Variational Construction of Heteroclinic Orbits For The Monotone Twist Maps

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Abstract

In this paper, we construct heteroclinic orbits among locally minimal fixed points for the monotone twist maps. Our construction uses the variational method developed by John Mather and others.

1 Preparation

1.1 The propositions

Take a function $h(x,x') : R^2 \mapsto R$. We assume that h is C^2, $h(x+1,x'+1) = h(x,x')$, $\partial_{12}h(x,x') < -\delta$. Where δ is a positive constant. The function h generates a monotone twist diffeomorphism of the infinite cylinder $(R/Z) \times R$.

Let $f(x) = h(x,x)$. $f(x)$ is a C^2 periodic function with period one. The critical points of the function $f(x)$ corresponds to the fixed points of the monotone twist map generated by $h(x,x')$. Denote the set of all the locally minimum critical points of $f(x)$ as S. We will use variational method to construct heteroclinic connections among the elements of S. In the writing of this paper, we assume our reader's knowledge on the basic theory of Aubry-Mather set presented in Bangert [2].

Take $c_1, c_2 \in S$, $c_1 < c_2$. We have

Definition 1 *$c_1, c_2 \in S$ are an adjacent pair if there is a c, $c_1 < c < c_2$, such that $f(x) > f(c_1)$ if $x \in (c_1, c]$, and $f(x) > f(c_2)$ if $x \in [c, c_2)$. We call c a separating point for this pair.*

For a given adjacent pair in S, let

$$K = max\{\mid \partial_{12}h(x,x') \mid : c_1 \leq x < x' \leq c_2\}$$

Definition 2 *A separating point c for an adjacent pair $c_1, c_2 \in S$ is a separating barrier if*

$$f'(c) > 2K(c_2 - c).$$

We will prove the following two propositions in this paper:

Proposition 1 *For an adjacent pair c_1, c_2, if $f(c_1) = f(c_2)$, then there are monotone heteroclinic connections from c_1 to c_2. We also have such connections from c_2 to c_1.*

Proposition 2 *Take any adjacent pair $c_1, c_2 \in S$ and assume $f(c_1) > f(c_2)$. If we have a separating barrier for (c_1, c_2), then there are monotone heteroclinic connections from c_1 to c_2.*

It is well-known, by the Aubry-Mather theory, that there exists heteroclinic connections between adjacent global minimal fixed points [2]. Our first proposition is a natural generalization of that statement. In the second proposition, the existence of a separating barrier is necessary for our proof work. We do not know if the heteroclinic connection still exist in general without this restriction.

To see that there are monotone twist maps satisfying the condition in the second proposition, we take a C^2 function $f(x)$ and let

$$h(x,x') = f(x) + \delta(x-x')^2.$$

Of course we have to assume $f(x+1) = f(x)$. For $h(x,x')$, the constant $K = 2\delta$. It is now trivial to choose $f(x)$ such that local minimals and separating barriers exist.

1.2 Mather's energy formula

For a given configuration $x = \{x_i\}_{i=1}^{n}$ and the given generating function $h(x, x')$, let

$$H(x) = \sum_{i=1}^{n-1} h(x_i, x_{i+1}).$$

$H(x)$ is the energy of the given configuration.

In his study of monotone twist maps [3, 4, 5, 6, 7], Mather introduced the following formula for the energy function $H(x)$;

$$H(x) = \sum_{i=1}^{n-1} h(x_i, x_i) - \int_{x_1}^{x_n} \partial_2 h(y, y) dy + \sum_{i=1}^{n-1} \mu(\Delta_i)$$

where μ is the positive measure defined by

$$\mu([\xi, x] \times [\xi', x']) = -\int_{\xi}^{x} \int_{\xi'}^{x'} \partial_{12} h(x, x')$$

and Δ_i is the planar region on the (x, x') plane defined by

$$\Delta_i = \{(x, x') : \quad x_i \le x \le x' \le x_{i+1}\} \quad if \; x_i \le x_{i+1}$$

or

$$\Delta_i = \{(x, x') : \quad x_i \ge x \ge x' \ge x_{i+1}\} \quad if \; x_i \ge x_{i+1}$$

This energy decomposition played a central role in Mather's study on monotone twist maps. For the detailed derivation of this formula, refer to Mather's original paper [6].

1.3 More on the energy function

The goal of this paper, (as well as all the studies based on the variational method, roughly speaking) is to show the existence of configurations which minimizes (globally or locally) the energy function. So techniquely, it is crucial to see how the energy changes if one changes a given configuration slightly, say, by dropping or adding one point, or moving one of its points a little bit. Fortunately, we can understand the change according to Mather's energy formula.

1.3.1 Adding a point

To simplify the picture, first take a configuration with two points x_1, x_2, $x_1 < x_2$. Let x be another point, $x_1 \le x \le x_2$. Adding x into the first configuration gives a new configuration x_1, x, x_2. By definition,

$$H(x_1, x_2) = h(x_1, x_2); \qquad H(x_1, x, x_2) = h(x_1, x) + h(x, x_2)$$

The energy difference of these two configurations, by Mather's formula, can be written as

$$H(x_1, x_2) - H(x_1, x, x_2) = \mu([x_1, x] \times [x, x_2]) - h(x, x)$$

This formula also gives the change of energy if one drops x from the configuration (x_1, x, x_2).

1.3.2 Moving the middle point

Again take a configuration x_1, x, x_2. we consider the change of energy when x is replaced by $x+\varepsilon$. Where ε is small. By Mather's formula

$$H(x_1, x+\varepsilon, x_2) - H(x_1, x, x_2) = f(x+\varepsilon, x+\varepsilon) - f(x, x)$$

$$+\mu([x_1, x] \times [x, x+\varepsilon]) - \mu([x, x+\varepsilon] \times [x+\varepsilon, x_2]).$$

We will repeatedly use these two estimations in the proof of both of our propositions.

2 The Proof of Proposition 1

2.1 Outline

Without loss of generality, we assume $f(c_1) = f(c_2) = 0$. The idea of the construction of the heteroclinc orbits is the following.

- For any two points a, b, $c_1 < a < b < c_2$. construct the *local minimal orbit segment* between a and b.
- Take a monotonically decreasing sequence $a_n \to c_1$ and a monotonically increasing sequence $b_n \to c_2$. For every pair (a_n, b_n), find the the minimal orbit segment $x_{a_n b_n}$.
- To show that $x_{a_n b_n}$ has a convergent subsequence, and the limit indeed gives a heteroclinic connection between c_1 and c_2.

Therefore, the detailed proof consists

- Define the *local minimal orbit segment* and show its existence. (Lemma 1)
- Show that, for all the positive integer $n > 1$, there always be a point of $x_{a_n b_n}$ in the middle part of the interval (c_1, c_2). This will allow us to start the construction of the convergent subsequences. (lemma 2)
- Denote the number of points in $x_{a_n b_n}$ as $N(n)$. We show that $N(n) \to \infty$. This is necessary because we need infinitely many points for the heteroclinic connection. We also have to show, in fact, that there are many points around c_1, and many points around c_2. (lemma 3, 4)
- The technical estimation in Lemma 3 also guarantees that the limit sequence we construct stays inside of (c_1, c_2) and no limit value touched the end points of the interval.

2.2 The proof

2.2.1 The minimal connection

For $c_1 < c_2$ in S, we have $f(c_1) = f(c_2) = 0$ and $f(x) > 0$ for all $x \in (c_1, c_2)$. Take a, b, $c_1 < a < b < c_2$. Define

$$X^n_{ab} = \{x = \{\xi_i\}^n_{i=1}; \quad a = x_1 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b\}$$

and let

$$H^n(x) = \sum_{i=1}^{n-1} h(x_i, x_{i+1})$$

for $x \in X^n_{ab}$.

Since X^n_{ab} is compact and $H^n(x)$ is a continues functional on X^n_{ab}, there exists a configuration $x \in X^n_{ab}$ that minimizes $H^n(x)$. Denote this sequence as x^n.

For any given $n > 1$, take the minimal energy sequence x^n. First observe that

$$\lim_{n\to\infty} H^n(x^n) = +\infty.$$

This is because the second and the third terms in Mather's energy formula are both bounded while the first summation goes to positive infinity as n goes to infinity. So we have an index I such that

$$H^I(x^I) \leq H^n(x^n)$$

for all $n > 1$. We call the configuration x^I the *minimal orbit connecting a and b*. Denote this sequence as x_{ab}, and $I = I(a, b)$.

Lemma 1 *$x_{ab} = \{\xi_i\}^I_{i=1}$ is a strictly monotone sequence, i.e.*

$$a = \xi_1 < \xi_2 < \cdots < \xi_{I-1} < \xi_I = b.$$

Proof: The sequence is monotone according to the definition of X^I_{ab}. Assume that it is not strictly increasing, we will get a new sequence of length $I - 1$ by dropping one duplicated element. Denote this sequence as w. We see that $H^{I-1}(w) < H(x^I)$. Therefore

$$H^{I-1}(x^{I-1}) \leq H^{I-1}(w) < H(x^I).$$

This inequality contradicts to the definition of x^I.

Now take a sufficiently small constant $\varepsilon > 0$. From now on, we will always take $a = c_1 + \varepsilon, b = c_2 - \varepsilon$. Accordingly, we will denote the minimal orbit connecting a and b so defined as $x(\varepsilon)$ and $I(a, b)$ as $I(\varepsilon)$.

2.2.2 The point in the middle part

Since $f(x)$ is a C^2 function and c_1 is a local minimal, there are constants $K_1 > 0$ and $\delta_1 > 0$ such that

$$f(x) \leq K_1(x - c_1)^2$$

for $c_1 < x < c_1 + \delta_1$. In fact, we can take $K_1 = 1$ if $f''(c_1) = 0$ or $K_1 = \frac{3}{2} f''(c_1)$ if $f''(c_1) > 0$. Also, we will take δ_1 sufficiently small such that

$$\delta_1 < min\{\frac{c_2 - c_1}{4}; \quad \frac{\delta(c_2 - c_1)}{4K_1}\}$$

where δ is the twist constant. Further let $\delta_2 = \frac{\delta_1}{2}$. Note that δ_2 only depends on the function $h(x, x')$. We have

Lemma 2 *For any* $0 < \varepsilon < \delta_2$, $x(\varepsilon)$ *has a point in the interval* $[c_1 + \delta_2, c_2 - \delta_2]$.

Proof: Let $x(\varepsilon) = \{\xi_i\}_{i=1}^{I(\varepsilon)}$. If $x(\varepsilon)$ has no element in the interval $[c_1 + \delta_2, c_2 - \delta_2]$, there would be an index i_0 such that

$$c_1 + \varepsilon = \xi_1 < \cdots < \xi_{i_0} < c_1 + \delta_2 < c_2 - \delta_2 < \xi_{i_0+1} < \cdots < \xi_{I(\varepsilon)} = c_2 - \varepsilon.$$

Let

$$\hat{\xi} = \xi_{i_0} + \frac{2K_1\delta_1^2}{\delta(c_2 - c_1)}$$

we see that

$$\xi_{i_0} < \hat{\xi} < \xi_{i_0} + \frac{2K_1\delta_1}{\delta(c_2 - c_1)}\delta_1 < \xi_{i_0} + \frac{\delta_1}{2} < c_1 + \delta_1.$$

Now form a new sequence by adding the point $\hat{\xi}$ in the middle of ξ_{i_0} and ξ_{i_0+1}. Denote the new sequence as $\hat{x}$. According to Mather's energy formula

$$H(\hat{x}) - H(x(\varepsilon)) = f(\hat{\xi}) - \int\int_R | \partial_{12} h |$$

where $R = [\xi_{i_0}, \hat{\xi}] \times [\hat{\xi}, \xi_{i_0+1}]$. We see

$$H(\hat{x}) - H(x(\varepsilon)) < K_1(\hat{\xi} - c_1)^2 - \delta(\hat{\xi} - \xi_{i_0})(\xi_{i_0+1} - \hat{\xi})$$

$$< K_1\delta_1^2 - \delta\frac{c_2 - c_1}{2}\frac{2K_1\delta_1^2}{\delta(c_2 - c_1)} = 0$$

Therefore, we conclude $H(\hat{x}) < H(x(\varepsilon))$. This is impossible since we would have

$$H^{I(\varepsilon)+1}(x^{I(\varepsilon)+1}) \leq H(\hat{x}) < H(x(\varepsilon)).$$

This inequality contradicts the definition of $x(\varepsilon)$.

2.2.3 More on energy estimation

We will need the following technical lemma to estimate the size of $I(\varepsilon)$.

Lemma 3 *Let* $K_2 = \frac{K_1}{\delta}$ *and*

$$\hat{K} = 2(\sqrt{K_2(1 + K_2)} + K_2)$$

If $\xi_1 < \xi_2 < c_1 + \delta_1$ *satisfies*

$$\xi_2 - \xi_1 > \hat{K}(\xi_1 - c)$$

then there is an ε, $0 < \varepsilon < \xi_2 - \xi_1$, *such that*

$$H(\xi_1, \xi_1 + \varepsilon, \xi_2) < H(\xi_1, \xi_2)$$

Proof: According to Mather's energy formula

$$E = H(\xi_1, \xi_1 + \varepsilon, \xi_2) - H(\xi_1, \xi_2) = f(\xi_1 + \varepsilon) - \int\int_R \mid \partial_{12} h \mid$$

where $R = [\xi_1 + \varepsilon, \xi_2] \times [\xi_1, \xi_1 + \varepsilon]$.

So we have

$$E < K_1(\xi_1 + \varepsilon - c_1)^2 - \delta(\xi_2 - \xi_1 - \varepsilon)\varepsilon.$$

$$\frac{E}{\delta} < K_2(\xi_1 + \varepsilon - c_1)^2 - (\xi_2 - \xi_1 - \varepsilon)\varepsilon.$$

To prove the lemma, it is enough to show that there is an ε, $0 < \varepsilon < \xi_2 - \xi_1$, such that

$$K_2(\xi_1 + \varepsilon - c_1)^2 - (\xi_2 - \xi_1 - \varepsilon)\varepsilon = 0$$

Rewrite the equation as

$$(1 + K_2)\varepsilon^2 + (2K_2(\xi_1 - c_1) - (\xi_2 - \xi_1))\varepsilon + K_2(\xi_1 - c_1)^2 = 0$$

This equation has a real solution for ε if and only if

$$(2K_2(\xi_1 - c_1) - (\xi_2 - \xi_1))^2 - 4(1 + K_2)K_2(\xi_1 - c_1)^2 \geq 0$$

So we either need

$$(2K_2(\xi_1 - c_1) - (\xi_2 - \xi_1)) \geq 2\sqrt{(1 + K_2)K_2}(\xi_1 - c_1)$$

or

$$(2K_2(\xi_1 - c_1) - (\xi_2 - \xi_1)) \leq -2\sqrt{(1 + K_2)K_2}(\xi_1 - c_1).$$

The second inequality is

$$(\xi_2 - \xi_1) \geq 2(\sqrt{K_2(1 + K_2)} + K_2)(\xi_1 - c_1)$$

which has be assumed in the statement of this lemma.

Also observe that

$$\xi_2 - \xi_1 \geq \xi_2 - \xi_1 - 2K_2(\xi_1 - c_1) \geq (\hat{K} - 2K_2)(\xi_1 - c_1) > 0$$

By the quadratic formulas we see one of the solutions for ε satisfies $0 < \varepsilon < \xi_2 - \xi_1$.

2.2.4 The size of $I(\varepsilon)$

Now take a decreasing sequence of positive numbers $\varepsilon_n \to 0$. Let $\varepsilon_1 < \delta_2$. For each given ε_i we have a minimal orbit segment $x(\varepsilon_i)$ connecting $c_1 + \varepsilon_i$ and $c_2 - \varepsilon_i$. According to Lemma 2, there is a point of $x(\varepsilon_i)$ locates in the interval $[c_1 + \delta_2, c_2 - \delta_2]$.

Re-indexing the sequence $x(\varepsilon_i)$ by giving the index zero to one of its points inside of the interval $[c_1 + \delta_2, c_2 - \delta_2]$. Denote the newly index $x(\varepsilon_i)$ as

$$x(\varepsilon_i) = \{\xi_j(\varepsilon_i)\}_{j=n_1(\varepsilon_i)}^{n_2(\varepsilon_i)}$$

We have

Lemma 4

$$limsup_{i\to\infty} n_1(\varepsilon_i) = -\infty \quad limsup_{i\to\infty} n_2(\varepsilon_i) = +\infty$$

Proof: Let

$$\delta_3 = min\{\frac{\delta_2}{2}, \quad \frac{\delta\delta_2}{2K_1 + \delta}\}.$$

and $\delta_4 = \frac{\delta_3}{2}$. First we show that if $\varepsilon < \delta_4$, then $x(\varepsilon)$ must have a point in the interval $[c_1 + \delta_4, c_1 + \delta_2]$. The proof of this statement follows the same argument in the proof of Lemma 2. If no such a point, we have index i, such that

$$\xi_i < c_1 + \delta_4 < c_1 + \delta_2 < \xi_{i+1}.$$

Now take

$$\hat{\xi} = \xi_i + \frac{K_1\delta_3^2}{\delta(\delta_2 - \delta_3)}$$

we see

$$\hat{\xi} < c_1 + \delta_4 + \frac{K_1\delta_3}{\delta(\delta_2 - \delta_3)}\delta_3 < c_1 + \delta_4 + \frac{\delta_3}{2} \le c_1 + \delta_3$$

Therefore

$$H(\xi_i, \hat{\xi}, \xi_{i+1}) - H(\xi_i, \xi_{i+1}) < f(\hat{\xi}) - \delta(\xi_{i+1} - \hat{\xi})(\hat{\xi} - \xi_i)$$
$$< K_1\delta_3^2 - \delta(\delta_2 - \delta_3)(\hat{\xi} - \xi_i) = 0$$

This gives a contradiction.

Now, the proof of the Lemma. Suppose the first limit is not true, there would be a N_0, such that $| n_1(\varepsilon_i) | < N_0$ for all $i > 0$. Since $\varepsilon_i \to 0$, there is an i_0, such that

$$\varepsilon_{i_0} < \frac{\delta_4}{2(\hat{K}^{N_0} + 2\hat{K}^{N_0-1} + \cdots + j\hat{K}^{N_0-j+1} + \cdots + N_0\hat{K})}.$$

Let $x(\varepsilon_{i_0}) = \{\xi_j\}_{j=n_1}^{n_2}$. By the definition of $x(\varepsilon_{i_0})$, one can not reduce its energy by adding points into this sequence. Note that, by the proceeding argument, we always have a point of $x(\varepsilon_{i_0})$ in the interval $[c_1 + \delta_4, c_1 + \delta_2]$. So, both ξ_{n_1+1} and ξ_{n_1} are smaller than $c_1 + \delta_1$. By Lemma 3 we must have

$$\xi_{n_1+1} - \xi_{n_1} < \hat{K}(\xi_{n_1} - c_1) < \hat{K}\varepsilon_{i_0} < \frac{\delta_4}{2}$$

We conclude that $\xi_{n_1+1} < c_1 + \delta_4$. In turn, we have $\xi_{n_1+2} < \delta_3$. Applying Lemma 3 again we get

$$\xi_{n_1+2} - \xi_{n_1} = \xi_{n_1+2} - \xi_{n_1+1} + \xi_{n_1+1} - \xi_{n_1}$$

$$< \hat{K}(\xi_{n_1+1} - c_1) + \hat{K}(\xi_{n_1} - c_1) < (\hat{K}^2 + 2\hat{K})(\xi_{n_1} - c_1) < \frac{\delta_4}{2}$$

inductively, we will finally get that

$$\xi_0 - \xi_{n_1} < (\hat{K}^{N_0} + 2\hat{K}^{N_0-1} + \cdots + j\hat{K}^{N_0-j+1} + \cdots + N_0\hat{K})(\xi_{n_1} - c_1) < \frac{\delta_4}{2}$$

This is a contradiction since by definition, $\xi_0 > c_1 + \delta_2$.

The second limit can be proved similarly.

2.3 The heteroclinic orbit

Now for the fixed sequence $\varepsilon_i \to 0$, get the corresponding sequence of minimal orbit segment $x(\varepsilon_i)$. We have a subsequence such that the index zero points converge to, say, $\xi_0 \in [c_1 + \delta_2, c_2 - \delta_2]$.

Take a subsequence of this concluded subsequence, such that the index one elements converge. Denote the limit as ξ_1. Inductively, we created a limit configuration $\xi = \{\xi_i\}_{i=-\infty}^{+\infty}$.

According to the estimation in Lemma 4, for a fixed integer $N_0 > 0$, take

$$\varepsilon^{N_0} = \frac{\delta_4}{2(\hat{K}^{N_0} + 2\hat{K}^{N_0-1} + \cdots + j\hat{K}^{N_0-j+1} + \cdots + N_0\hat{K})}.$$

Any minimal orbit segment $x(\varepsilon)$ with $\varepsilon < \varepsilon^{N_0}$ must have N_0 elements between $[c_1 + \varepsilon^{N_0}, c_1 + \delta_4]$. Therefore we have $\xi_{-N_0} > c_1 + \varepsilon_{N_0}$. Similar estimation can be done around c_2. Also, it is obvious that the configuration ξ is monotone and $\xi_i \to c_1$ as $i \to -\infty$, $\xi_i \to c_2$ as $i \to +\infty$.

Since all the minimal segment $x(\varepsilon_i) = \{\xi_j(\varepsilon_i)\}_{j=n_1}^{n2}$ are stationary, we have

$$\partial_2 h(\xi_{j-1}(\varepsilon_i), \xi_j(\varepsilon_i)) + \partial_1 h(\xi_j(\varepsilon_i), \xi_{j+1}(\varepsilon_i)) = 0.$$

Now for any fixed integer j, take the subsequence in $\{x(\varepsilon_i)\}$ which defines $\xi_{j-1}, \xi_j, \xi_{j+1}$. The limit of the last equation on this subsequence gives

$$\partial_2 h(\xi_{j-1}, \xi_j) + \partial_1 h(\xi_j, \xi_{j+1}) = 0.$$

So the configuration ξ is indeed an orbit. This concludes our proof for the first proposition.

3 The Proof of Proposition 2

Take an adjacent pair (c_1, c_2) from the set of local minimal points of the function $f(x) = h(x, x)$. In this section we drop the condition $f(c_1) = f(c_2)$. Without loss of generality, we assume that $f(c_1) > f(c_2)$. Remember that, in the statement of Proposition 2, we have a separating barrier

c between c_1 and c_2. Our proof of the proposition is a modification of what we presented in the last section. To play the energy reducing game around c_1, we can not use the old energy function since adding a point around c_1 automatically increase the value of the energy by at least $f(c_1)$. A nature way to overcome this difficulty is to define the new energy for the points around c_1 by subtracting $f(c_1)$ from the old energy formula. However, since we have to do the same around c_2, discontinuity will occur for the modified energy function. What we will do in this section is to show that, thanks to the assumed separating barrier, the discontinuity of the energy function does not destroy the argument we developed earlier.

Again we start from defining the minimal orbit segment. Take a, b

$$c_1 < a < c < b < c_2.$$

We define

$$X_{ab}^n = \{x = \{\xi_i\}_{i=1}^n; \quad a = \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n-1} \leq \xi_n = b\}$$

and let

$$H^n(x) = \sum_{i=1}^{n-1} (h(\xi_i, \xi_{i+1}) - f(c^*))$$

for $x \in X_{ab}^n$. Where $c^* = c_1$ if $\xi_i \leq c$ and $c^* = c_2$ if $\xi_i > c$. Notice that the energy functional $H^n(x)$ is not continuous on X_{ab}^n.

Now define

$$H_l^n(a, b) = \inf_{x \in X_{ab}^n} \{H^n(x)\}$$

$H_l^n(a, b)$ exists according to Mather's energy formula. We further claim that this minimal energy can be reached by an configuration in X_{ab}^n:

Lemma 5 *There is a configuration $x_l^n \in X_{ab}^n$ such that $H(x_l^n) = H_l^n(a, b)$.*

Proof: Take a sequence $x(m) \in X_{ab}^n$ such that $H(x(m))$ monotonically decreasing as m increasing and

$$\lim_{m \to \infty} H(x(m)) = H_l^n(a, b)$$

We can certainly take a subsequence of $x(m)$, such that every configuration in this subsequence has the same number of points in the interval $[a, c]$. So, without loss of generality, assume that every configuration of the sequence $x(m)$ has n_1 points in $[a, c]$.

The sequence $x(m)$ must have a convergent subsequence in X_{ab}^n since X_{ab}^n is compact. Re-index the convergent subsequence as $x(m)$, we have $x_l \in X_{ab}^n$ such that

$$\lim_{m \to \infty} x(m) = x_l = \{\xi_j\}_{j=1}^n$$

Let n' be the number of points of x_l in $[a, c]$, $n' \geq n_1$ because $[a, c]$ is compact. In fact, $n' = n_1$.

To show the equality, take a positive number $\hat{\delta}$, such that $f(x)$ increases in the interval $(c, c+\hat{\delta}]$ and

$$\hat{\delta} < \frac{f(c_1) - f(c_2)}{2K(c_2 - c)}$$

where K is the constant introduced in the definition of the separating barrier.

We claim that, there is a sufficiently large m_0 such that $x(m)$ has no points in $(c, c+\hat{\delta})$ for all $m > m_0$. First choice m_0 such that for all $m > m_0$

$$H^n(x(m)) - H_l^n < \frac{1}{2}(f(c_1) - f(c_2)).$$

If there is a $x(m) = \{\xi_j\}_{j=1}^n$, such that $\xi_{n_1+1} \in (c, c+\hat{\delta})$, we form a new sequence $\hat{x}$ by replacing ξ_{n_1+1} with c. Now

$$H^n(\hat{x}) - H^n(x(m)) = f(c) - f(\xi_{n_1+1}) +$$

$$\mu([c, \xi_{n_1+1}] \times [\xi_{n_1+1}, \xi_{n_1+2}]) - \mu([\xi_{n_1}, c] \times [c, \xi_{n_1+1}]) + f(c_2) - f(c_1)$$
$$< \mu([c, \xi_{n_1+1}] \times [\xi_{n_1+1}, \xi_{n_1+2}]) + f(c_2) - f(c_1)$$

since $f(c) < f(\xi_{n_1+1})$. So

$$H^n(\hat{x}) - H^n(x(m)) < f(c_2) - f(c_1) + K(\xi_{n_1+1} - c)(\xi_{n_1+2} - \xi_{n_1+1})$$

$$\leq \frac{1}{2}(f(c_2) - f(c_1))$$

Therefore

$$H^n(\hat{x}) \leq H^n(x(m)) + \frac{1}{2}(f(c_2) - f(c_1)) < H_l^n$$

The last inequality contradicts to the definition of H_l^n.

Now we have

$$\lim_{m \to \infty} H^n(x(m)) = H^n(x_l) = H_l^n.$$

This finishes the proof of the lemma.

As in the last section, now denote the minimal energy sequence concluded in the last lemma as x^n for the given integer n. Take the sequence of configurations x^n for all positive integer $n > 1$. Again we have

$$\lim_{n \to \infty} H^n(x^n) = +\infty.$$

according to Mather's energy formula. So there is an index I such that

$$H^I(x^I) \leq H^n(x^n)$$

for all $n > 1$. We call the configuration x^I the *minimal segment connecting a and b*. Denote this configuration as x_{ab}, and $I = I(a, b)$. We have

Lemma 6 *$x_{ab} = \{\xi_j\}_{j=1}^I$ is a strictly monotonic configuration. Furthermore, for all the index j, $1 \leq j \leq I$, $\xi_j \neq c$.*

Proof: By the same argument in the proof of lemma 1, we conclude that x_{ab} is a strictly monotone configuration.

For the second part of the claim, if x_{ab} hit the point c, say,

$$\xi_{i_0-1} < \xi_{i_0} = c < \xi_{i_0+1}$$

we will use the fact that c is a separating barrier to induce a contradiction. All we need is to show that, by moving ξ_{i_0} toward the left a little bit, we indeed reduce the energy of the configuration.

Denote

$$\hat{x} = \{a = \xi_1 < \cdots < \xi_{i_0-1} < c - \varepsilon < \xi_{i_0+1} < \cdots < \xi_I = b\}$$

where $\varepsilon > 0$. Take ε so small such that

$$f(c) - f(c-\varepsilon) \geq \frac{1}{2} f'(c)\varepsilon$$

According to Mather's energy formula,

$$H^I(\hat{x}) - H^I(x_{ab}) < -\frac{1}{2} f'(c)\varepsilon + \mu(\Delta)$$

where

$$\mu(\Delta) = \int_{c-\varepsilon}^{c} \int_{c}^{\xi_{i_0+1}} \mid \partial_{12} h \mid < K\varepsilon(\xi_{i_0+1} - c) < K\varepsilon(c_2 - c).$$

Therefore,

$$H^I(\hat{x}) - H^I(x_{ab}) < (-\frac{1}{2} f'(c) + K(c_2 - c))\varepsilon.$$

Since c is a separating barrier, we have

$$H^I(\hat{x}) - H^I(x_{ab}) < 0$$

which, of course, is against the definition of x_{ab}. This proves the lemma.

The last lemma implies the fact that the minimal energy segment x_{ab} according to our construction is indeed a segment of orbit of the monotone twist map.

From this point on we can copy the corresponding part of the proof of proposition 1 line by line to get the heteroclinic connection without any extra trouble. We will save all the writing.

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