\]

We have

\[
\left| \frac{d}{ds} \nabla_a (\gamma_{n+1}(s) - \gamma_n(s)) \right| \leq |\nabla_a (e_{n+1}(\gamma_{n+1}(s)) - e_n(\gamma_{n+1}(s)))| \\
+ |\nabla_a (e_n(\gamma_{n+1}(s)) - e_n(\gamma_n(s)))| \\
\leq |e_{n+1} - e_n| + |\nabla_a e_n||\gamma_{n+1}(s) - \gamma_n(s)| \leq K\eta^n.
\]

From this the second item of Lemma B.3(b) follows. \(\square\)

**B.4. The proof of Claim A.6(b).** We are ready to prove Claim A.6(b). First we note that at the point of tangency, \(|z| < b^{\frac{1}{4}}\) so it suffices for us to consider \(A_\eta\) in the place of \(A\) with \(\eta = b^{\frac{1}{4}}\).

From Lemma B.3(c), we know that \(\gamma_n \to \gamma_\infty\) uniformly as \(n \to \infty\), and from Lemma B.3(a) we know that \(\gamma_\infty\) is the stable manifold of \(q_0 = q_m\), which we write as (B.15)

\[\theta = \theta_\infty(s, a), \quad z = z_\infty(s, a).\]
We also have
\[ (B.16) \quad |\partial_a \theta_\infty(s, \mathbf{a}) - \partial_a \theta_1(s, \mathbf{a})|, \quad |\partial_a \theta_\infty(s, \mathbf{a}) - \partial_a \theta_1(s, \mathbf{a})| < 2b^{\frac{1}{10}} \]
from Lemma B.2(c).

Write \( \gamma_\infty \) using \( \theta = w^s(z, \mathbf{a}) \). We first solve \( s = s_\infty(z, \mathbf{a}) \) from the second item in (B.15), then substitute to the first item in (B.15) to obtain
\[ w^s(z, \mathbf{a}) = \theta_\infty(s_\infty(z, \mathbf{a}), \mathbf{a}). \]
Differentiating on both side, we obtain
\[ \partial_a w^s(z, \mathbf{a}) = \partial_s \theta_\infty(s_\infty(z, \mathbf{a}), \mathbf{a}) \partial_s s_\infty(z, \mathbf{a}) + \partial_a \theta_\infty(s, \mathbf{a}) \]
\[ = -\partial_s \theta_\infty(s, \mathbf{a}) \frac{\partial_a z_\infty(s, \mathbf{a})}{\partial_s z_\infty(s, \mathbf{a})} + \partial_a \theta_\infty(s, \mathbf{a}) \]
\[ = \partial_a w^s_1(z, \mathbf{a}) + O(b^{\frac{11}{10}}) \]
where \( \theta = w^s_1(z, \mathbf{a}) \) is the equation for \( \gamma_1 \). To obtain the last estimate we use (B.16) and the fact that \( \partial_s z_1(s, \mathbf{a}) > \frac{1}{2} \).

To prove Claim A.6(b), it now suffices for us to confirm that
\[ \partial_a w^s_1(z, \mathbf{a}) < \frac{1}{50}. \]
To verify this estimate we observe
\[ (B.17) \quad |\partial_a \frac{d}{dz} w^s_1(z, \mathbf{a})| = |\partial_a s^{-1}(e_1)| < Kb, \]
where \( s(e_1) \) is the slope for \( e_1 \). This follows from a direct computation using (B.3). From (B.17),
\[ |\partial_a w^s_1(z, \mathbf{a})| < |\partial_a w^s_1(z_m, \mathbf{a})| + Kb \eta \]
where \( \partial_a w^s_1(z_m, \mathbf{a}) \) is the value of \( \partial_a w^s_1(z, \mathbf{a}) \) at \( q_0 = q_m \). We now use Claim A.2 for \( \partial_a w^s_1(z_m, \mathbf{a}) \). □

References


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