

# PERIODICALLY FORCED DOUBLE HOMOCLINIC LOOPS TO A DISSIPATIVE SADDLE

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ABSTRACT. In this paper we present a comprehensive theory on the dynamics of strange attractors in periodically perturbed second order differential equations assuming that the unperturbed equations have two homoclinic loops to a dissipative saddle fixed point. We prove the existence of many complicated dynamical objects for given equations, ranging from attractive quasi-periodic torus, to Newhouse sinks and Hénon-like attractors, and to rank one attractors with SRB measures and full stochastic behavior. This theory enables us to apply rigorously many profound dynamics theories on non-uniformly hyperbolic maps developed in the last forty years, including the Newhouse theory, the theory of SRB measures, the theory of Hénon-like attractors and the theory of rank one attractors, to the analysis of the strange attractors in a periodically perturbed Duffing equation.

## 1. INTRODUCTION

We study the complicated dynamics of strange attractors in periodically perturbed second order differential equations with two homoclinic orbits to a dissipative saddle fixed point through a return map originally proposed by Afraimovich and Shilnikov in [AS]. An explicit formula for this return map has been rigorously derived for given equations with one homoclinic loop and a comprehensive theory has been recently developed for the one loop case in [WO] and [WOk]. In this paper we extend this theory to the two loops case, obtaining a generic overview on various admissible dynamical scenarios for the associated attractors. We rigorously prove the existence of many complicated dynamical objects for given equations, ranging from attractive torus of quasi-periodic solutions, to Newhouse sinks [N] and Hénon-like attractors [BC], [BY], [MV], and to rank one attractors with SRB measures and full stochastic behavior [WY1], [WY2].

As a complicated dynamical object, periodically perturbed homoclinic solutions have been studied extensively in history. It is among the few that have occupied the center stage of the chaos theory since the time of Henry Poincaré [P], [B], [S], [M]. Literatures on rigorous mathematical analysis and on numerical simulations are numerous and are both too vast to be systematically reviewed. Here we mention a few that are closely related to the studies of this paper: The theory of Smale's horseshoes [Sm1], [Sm2] and its applications to differential equations through the Melnikov method [M], [GH]; the large body of work from Shilnikov's school [SSTC1], [SSTC2], [AS], [Shi1]-[Shi5]; and those from Hale's school [CHM-P], [CH], concerning chaos and homoclinic bifurcations in ordinary differential equations.

Duffing [D] introduced a non-linear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical problems in 1918. Since then non-autonomously perturbed Duffing equations have been frequently used in literature to justify the practical importance of new dynamics theories to real world problems [HM], [HR], [IW], [MS], [PS]. With the theory introduced in this paper, we are able to rigorously apply many profound dynamics theories on non-uniformly hyperbolic maps gradually developed in the last forty years, including the Newhouse theory [N], [PT], the theory of SRB measures [Si], [R], [Bo], the theory of Hénon-like maps [BC], [BY], [MV] and the theory of rank one attractor [WY1], [WY2], to the analysis of the strange attractors in periodically perturbed Duffing equations.

**Synopsis of results** We start with a 2D autonomous equation

$$(1.1) \quad \frac{dx}{dt} = -\alpha x + f(x, y), \quad \frac{dy}{dt} = \beta y + g(x, y)$$

where  $f(x, y), g(x, y)$  are high order terms at  $(x, y) = (0, 0)$ . We assume that  $\alpha > \beta > 0$ . We also assume that the stable and unstable manifold of  $(0, 0)$  of equation (1.1) form two homoclinic loops, which we denote as

$$\ell_+(t) = (a_+(t), b_+(t)), \quad \ell_-(t) = (a_-(t), b_-(t)).$$

To equation (1.1) we add a time periodic perturbation of period  $T$  to obtained

$$(1.2) \quad \frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y, t), \quad \frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y, t).$$

We rewrite equation (1.2) as

$$(1.3) \quad \frac{dx}{dt} = -\alpha x + f + \mu P(x, y, \theta/\omega), \quad \frac{dy}{dt} = \beta y + g + \mu Q(x, y, \theta/\omega), \quad \frac{d\theta}{dt} = \omega$$

where  $\theta \in S^1$  and  $\omega = T^{-1}$  is the forcing frequency. The space of  $(x, y, \theta)$  is the extended phase space. Additional assumptions of technical nature on equation (1.3) are detailed in the first paragraph of Section 3. Let  $W_\pm^s$  and  $W_\pm^u$  be the corresponding stable and unstable manifolds from  $\ell_\pm$  for equation (1.3). First we prove that, for  $\mu$  sufficiently small, there exists a trapping region  $V_\mu$  such that  $W_-^u \cup W_+^u \subset V_\mu$ .

Denote  $q = (x, y, \theta)$  and let  $q = q(t, q_0)$  be the solution of equation (1.3) satisfying  $q(0, q_0) = q_0$ . Let

$$\Lambda = \bigcap_{s>0} \bigcup_{q_0 \in V_\mu} \{q(t, q_0), t > s\}.$$

$\Lambda$  is the strange attractor we study in this paper.

Expansions are created by two independent mechanisms, inducing complicated dynamics in  $\Lambda$ . They are

- (a) The expansion induced by the intersections of the broken homoclinic loops  $\ell_\pm$ .
- (b) The expansion induced by large forcing frequency  $\omega$ .

(a) is a well-noted *singular* expansion responsible for Smale's horseshoe [S]. (b) is a *regular* expansion responsible for rank one attractors of [WY1]-[WY2] with SRB measures and fully stochastic behavior [WO].

Let  $W_\pm^s$  and  $W_\pm^u$  be the corresponding stable and unstable branches from  $\ell_\pm$  for equation (1.3).  $W_\pm^s$  and  $W_\pm^u$  are either pulled apart or intersect. Depending on the

ways they are inter-connected, which we detect through the Melnikov functions,  $\Lambda$  are classified into the following five generic types.

- (I) Two rank one attractors.    (II) One rank one attractor.
- (III) One tangle and one rank one attractor.    (IV) Mixture of two tangles.
- (V) One tangle mixed with one rank one map.

See Fig. 1 for the corresponding configurations of  $W_{\pm}^s$  and  $W_{\pm}^u$  for each of the five types.

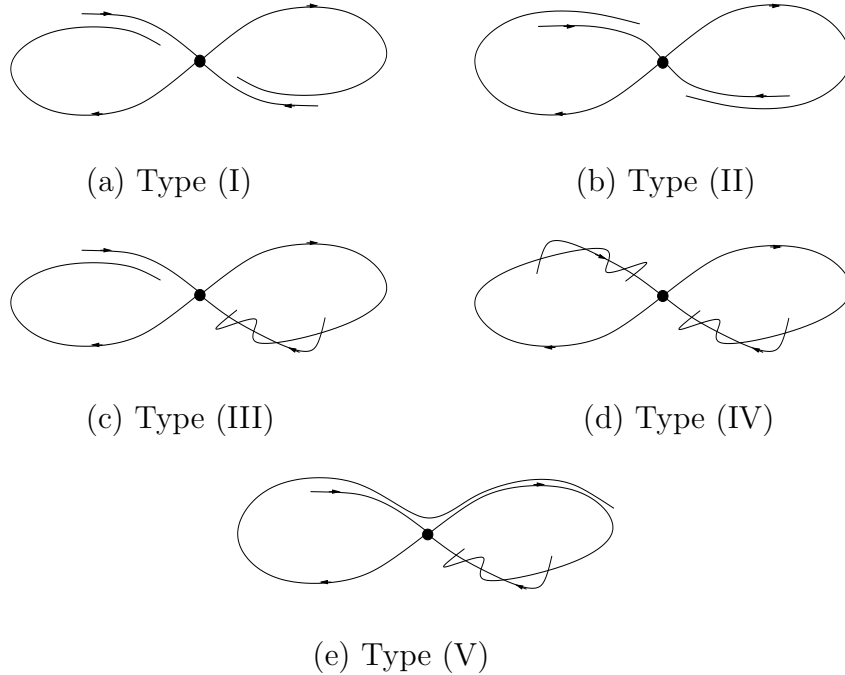


Fig. 1  $W_{\pm}^s$  and  $W_{\pm}^u$  for the five types of strange attractors

Among the five, the singular expansion (a) is missing in action for (I) and (II). A type (I) attractor  $\Lambda$  consists of three disjoint components. One is the saddle point in the middle and the other two, which we denote as  $\Lambda_+$  and  $\Lambda_-$ , are around  $\ell_+$  and  $\ell_-$  respectively. Both  $\Lambda_+$  and  $\Lambda_-$  are rank one attractors, systematically studied by Wang and Young in [WY1]-[WY3]. If  $\omega$  is small, then  $\Lambda_+$  and  $\Lambda_-$  are both 2D quasi-periodic torus representing non-chaotic dynamics. If  $\omega$  is large, then there is a positive measure set of  $\mu$ , so that  $\Lambda_{\pm}$  are both rank one attractors of [WY1]-[WY2] with SRB measures and full stochastic behavior, representing chaotic dynamics. There are also transitional dynamical scenarios from non-chaotic to chaotic dynamics as  $\omega$  gradually increases. A type (II) attractor is the union of the saddle in the middle and one rank one component that goes around the two loops.

Singular expansions from (a) come into play for (III)-(V), creating not only Smale's horseshoe, but also Newhouse sinks and Hénonn-like attractors. For (III), we can apply separately the studies of [WY3] and [WO] to the perturbed loop on the left,

and the studies of [WOk] to the perturbed loop on the right. So on the left-hand side we have an attractive component, the dynamics of which are completely determined by a family of rank one maps (see Sect. 2.2). On the right-hand side, we have an invariant subset the dynamics of which are completely determined by an infinitely wrapped horseshoe map (see Sect. 2.1).

For the analysis of the types (IV) and (V), singular expansions from (a) introduce, unfortunately, singularities that prevent a direct application of the theory of rank one maps in [WY1]-[WY2]. Though there is nothing *a priori* against extending the theory on rank one maps to maps with singularities, a substantial overhaul is called for before we could claim rigorously the corresponding results for these two types.

Finally we apply the theory developed to the analysis of the periodically perturbed Duffing equation

$$(1.4) \quad \frac{d^2q}{dt^2} + (\lambda - \gamma q^2 - \hat{\gamma} q^3) \frac{dq}{dt} - q + q^3 = \mu q^2 \sin 2\pi\omega t.$$

**Organizations** In Section 2 we study the dynamics of a specific family of 2D maps. In Section 3 we present the fore-mentioned return maps in the extended phase space for equation (1.3), through which we could then relate the results of Section 2 to the analysis of the strange attractors of equation (1.3). The details of this analysis are presented in Section 4. In Section 5 we apply the theory developed in Sections 2-4 to equation (1.4).

**Acknowledgement:** The author would like to thank Kening Lu for numerous conversations that have helped shaping the outlook of this and a few other related researches.

## 2. DYNAMICS OF A RANK ONE FAMILY OF 2D MAPS

We study the dynamics of a specific family of 2D maps in this section. These maps, in slightly restricted forms, have been studied systematically in [WY3] and [WOk]. Here our purpose is to summarize the related results from [WY3] and [WOk], painting a comprehensive dynamical picture. For the many statements made in this section, detailed proofs in slightly restricted forms have been previously presented in [WY3] and [WOk]. We will often refer to these two papers instead of re-writing essentially identical proofs. The maps studied in this section will be linked to the attractors of equation (1.3) through the Main Theorem in Section 3.

The maps we study are as follows. Let  $\mathcal{A} = S^1 \times I$  where  $S^1 = \mathbb{R}/\mathbb{Z}$  is the unit circle and  $I = [-1, 1]$ . Let  $\mathcal{F}_a : \mathcal{A} \rightarrow \mathcal{A}$  be a family of 2D map in the form of

$$(2.1) \quad \begin{aligned} \theta_1 &= \theta + a + bg_1(\theta, r, a) + \omega \ln(\varepsilon r + \Phi(\theta) + bg_2(\theta, r, a)) \\ r_1 &= bf(\theta, r, a) \end{aligned}$$

where  $a \in (a_0, +\infty)$  is a parameter and  $f(\theta, r, a), g_1(\theta, r, a), g_2(\theta, r, a)$  are  $C^k$ -functions in  $(\theta, r, a)$  defined on  $\mathcal{A} \times (a_0, +\infty)$  for some  $k \geq 3$ , and their  $C^k$ -norms are bounded from above by a constant  $K$ .  $\Phi(\theta) : S^1 \rightarrow S^1$  is also  $C^k$ . We assume

- (H1) All critical point of  $\Phi(\theta) : S^1 \rightarrow S^1$  are non-degenerate and the graph of  $r = \Phi(\theta)$  in  $\mathcal{A}$  is transversal to the  $\theta$ -axis.
- (H2)  $b, \varepsilon, \omega$  are fixed quantities satisfying  $b \ll \varepsilon \ll \min\{K^{-1}, \omega^{-1}\}$  and  $\varepsilon \ll \min\{|\Phi'(\theta_0)| : \Phi(\theta_0) = 0\}$ .

For the moment let us also assume that  $\Phi(\theta)$  has exactly *two* critical points. Let

$$\Lambda_a = \bigcap_{n>1} \mathcal{F}_a^n(\mathcal{A}).$$

In this section we present a comprehensive analysis on the dynamics of  $\Lambda_a$  for  $\mathcal{F}_a$ . Let

$$M = \max_{\theta \in S^1} \Phi(\theta), \quad m = \min_{\theta \in S^1} \Phi(\theta).$$

We assume  $M > 0$  and consider separately the two cases of  $m > 0$  and  $m < 0$ .

**2.1. The dynamics of  $\Lambda_a$  for the case of  $m < 0$ .** We refer the direction of  $\theta$  in  $\mathcal{A}$  as the *horizontal direction* and the direction of  $r$  the *vertical direction*. For  $q = (\theta, r) \in \mathcal{A}$ , let  $\mathbf{v} = (u, v)$  be a tangent vector of  $\mathcal{A}$  at  $q$  and let  $s(\mathbf{v}) = vu^{-1}$ .  $\mathbf{v}$  is *horizontal* if  $|s(\mathbf{v})| < \frac{1}{100}$  and  $\mathbf{v}$  is *vertical* if  $|s(\mathbf{v})| > 100$ . A smooth curve in  $\mathcal{A}$  is a *horizontal curve* if all its tangent vectors are horizontal and it is a *vertical curve* if all its tangent vectors are vertical. A vertical curve is *fully extended* if it reaches both boundaries of  $\mathcal{A}$  in  $r$ -direction. A region in  $\mathcal{A}$  bounded by two non-intersecting, fully extended vertical curves is a *vertical strip*.

Let

$$(2.2) \quad \mathbb{F}_a(\theta, r) = \varepsilon r + \Phi(\theta) + bg_2(\theta, r, a).$$

By  $m < 0 < M$  and  $\varepsilon \ll \min\{|\Phi'(\theta_0)|, \Phi(\theta_0) = 0\}$ ,

$$\mathbb{F}_a(\theta, r) = 0$$

defines two fully extended vertical curves that divide  $\mathcal{A}$  into two vertical strips we denote as  $U$  and  $V$ .  $\mathbb{F}_a(\theta, r) < 0$  on, say  $U$ , and  $\mathbb{F}_a(\theta, r) > 0$  on  $V$ . In this case  $\mathcal{F}_a : V \rightarrow \mathcal{A}$  is only partially defined on  $\mathcal{A}$  and the dynamics of  $\mathcal{F}_a$  on  $V$  is as follows. It compresses  $V$  in the vertical direction and stretches it in the horizontal direction, making the image infinitely long towards both ends.  $\mathcal{F}_a$  then folds it and wraps it around  $\mathcal{A}$  infinitely many times. See Fig. 2.  $\Lambda_a$  obviously contain a horseshoe of infinitely many branches for all  $a$ , which we call as *the* Smale horseshoe.<sup>1</sup>

The geometric and dynamic structure of  $\Lambda_a$  depend on the location of the folded part of  $\mathcal{F}_a(V)$  in  $\mathcal{A}$ . If this part is deep inside of  $U$ , then  $\Lambda_a$  contains nothing else but the Smale horseshoe. Observe that the image  $\mathcal{F}_a(V)$  moves horizontally towards  $\theta = +\infty$  with a roughly constant speed with respect to  $a$  as  $a \rightarrow +\infty$ . It crosses  $V$  and  $U$  infinitely many times as  $a \rightarrow +\infty$ . As the folded part of  $\mathcal{F}_a(V)$  traverses  $V$ , we encounter complicated dynamics caused by our allowing the folded images of the unstable manifold of the Smale horseshoe to come back to form tangential intersections to the stable manifold of the same horseshoe. In particular, there exists

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<sup>1</sup>As we will see in Sect. 4.2, this horseshoe is *the* one that is induced by transversal homoclinic intersections.

transversal, non-degenerate homoclinic tangency formed by the stable and unstable manifolds of periodic saddles in the Smale horseshoe.

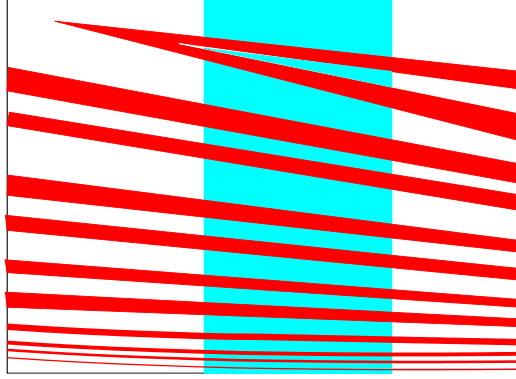


Fig. 2 The geometry of  $\mathcal{F}(V)$

Let us now put the discussion above in more precise terms. We start with the Smale horseshoe. Denote

$$V_f = \{(\theta, r) \in V : \left| \frac{\partial \theta_1}{\partial \theta} \right| < 10\},$$

**Proposition 2.1** (The Smale horseshoe).  *$S_a = \bigcap_{n=-\infty}^{+\infty} \mathcal{F}_a^n(V \setminus V_f)$  is a uniformly hyperbolic invariant subset of  $\Lambda_a$  for all  $a \in (a_0, +\infty)$ . In particular,  $S_a$  contains an invariant subset  $\hat{S}_a$ , on which  $\mathcal{F}_a$  conjugates topologically to a full shift of infinitely many symbols.*

We also have

**Proposition 2.2** (Homoclinic tangency). *There exists an  $\omega_0$  sufficiently large, such that the following hold under the assumption  $\omega > \omega_0$ . For any given parameter interval  $I$  of unit length, there exist infinitely many  $\hat{a} \in I$  and, for each of these  $\hat{a}$ , a periodic saddle  $q(\hat{a}) \in \hat{S}_{\hat{a}}$ , such that*

- (i) *Let  $W^s(q(\hat{a}))$  and  $W^u(q(\hat{a}))$  be the stable and unstable manifold of  $q(\hat{a})$ . Then  $W^u(q(\hat{a})) \cap W^s(q(\hat{a}))$  contains a point of quadratic tangency.*
- (ii) *Let  $q(a)$  be the continuous extension of  $q(\hat{a})$  on a small parameter interval around  $\hat{a}$ . Then, as  $a$  passes through  $\hat{a}$ ,  $W^u(q(a))$  crosses  $W^s(q(a))$  at the tangential intersection point of (i) with a relative speed  $> \frac{1}{2}$  with respect to  $a$  in  $\theta$ -direction.*

We have immediately the following corollary from Proposition 2.2.

**Corollary 2.1.** *Let  $\hat{a}$  be as in Proposition 2.2. Then*

- (i) (Persistence of homoclinic tangency) *There exists a parameter interval  $\hat{I}$  around  $\hat{a}$ , so that for every  $a \in \hat{I}$ ,  $\mathcal{F}_a$  admits a transversal homoclinic tangency of a periodic saddle.*
- (ii) (Newhouse sinks) *There exist a sequence of parameters  $a_n \rightarrow \hat{a}$ , so that for all  $n$ ,  $\mathcal{F}_{a_n}$  admits a periodic sink.*

- (iii) (Hénon-like attractors) *There exists a positive measure set  $\Delta$  of parameter  $a$  in  $\hat{I}$ , so that for every  $a \in \Delta$ ,  $\mathcal{F}_a$  admit a strange attractor with an ergodic SRB measure.*

**Proofs of Propositions 2.1, 2.2 and Corollary 2.1:** Proof for Proposition 2.1 is elementary. Proof for Proposition 2.2 goes as follows. For  $a \in I$  let  $q(a)$  be a fixed point in  $\hat{S}_a$ ,  $W^u(q(a))$  be the unstable manifold and  $W^s(q(a))$  be the stable manifold of  $q(a)$ . First we prove that  $W^u(q(a))$  traverses  $V_f$  in horizontal direction and it has a horizontal segment inside of  $V_f$ , which we denote as  $\ell^u(a)$ . We also prove that  $W^s(q(a))$  has a fully extended vertical segment in  $V$ , which we denote as  $\ell^s(a)$ . We then prove that  $q(a)$ ,  $\ell^u(q(a))$  and  $\ell^s(q(a))$  extend continuously on the parameter interval  $I$ . Observe that  $\mathcal{F}_a(\ell^u(a))$  has a sharp quadratic turn, and as  $a$  varies over  $I$ , it moves from one side of  $V$  to the other, transversally crossing  $\ell^s(a)$ . See Fig. 3.

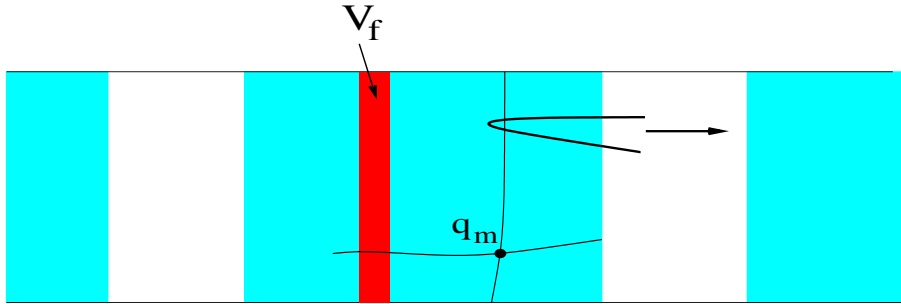


Fig. 3 Transversal homoclinic tangency.

Detailed proof for Proposition 2.2 is long and involves tedious computations. A complete proof in a slightly restricted case is presented in Appendix B and C of [WOk]. Corollary 2.1 follows directly from the Newhouse theory ([PT], [N]) and the theory of Hénon-like maps ([BC], [BY], [MV]).  $\square$

We have two more cases where we understand completely the dynamics of  $\Lambda_a$ .

**Proposition 2.3.** *There exists an  $\omega_0$  sufficiently large, so that under the assumption that  $\omega > \omega_0$ , we have as follows: let  $I$  be a parameter interval of unit length in  $(a_0, +\infty)$ , then*

- (i) *there exists a subinterval  $I_1 \subset I$ , such that for all  $a \in I_1$ ,  $\mathcal{F}_a$  on  $\Lambda_a$  conjugates topologically to a full shift of countably many symbols;*
- (ii) *there exists also a sub-interval  $I_2 \subset I$ , such that for all  $a \in I_2$ ,  $\mathcal{F}_a$  on  $\Lambda_a$  conjugates topologically to the union of one sink and one full shift of countably many symbols.*

**Proof:** Proof for this proposition is again elementary. The details for a slightly restricted case is presented in Appendix A of [WOk].  $\square$

**2.2. The dynamics of  $\Lambda$  for the case of  $m > 0$ .** When  $m > 0$ ,  $\mathcal{F}_a$  is well defined on the entire annulus  $\mathcal{A}$ . In this case, maps such as  $\mathcal{F}_a$  has been studied systematically in [WY1]-[WY3]. The dynamics of  $\Lambda_a$  is by large determined by the magnitude of  $\omega$ .

When  $\omega$  is small,  $\Lambda_a$  is a circle, on which  $\mathcal{F}_a$  induces a circle diffeomorphism. As  $\omega$  increases, this invariant circle disintegrates into isolated periodic sinks and saddles. Increasing the magnitude of  $\omega$  further,  $\mathcal{A}$  is stretched and folded, creating horseshoes and strange attractors. The very rich array of dynamical phenomena proved to exist for (2.1) in [WY3] include

- (i) invariant curves with quasi-periodic behavior,
- (ii) gradient-like dynamics with stable and unstable equilibria,
- (iii) transient chaos caused by the presence of horseshoes, with almost every trajectory eventually tending to a stable equilibrium, and
- (iv) strange attractors with SRB measures and fully stochastic behavior.

As abstract dynamical phenomena, (i)–(iii) are relatively simple and are fairly well understood. (iv) is proved by using the theory of rank one maps developed by Wang and Young [WY1]–[WY2] based on the theory of Benedicks and Carleson on strongly dissipative Hénon maps [BC]. Assume that  $\omega$  is sufficiently large, we have, in addition, a globally defined horseshoe that resembles  $S_a$  in Proposition 2.1 and transversal homoclinic tangency resembles the ones in Proposition 2.2.

We now present these results in more precise terms. Here we focus on the two extreme cases of small and large  $\omega$ . Isolated statements are also made in [WY3] on the transitional cases, that is, the cases for (ii) and (iii) among the listed phenomena above. We refer the interested reader to [WY3] for these statements.

We start with the case where  $\omega$  is small.

**Proposition 2.4.** *Assume that  $\omega$  is such that*

$$(2.3) \quad \omega \cdot \sup_{\theta \in S^1} \left| \frac{\Phi'(\theta)}{\Phi(\theta)} \right| < 1.$$

*Then for every  $a \in (a_0, +\infty)$ ,  $\mathcal{F}_a : \mathcal{A} \rightarrow \mathcal{A}$  admits an invariant circle  $h$  defined by a  $C^r$  function  $h(\theta) : S^1 \rightarrow \mathcal{A}$ . Furthermore,*

- (i)  *$h$  is globally attracting in the sense that, for every  $p \in \mathcal{A}$ , there exists a point  $p_0 \in h$  such that*

$$\lim_{n \rightarrow +\infty} |\mathcal{F}_a^n(p) - \mathcal{F}_a^n(p_0)| \rightarrow 0.$$

- (ii) *The dynamics of  $\mathcal{F}_a$  induced on  $h$  is a circle diffeomorphism, and for any given parameter interval  $I$  of unit length, there exists a positive measure set  $\Delta \subset I$ , so that the rotation number on  $h$  is irrational for all  $a \in \Delta$ .*

**Proof:** A detailed proof for this proposition in a slightly restricted case is presented in [WY3] (Theorem 1(a) in [WY3]). The proof for the invariant circle and (i) uses the standard invariant cone construction. The proof of (ii) uses a theorem of Herman [H] on circle diffeomorphisms.  $\square$

We now move to the case where  $\omega$  is large.

**Proposition 2.5.** *There exists  $\omega_0 > 0$  sufficiently large, depending on  $\Phi(\theta)$ , so that under the assumption that  $\omega > \omega_0$ , there exists, for every parameter interval  $I$  of*

length  $\mathcal{O}(\omega^{-1})$ , a subset of positive measure  $\Delta$ , such that for every  $a \in \Delta$ ,  $\mathcal{F}_a$  on  $\Lambda_a$  is with an SRB measure and full stochastic behavior.<sup>2</sup>

**Proof:** To prove this proposition we apply the long theory developed in [WY1] and [WY2]. It suffices to prove that  $\mathcal{F}_a$  is an admissible family of rank one maps. Again, detailed verifications in slightly different cases have been presented not only in [WY3], but also in [WY4], [LWY] and [WO]. These arguments apply directly to  $\mathcal{F}_a$ .  $\square$

We remark that the SRB measure  $\nu$  obtained in this proposition is *global* in the sense that almost every point in  $\mathcal{A}$  is generic with respect to  $\nu$ .

We continue with the correspondences of Proposition 2.1 and 2.2.

**Proposition 2.6.** *There exists an  $\omega_0$  sufficiently large, so that for any given  $\omega > \omega_0$ ,*

- (i) (Afraimovich-Shilnikov horseshoe)<sup>3</sup> *For any given  $a \in [a_0, +\infty)$ ,  $\Lambda_a$  contains a uniformly hyperbolic invariant subset  $S_a$  conjugating to a full shift of two symbols;*
- (ii) (Homoclinic tangency) *for any given parameter interval  $I$  of unit length, there exists  $\hat{a} \in I$  and a periodic saddle  $q(\hat{a}) \in S_{\hat{a}}$  so that  $W^u(q(\hat{a})) \cap W^s(q(\hat{a}))$  contains a point of quadratic tangency. In addition, let  $q(a)$  be the extension of  $q(\hat{a})$  on a small interval around  $\hat{a}$ . Then, as  $a$  passes through  $\hat{a}$ ,  $W^u(q(a))$  crosses  $W^s(q(a))$  at the point of tangency with a relative speed  $> \frac{1}{2}$  with respect to  $a$  in  $\theta$ -direction.*

**Proof:** The proof for (i) is again elementary. Proof for (ii) is essentially the same as the proof of Theorem 3 in [WOk].  $\square$

Homoclinic tangency of Proposition 2.6(ii) again lead us to rigorous applications of the Newhouse theory and the theory of Hénon-like attractors to  $\mathcal{F}_a$ .

**Corollary 2.2.** *Assume that  $m > 0$  and let  $\hat{a}$  be as in Proposition 2.6(ii). Then*

- (i) (Persistence of homoclinic tangency) *There exists a parameter interval  $\hat{I}$  around  $\hat{a}$ , so that for every  $a \in \hat{I}$ ,  $\mathcal{F}_a$  admits a transversal homoclinic tangency of a periodic saddle.*
- (ii) (Newhouse sinks) *There exist a sequence of parameters  $a_n \rightarrow \hat{a}$ , so that for all  $n$ ,  $\mathcal{F}_{a_n}$  admits a periodic sink.*
- (iii) (Hénon-like attractors) *There exists a positive measure set  $\Delta$  of parameter  $a$  in  $\hat{I}$ , so that for every  $a \in \Delta$ ,  $\mathcal{F}_a$  admit a strange attractor with an ergodic SRB measure.*

We note that though this corollary is almost identical to Corollary 2.1 in wording, it is for an entirely different dynamical scenario. We also note that the Hénon-like attractors of Corollaries 2.1 and 2.2 are very different from the rank one attractors of Proposition 2.5 in the sense that they are local dynamical objects around one specific periodic saddle with much weaker expansions.

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<sup>2</sup>By full stochastic behavior here we mean that it satisfies (SA1)-(SA3) in [WY4].

<sup>3</sup>To the best of my knowledge, horseshoes created by large forcing frequency, not related to transversal homoclinic intersections, are first observed by Afraimovich and Shilnikov in [AS].

**2.3. More general considerations.** In this subsection we discuss the more general case in which  $\Phi(\theta)$  has more than two critical points. We start with the case  $m > 0$  discussed in Sect. 2.2. In the two extreme cases where  $\omega$  is either small or very large, the number of critical points of  $\Phi(\theta)$  is not relevant, and Propositions 2.4-2.6 and Corollary 2.2 remain valid for  $\mathcal{F}_a$ . For the transitional cases in between, in particular, the statement made in Theorem 2(a)(b) of [WY3], the correspondence of (a) remains valid but there is no correspondence for (b).

For the case of  $m < 0$  discussed in Sect. 2.1,

$$\mathbb{F}_a(\theta, r, a) > 0$$

defines a finite collection of vertical strips, say  $V = \cup V_i$ , in  $\mathcal{A}$ . The actions of  $\mathcal{F}_a$  on each  $V_i$  are similar to the ones on  $V$  described in Sect. 2.1: First it compresses  $V_i$  in the vertical direction and stretches it in the horizontal direction. It then folds it, but probably more than one time. The number of turns of this image is determined by the number of critical points of the 1D function  $\theta + \omega \ln \Phi(\theta)$  on the corresponding  $\theta$ -interval for  $V_i$ . This image is then put back into  $\mathcal{A}$ , wrapping around  $\mathcal{A}$  infinitely many time in  $\theta$ -direction. Finally, the images of all  $V_i$  are tied together, moving towards  $\theta = +\infty$  in a roughly constant speed with respect to  $a$  as  $a \rightarrow +\infty$ . The rich possibility on the number of folds for each vertical strip, and the possibilities on the different locations of all these folded part create a rich array of complicated structures where the three dynamical scenarios of horseshoes, sinks, and the Hénon-like attractors co-exist.

As to the precise statements. Propositions 2.1 and 2.2 and Corollary 2.1 remain valid. Proposition 2.3 does not necessarily hold. Since all we could manage is to put one critical value out of  $V$  for some  $a$ , the corresponding statement for Proposition 2.2(i) would be that, under the assumption that  $\Phi(\theta)$  has only one critical point for a vertical strip  $V_i$ , the part of  $\Lambda_a$  that stays inside of  $V_i$  for all time conjugates to a horseshoe of infinitely many branches. No corresponding statement in general could be made for the entire attractor  $\Lambda_a$ . Similar modifications are in place for the correspondence of Proposition 2.2(ii).

### 3. RETURN MAPS IN EXTENDED PHASE SPACE

We assume  $(x, y) = (0, 0)$  is a *dissipative, non-resonant* saddle for equation (1.1). This is to say that (i)  $\alpha > \beta > 0$  and (ii)  $n\alpha + m\beta \neq 0$  for all integers satisfying  $|m| + |n| \leq k$  for some  $k > 3$ .  $k$  is an arbitrary integer fixed throughout. The two homoclinic loops are denoted as

$$\ell_+(t) = (a_+(t), b_+(t)), \quad \ell_-(t) = (a_-(t), b_-(t)).$$

$f(x, y), g(x, y)$  are  $C^k$  functions in an open neighborhood  $U$  of  $\ell_{\pm}$ . For equation (1.3) we further assume that  $P(x, y, t), Q(x, y, t)$  are periodic in  $t$  of period  $T$  and they are  $C^k$  on  $U$ . In addition we assume that both  $P$  and  $Q$  are high order terms for all  $t$  in  $(x, y)$  at  $(x, y) = (0, 0)$ .

**A. Poincaré sections** We start with a standard proposition on local linearizations.

**Proposition 3.1 (Local linearization).** *Assume that  $\alpha$  and  $\beta$  are non-resonant up to order  $k$ . Then there exists a  $C^k$ -coordinate transformation in the form of*

$$(3.1) \quad \begin{aligned} x &= X + \mathbb{P}(X, Y) + \mu \tilde{\mathbb{P}}(X, Y, \theta; \mu) \\ y &= Y + \mathbb{Q}(X, Y) + \mu \tilde{\mathbb{Q}}(X, Y, \theta; \mu) \end{aligned}$$

where  $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$  are high order terms in  $(X, Y)$  at  $(0, 0)$  for all  $\theta \in S^1$ , such that for all  $\mu \geq 0$  sufficiently small, (3.1) transforms equation (1.3) into

$$\frac{dX}{dt} = -\alpha X, \quad \frac{dY}{dt} = \beta Y, \quad \frac{d\theta}{dt} = \omega$$

on  $\mathcal{B}_{2\varepsilon} = \{(X, Y, \theta) : |(X, Y)| < 2\varepsilon\}$  with an  $\varepsilon$  that is independent of  $\mu$ .

We define  $\Sigma_{\pm}$  by letting

$$\Sigma_{\pm} = \{(X, Y, \theta) : Y = \pm\varepsilon, |X| < \mu, \theta \in S^1\}.$$

We study the dynamics of equation (1.3) through the return map  $\mathcal{R}$  on  $\Sigma = \Sigma_+ \cup \Sigma_-$  induced by the solutions in the extended phase space  $(X, Y, \theta)$ .  $\Sigma$  is the union of two disjoint annulus. Following [WO] and [WOk], we first drive a formula for  $\mathcal{R}$  from equation (1.3).

**Two small scales:**  $\mu \ll \varepsilon \ll 1$  represent two small scales of different magnitude.  $2\varepsilon$  represents the size of a small neighborhood of  $(x, y) = (0, 0)$ , on which Proposition 3.1 holds. Let  $L_{\pm}^{pos}, L_{\pm}^{neg}$  be such that

$$|\ell_+(L_+^{pos})|, |\ell_+(L_+^{neg})|, |\ell_-(L_-^{pos})|, |\ell_-(L_-^{neg})| = \varepsilon.$$

$L_{\pm}^{pos}, L_{\pm}^{neg}$  are the respective times  $\ell_{\pm}$  intersect  $|(x, y)| = \varepsilon$  in the space of  $(x, y)$  in the positive and negative time directions respectively.  $L_{\pm}^{pos}$  and  $L_{\pm}^{neg}$  are completely determined by  $\ell_{\pm} = (a_{\pm}(t), b_{\pm}(t))$  and  $\varepsilon$ . The parameter  $\mu \ll \varepsilon$  controls the magnitude of the time-dependent perturbation.

**Notations:** 1. (*Generic constant*) Quantities that are independent of the phase variables, time and  $\mu$  are regarded as constants and  $K$  is used to denote a generic constant, the precise value of which is allowed to change from line to line. We will also make distinctions between constants depend on  $\varepsilon$  and those do not by making such dependencies explicit. A constant that depends on  $\varepsilon$  is written as  $K(\varepsilon)$ . A constant written as  $K$  is independent of  $\varepsilon$ .

2. (*The  $\mathcal{O}$ -terms*) The intended formula for the return maps would inevitably contain terms that are explicit and terms that are implicit. Implicit terms are usually “error” terms, and the usefulness of a derived formula would depend completely on how well the error terms are controlled. We aim on  $C^k$ -control on all error terms and we adopt the following conventions indicating controls on magnitude. For a given constant, we write  $\mathcal{O}(1)$ ,  $\mathcal{O}(\varepsilon)$  or  $\mathcal{O}(\mu)$  to indicate that the magnitude of this constant is bounded by  $K$ ,  $K\varepsilon$  or  $K(\varepsilon)\mu$ , respectively. For a function of a set  $V$  of variables on a specific domain, we write  $\mathcal{O}_V(1)$ ,  $\mathcal{O}_V(\varepsilon)$  or  $\mathcal{O}_V(\mu)$  to indicate that the  $C^k$ -norm of the function on a specified domain is bounded by  $K$ ,  $K\varepsilon$  or  $K(\varepsilon)\mu$ , respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in this notation. For example,  $\mathcal{O}_{Z, \theta}(\mu)$  represents a function

of  $Z, \theta$ , the  $C^k$ -norm of which is bounded by  $K(\varepsilon)\mu$  on a domain specified in the proceeding text.

3. (*The new parameter  $p = \ln \mu^{-1}$* ) We also need to study the dependency of  $\mathcal{R}$  on parameter  $\mu$ . Here difficulties arise because taking derivative with respect to  $\mu$  would remove a copy of  $\mu$  from the forcing term, creating non-perturbation terms in variational equations from the terms of perturbation. To avoid this problem we use a slower parameter  $p = \ln \mu^{-1}$  in the place of  $\mu$ . Observe that

$$\partial_p F(\mu) = -\mu \partial_\mu F(\mu).$$

By regarding all functions of  $\mu$  as functions of  $p$ , and by taking derivatives only with respect to  $p$ , not to  $\mu$ , we give a copy of  $\mu$  back to  $\partial_\mu F(\mu)$ .

**B. Melnikov functions** We define Melnikov functions  $\mathcal{W}_+(\theta)$  and  $\mathcal{W}_-(\theta)$  for  $\ell_+$  and  $\ell_-$  respectively as follows. Let

$$(3.2) \quad \mathcal{W}_+(\theta) = \int_{-\infty}^{\infty} (v_+(s)P(\ell_+(s), s+\omega^{-1}\theta) - u_+(s)Q(\ell_+(s), s+\omega^{-1}\theta)) e^{-\int_0^s E_+(\tau) d\tau} ds$$

where

$$(u_+(t), v_+(t)) = \left| \frac{d}{dt} \ell_+(t) \right|^{-1} \frac{d}{dt} \ell_+(t)$$

is the unit vector tangent to  $\ell_+$  at  $\ell_+(t)$  and

$$(3.3) \quad \begin{aligned} E_+(t) = & v_+^2(t)(-\alpha + \partial_x f(a_+(t), b_+(t))) + u_+^2(t)(\beta + \partial_y g(a_+(t), b_+(t))) \\ & - u_+(t)v_+(t)(\partial_y f(a_+(t), b_+(t)) + \partial_x g(a_+(t), b_+(t))) \end{aligned}$$

is the rate of expansion of the solutions of equation (1.1) in the direction normal to  $\ell_+$  at  $\ell_+(t)$ . We also denote

$$(3.4) \quad C_{+,L} = e^{\int_{L_+^{neg}}^{L_+^{pos}} E_+(s) ds}, \quad \tilde{C}_{+,L} = e^{\int_0^{L_+^{pos}} E_+(s) ds},$$

and

$$(3.5) \quad \mathcal{W}_{+,L}(\theta) = \int_{L_+^{neg}}^{L_+^{pos}} (v_+(s)P(\ell_+(s), s+\omega^{-1}\theta) - u_+(s)Q(\ell_+(s), s+\omega^{-1}\theta)) e^{-\int_0^s E_+(\tau) d\tau} ds.$$

It holds from definition that

$$C_{+,L} \sim \varepsilon^{\frac{\alpha}{\beta} - \frac{\beta}{\alpha}} \ll 1, \quad \tilde{C}_{+,L} \sim \varepsilon^{-\frac{\beta}{\alpha}} \gg 1.$$

See Lemma 5.1 of [WO]. We define  $\mathcal{W}_-(\theta)$  and the related quantities the same way using  $\ell_-$  in the place of  $\ell_+$ .

**C. Main Theorem** Let

$$\mathbb{X} = \mu^{-1}X, \quad \mathbb{Y} = \mu^{-1}Y.$$

On  $\Sigma_\pm$  we have  $\mathbb{Y} = \pm \mu^{-1}\varepsilon$  and all points are denoted by using the  $(\theta, \mathbb{X})$ -coordinates. In what follows,  $\sigma, \tilde{\sigma} \in \{+, -\}$ . Let

$$\Sigma_\sigma = \{(\theta^\sigma, \mathbb{X}^\sigma), \quad \theta^\sigma \in S^1, \quad |\mathbb{X}^\sigma| < 1\}.$$

By definition

$$\Sigma = \Sigma_+ \cup \Sigma_-.$$

Our next theorem is about the return map  $\mathcal{R} : \Sigma \rightarrow \Sigma$  induced by the solutions of equation (1.3). Recall that we regard  $p = \ln \mu^{-1}$  as the parameter of equation (1.3).

**Main Theorem (Formula for  $\mathcal{R}$ ).** *Let  $(\theta^\sigma, \mathbb{X}^\sigma) \in \Sigma_\sigma$ , and  $(\theta_1^{\tilde{\sigma}}, \mathbb{X}_1^{\tilde{\sigma}}) = \mathcal{R}(\theta^\sigma, \mathbb{X}^\sigma)$ . We have*

$$(3.6) \quad \begin{aligned} \theta_1^{\tilde{\sigma}} &= \theta^\sigma + \mathbf{a}_\sigma - \frac{\omega}{\beta} \ln(\tilde{\sigma} \cdot \sigma \cdot \mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu)) + \mathcal{O}_{\theta^\sigma, \mathbb{X}^\sigma, p}(\mu) \\ \mathbb{X}_1^{\tilde{\sigma}} &= \mathbf{b}_\sigma [\tilde{\sigma} \cdot \sigma \cdot \mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu)]^{\frac{\alpha}{\beta}} \end{aligned}$$

where  $\tilde{\sigma}$  is such that

$$(3.7) \quad \tilde{\sigma} \cdot \sigma \cdot \mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu) > 0;$$

$$(3.8) \quad \begin{aligned} \mathbf{a}_\sigma &= \frac{1}{\beta} \ln \mu^{-1} + (L_\sigma^{pos} - L_\sigma^{neg}) + \frac{1}{\beta} \ln(\varepsilon(1 + \mathcal{O}(\varepsilon))\tilde{C}_{\sigma,L}) \\ \mathbf{b}_\sigma &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1} [(1 + \mathcal{O}(\varepsilon))\tilde{C}_{\sigma,L}]^{\frac{\alpha}{\beta}}; \end{aligned}$$

and

$$(3.9) \quad \mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu) = \mathcal{W}_\sigma(\theta^\sigma) + \mathbf{k}_\sigma \mathbb{X}^\sigma + \mathbb{E}_\sigma(\theta^\sigma, \mu) + \mathcal{O}_{\theta^\sigma, \mathbb{X}^\sigma, p}(\mu),$$

in which  $\mathcal{W}_\sigma(\theta)$  is as in (3.2),

$$(3.10) \quad \mathbf{k}_\sigma = C_{\sigma,L}(\tilde{C}_{\sigma,L})^{-1}(1 + \mathcal{O}(\varepsilon))$$

and

$$(3.11) \quad \mathbb{E}_\sigma(\theta^\sigma, \mu) = (\tilde{C}_{\sigma,L})^{-1}(1 + C_{\sigma,L})\mathcal{O}_{\theta,p}(1) + \mathcal{W}_{\sigma,L}(\theta^\sigma) - \mathcal{W}_\sigma(\theta^\sigma).$$

**D. Digestive Remarks** We offer the following remarks on (3.6) for  $\mathcal{R}$  presented above in the Main Theorem.

(1)  $\Sigma = \Sigma_+ \cup \Sigma_-$  is the union of two annulus. Points in  $\Sigma_+$  is represented by  $(\theta^+, \mathbb{X}^+)$  and points in  $\Sigma_-$  is represented by  $(\theta^-, \mathbb{X}^-)$ . Let  $\sigma = +$  in (3.6).  $\mathcal{R}(\theta^+, \mathbb{X}^+)$  for  $(\theta^+, \mathbb{X}^+) \in \Sigma_+$  is not defined if  $\mathbb{F}_+(\theta^+, \mathbb{X}^+, \mu) = 0$ . Otherwise  $\tilde{\sigma} = +$  if  $\mathbb{F}_+(\theta^+, \mathbb{X}^+, \mu) > 0$ . In this case  $\mathcal{R}(\theta^+, \mathbb{X}^+) \in \Sigma_+$ . If  $\mathbb{F}_+(\theta^+, \mathbb{X}^+, \mu) < 0$ , then  $\tilde{\sigma} = -$ . In this case  $\mathcal{R}(\theta^+, \mathbb{X}^+) \in \Sigma_-$ . The cases for  $\sigma = -$  are similar: for  $(\theta^-, \mathbb{X}^-) \in \Sigma_-$ ,  $\mathcal{R}(\theta^-, \mathbb{X}^-)$  is not defined if  $\mathbb{F}_-(\theta^-, \mathbb{X}^-, \mu) = 0$ ;  $\mathcal{R}(\theta^-, \mathbb{X}^-) \in \Sigma_-$  if  $\mathbb{F}_-(\theta^-, \mathbb{X}^-, \mu) > 0$ ; and  $\mathcal{R}(\theta^-, \mathbb{X}^-) \in \Sigma_+$  if  $\mathbb{F}_-(\theta^-, \mathbb{X}^-, \mu) < 0$ .

To understand why this is the case we let

$$\mathcal{S}_\pm = \{(\mathbb{X}, \mathbb{Y}, \theta), \mathbb{X} = \pm\mu^{-1}\varepsilon, |\mathbb{Y}| < K_1(\varepsilon), \theta \in S^1\}.$$

For  $q_0 \in \Sigma_+$ , let  $q(t, q_0)$  be the solution of equation (1.3) satisfying  $q(0, q_0) = q_0$ . Let  $\hat{t} > 0$  be the first time this solution reaches  $\mathcal{S}_+$  and let  $\hat{q} = q(\hat{t}, q_0)$ . If  $\hat{q} \in \mathcal{S}_+$  is such that  $\mathbb{Y} > 0$ , then  $q(t, q_0)$  will return to  $\Sigma_+$  later. If  $\mathbb{Y} < 0$  for  $\hat{q}$ , then  $q(t, q_0)$  will reach  $\Sigma_-$ . If  $\mathbb{Y} = 0$ , then  $q(t, q_0)$  is on the stable manifold of the solution  $(x, y, \theta) = (0, 0, \omega t)$ . It will approach to  $(x, y) = (0, 0)$  as  $t \rightarrow \infty$ , never return to  $\Sigma$ . See Fig. 4.

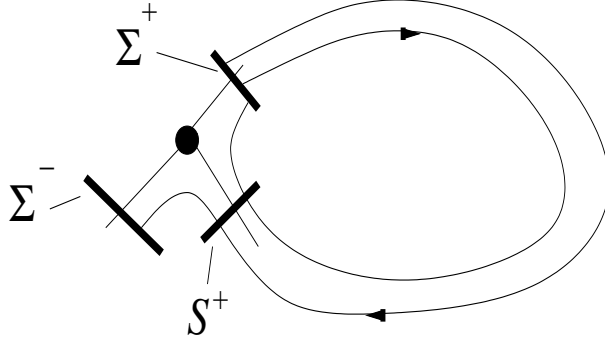


Fig. 4. Different destinations of  $\mathcal{R}(\Sigma_+)$

(2) Observe that if the unperturbed equation (1.1) is a Hamiltonian equation, then  $\alpha = \beta$ , and by definition,

$$-\alpha x + f(x, y) = \partial_y H(x, y), \quad \beta y + g(x, y) = -\partial_x H(x, y).$$

Denote  $E(t) = E_+(t)$ ,  $\mathcal{W}(\theta) = \mathcal{W}_+(\theta)$ , and  $\ell_+ = (a(t), b(t))$ . Then  $E(t)$  in (3.3) is reduced to

$$E(t) = -\frac{d}{dt} \left( (\partial_x H(a(t), b(t)))^2 + (\partial_y H(a(t), b(t)))^2 \right)^{\frac{1}{2}},$$

and  $\mathcal{W}(\theta) = \mathcal{W}_+(\theta)$  in (3.2) is reduced to the regular Melnikov function

$$(3.12) \quad \mathcal{W}(\theta) = -\left| \frac{d\ell(0)}{dt} \right|^{-1} \int_{-\infty}^{\infty} (\partial_y H \cdot P(a(s), b(s), s + \omega^{-1}\theta) + \partial_x H \cdot Q(a(s), b(s), s + \omega^{-1}\theta)) ds.$$

(3) Observe that  $\mathbf{b}_\sigma \sim \mu^{\frac{\alpha}{\beta}-1}$ . From  $\alpha > \beta > 0$ ,  $\mathbf{b}_\sigma \rightarrow 0$  as  $\mu \rightarrow 0$ . First we construct a trapping region for equation (1.3) by using (3.6).

**Proposition 3.2 (Trapping region).** *There exists  $\mu_0$  sufficiently small, such that for any given  $0 < \mu < \mu_0$ , there exists an open neighborhood  $V_\mu$  in the space of  $(x, y, \theta)$  inside of  $U \times S^1$ , so that (i) all solutions of (1.3) initiated in  $V_\mu$  stays in  $V_\mu$  for all  $t > 0$ , and (ii) the entire unstable manifold of the solution  $(0, 0, \omega t)$  is inside of  $V_\mu$ .*

**Proof:**  $V_\mu$  is constructed as follows. First we take the union of all segment of solutions from  $\Sigma_+$  and  $\Sigma_-$  to  $\mathcal{S}_+$  and  $\mathcal{S}_-$  respectively. Second we take the union of all segment of solutions from  $\mathcal{S}_\pm$  to  $\Sigma_\pm$  respectively. Together with the solution  $(x, y, \theta) = (0, 0, \omega t)$  and all solutions on the local stable manifold starting from  $\mathcal{S}_\pm$ , we obtain the trapping region  $V_\mu$  of Proposition 3.2. See Fig. 5.  $\square$

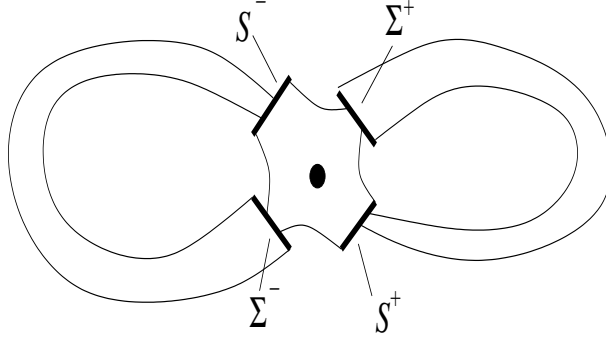


Fig. 5. Construction of the trapping region  $V_\mu$

(4) For  $\mu > 0$  sufficiently small, we can think of  $\mathcal{R}$  as a 2D family of maps unfolded from the following 1D maps  $f : S^1 \cup S^1 \rightarrow S^1 \cup S^1$ . Again, denote  $\sigma, \tilde{\sigma} \in \{+, -\}$ ,  $\theta^+, \theta^- \in S^1$ , and let  $\theta_1^{\tilde{\sigma}} = f(\theta^\sigma)$ . We have

$$(3.13) \quad \theta_1^{\tilde{\sigma}} = \theta^\sigma + \mathbf{a}_\sigma - \frac{\omega}{\beta} \ln \tilde{\sigma} \cdot \sigma \cdot (\mathcal{W}_\sigma(\theta^\sigma) + \mathbb{E}_\sigma(\theta^\sigma, 0))$$

where  $\tilde{\sigma}$  is such that

$$\tilde{\sigma} \cdot \sigma \cdot (\mathcal{W}_\sigma(\theta^\sigma) + \mathbb{E}_\sigma(\theta, 0)) > 0.$$

Following [WY1] and [WY2], we will regard  $f : S^1 \cup S^1 \rightarrow S^1 \cup S^1$  as the 1D singular limit for  $\mathcal{R} : \Sigma_+ \cup \Sigma_- \rightarrow \Sigma_+ \cup \Sigma_-$ .

(5) We also have the following for  $\mathbf{a}_\sigma, \mathbf{k}_\sigma$  and  $\mathbb{E}_\sigma(\theta^\sigma, \mu)$ :

(i) Regarding  $\mu$  as the parameter of equation (1.3),  $\mathbf{a}_\sigma \approx \mathbf{a} = \frac{\omega}{\beta} \ln \mu^{-1}$ .  $\mathbf{a} \rightarrow +\infty$  as  $\mu \rightarrow 0$ . This has two direct consequences. First, as parameters  $p = \ln \mu^{-1}$  and  $\mathbf{a} = \frac{\omega}{\beta} \ln \mu^{-1}$  are equivalent, and in estimating derivatives with respect to parameters, we can use  $\mathbf{a}$  for  $p = \ln \mu^{-1}$ . Second, since  $\mathbf{a}$  appears in the angular component in (3.6), the 1D singular limit  $f : S^1 \cup S^1 \rightarrow S^1 \cup S^1$  is periodic of period one in  $\mathbf{a}$ . This implies that, in the space of  $\mu$ , there is a roughly fixed pattern of dynamics behavior for the return maps  $\mathcal{R}$ , repeating in an accelerated fashion as  $\mu \rightarrow 0$ . We also know that the multiplicative ratio for one such period is roughly  $\approx e^{\beta T}$ . We use the word “roughly” here because the periodicity with respect to  $\mu$  as stated is for the 1D singular limit  $f$  on  $S^1 \cup S^1$ , not the 2D return map  $\mathcal{R}$  on  $\Sigma_+ \cup \Sigma_-$ .

(ii)  $\mathbf{k}_\sigma \sim \varepsilon^{\frac{\alpha}{\beta}}$ . Since  $\mathbf{k}_\sigma \gg \mu$ , the first derivative of  $\mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, p)$  with respect to  $\mathbb{X}^\sigma$  is approximately  $\mathbf{k}_\sigma$  and the unfolding from  $f$  to  $\mathcal{R}$  in  $\mathbb{X}^\sigma$ -direction is determined mainly by the linear term  $\mathbf{k}_\sigma \mathbb{X}$ .

(iii)  $\mathbb{E}_\sigma(\theta^\sigma, \mu) \sim \varepsilon^{\frac{\beta}{\alpha}} \mathcal{O}_{\theta^\sigma, p}(1)$ . When  $\varepsilon$  is sufficiently small,  $\mathbb{E}_\sigma(\theta^\sigma, \mu)$  is a  $C^k$ -small perturbation to  $\mathcal{W}_\sigma(\theta^\sigma)$ .

Let

$$\Lambda = \bigcap_{n \geq 1} \mathcal{R}^n(\Sigma).$$

The *ultimate objective* of our study is to understand the geometric and dynamic structure of  $\Lambda$  for the return maps  $\mathcal{R}$  in (3.6) derived from equation (1.3) in the Main Theorem.

**Proof of the Main Theorem:** To prove the Main Theorem we are to repeat the previous derivations of the return map  $\mathcal{R}$  around one homoclinic loop detailed in [Wok] (Also see [LW] and [WO]). Technical adjustments needed over previous derivations are minimal.  $\square$

#### 4. A GENERIC DYNAMICAL DESCRIPTION

In this section we give a generic overview on the dynamics of strange attractor of equation (1.3) by using the Main Theorem. The unperturbed equation (1.1) is fixed throughout and we have assumed that the saddle fixed point  $(x, y) = (0, 0)$  has two homoclinic loops.

**4.1. Attainable Melnikov functions.** Let  $U$  be a fixed open neighborhood of the closure of  $\ell_+ \cup \ell_-$  in the space of  $(x, y)$ , and  $H_T$  be the collection of all real values  $C^k$  functions in  $(x, y, t)$  defined on  $U \times \mathbb{R}$  such that (1) they are periodic in  $t$  of period  $T$  and (2) their first derivatives with respect to  $x$  and  $y$  are both zero at  $(x, y) = (0, 0)$  for all  $t$ . Let

$$H = \cup_{T>0} H_T \times H_T.$$

Every pair  $(P(x, y, t), Q(x, y, t)) \in H$  defines an equation in the form of (1.2); and for this equation we obtain (3.6) in the Main Theorem for the return map  $\mathcal{R}$ . The two characteristic quantities in (3.6) determined by  $(P(x, y, t), Q(x, y, t)) \in H$  are the forcing frequency  $\omega$  and the two Melnikov functions  $\mathcal{W}_+(\theta)$  and  $\mathcal{W}_-(\theta)$ .

In this subsection we consider the problem in reverse. We have

**Theorem 1.** *Let  $W$  be the collection of all  $C^k$ -functions from  $S^1$  to  $\mathbb{R}$ , and  $(F_+(\theta), F_-(\theta))$  be a given pair in  $W \times W$ . Then for almost all  $\omega \in \mathbb{R}$ , there exists  $(P(x, y, t), Q(x, y, t)) \in H_{\omega^{-1}}$  such that the return map  $\mathcal{R}$  for the corresponding equation (1.2) is written as in (3.6), in which  $F_{\pm}(\theta)$  is in the places of  $\mathcal{W}_{\pm}(\theta)$ .*

**Proof:** Let

$$\mathcal{W}_{\pm}(\theta) = \sum_{n=-\infty}^{\infty} a_{\pm}^{(n)} e^{2\pi n \theta i}$$

be the Fourier expansion of  $\mathcal{W}_{\pm}(\theta)$ . Let  $A(x, y), B(x, y)$  be two  $C^k$ -functions defined on  $U$  and assume that their values and the first derivatives at  $(x, y) = (0, 0)$  are all zero. Let  $P(x, y, t), Q(x, y, t)$  be in the form of

$$P(x, y, t) = A(x, y) \sum_{n=-\infty}^{\infty} f^{(n)} e^{2\pi n \omega t i}, \quad Q(x, y, t) = B(x, y) \sum_{n=-\infty}^{\infty} g^{(n)} e^{2\pi n \omega t i}.$$

We have from (3.2) that, for all  $n$ ,

$$(4.1) \quad a_{\pm}^{(n)} = J_{\pm,1}(n\omega) f^{(n)} - J_{\pm,2}(n\omega) g^{(n)}$$

where

$$(4.2) \quad \begin{aligned} J_{\pm,1}(\omega) &= \int_{-\infty}^{\infty} v(s)A(a_{\pm}(s), b_{\pm}(s))e^{-\int_0^s E_{\pm}(\tau)d\tau} \cdot e^{2\pi\omega si} ds, \\ J_{\pm,2}(\omega) &= \int_{-\infty}^{\infty} u(s)B(a_{\pm}(s), b_{\pm}(s))e^{-\int_0^s E_{\pm}(\tau)d\tau} \cdot e^{2\pi\omega si} ds. \end{aligned}$$

From (4.1) and (4.2),

$$(4.3) \quad \begin{aligned} f^{(n)} &= \frac{1}{J(n\omega)}(J_{+,2}(n\omega)a_-^{(n)} - J_{-,2}(n\omega)a_+^{(n)}), \\ g^{(n)} &= \frac{1}{J(n\omega)}(J_{+,1}(n\omega)a_-^{(n)} - J_{-,1}(n\omega)a_+^{(n)}) \end{aligned}$$

where

$$(4.4) \quad J(\omega) = J_{-,2}(\omega)J_{+,1}(\omega) - J_{+,2}(\omega)J_{-,1}(\omega).$$

Observe that  $J_{\pm,1}(\omega)$ ,  $J_{\pm,2}(\omega)$  are the respective Fourier transforms of the functions

$$v(s)A(a_{\pm}(s), b_{\pm}(s))e^{-\int_0^s E_{\pm}(\tau)d\tau}, \quad u(s)B(a_{\pm}(s), b_{\pm}(s))e^{-\int_0^s E_{\pm}(\tau)d\tau},$$

and as  $s \rightarrow \pm\infty$ , these two functions decay exponentially fast. From the Paley-Wiener theorem it follows that  $J_{\pm,1}(\omega)$  and  $J_{\pm,2}(\omega)$  are analytic in a strip containing the real  $\omega$ -axis.  $J(\omega)$  is then real analytic. Either  $J(\omega) = 0$  for at most a discrete set of real  $\omega$  or it is an identically zero-valued function. With the freedom of choices on  $A(x, y)$  and  $B(x, y)$ , we can easily stay away from the latter case. It then follows that the set

$$\{\omega \in \mathbb{R} : \exists n \in \mathbb{Z}, \text{ s.t. } J(n\omega) = 0\}$$

is countable. Therefore for almost all  $\omega \in \mathbb{R}$ ,  $J(n\omega) \neq 0$  for all  $n \in \mathbb{Z}$ , and we can solve for  $f^{(n)}$ ,  $g^{(n)}$  for all  $n \in \mathbb{Z}$ .

A seemingly dire problem for us is that  $P(x, y, t)$ ,  $Q(x, y, t)$  so obtained are not necessarily convergent. Here we take advantage of the fact that the (3.6) contains error terms. Starting from a given pair  $(F_+(\theta), F_-(\theta)) \in W \times W$ , let  $(\mathcal{W}_+(\theta), \mathcal{W}_-(\theta))$  be obtained by truncating the Fourier expansions of  $(F_+(\theta), F_-(\theta))$  so that

$$|\mathcal{W}_+(\theta) - F_+(\theta)|_{C^k}, \quad |\mathcal{W}_-(\theta) - F_-(\theta)|_{C^k}$$

are so small that we could put them into the  $\mathbb{E}_{\pm}(\theta, \mu)$  terms in (3.11). We then obtain  $P(x, y, t)$ ,  $Q(x, y, t)$  as in the above. Since we only have finitely many terms to manage, there is no more issue of convergence for  $P$  and  $Q$ .  $\square$

**4.2. A generic description on various dynamical scenarios.** It follows from the Main Theorem and Theorem 1 that, for the purpose of studying the dynamics of equation (1.2) collectively assuming  $(P(x, y, t), Q(x, y, t)) \in H$ , we can first take an arbitrary pair  $(F_+(\theta), F_-(\theta)) \in W \times W$  for the Melnikov functions, then almost all  $\omega > 0$  for the forcing frequency in (3.6) for  $\mathcal{R}$ . Let  $(\mathcal{W}_+(\theta), \mathcal{W}_-(\theta)) \in W \times W$  be an arbitrarily fixed pair. In the rest of this section we assume that  $\mathcal{W}_+(\theta)$  and  $\mathcal{W}_-(\theta)$  are both Morse functions and their graphs are transversal to the  $\theta$ -axis.

From (3.6) for  $\mathcal{R}$ , expansions are mainly emanated from the following two sources, inducing complicated dynamics in  $\Lambda$ .

- (a) The expansion induced by the singularity of  $\ln |\mathbb{F}_\sigma|$  at  $\mathbb{F}_\sigma = 0$ ;
- (b) The expansion induced by large forcing frequency  $\omega$ .

In between the two, (a) is a *singular* expansion responsible for the Smale horseshoe. (b) is a more regular expansion responsible for the rank one attractors of [WY1]-[WY2] with SRB measures and fully stochastic behavior. Let

$$M_\pm = \max_{\theta \in S^1} \mathcal{W}_\pm(\theta), \quad m_\pm = \min_{\theta \in S^1} \mathcal{W}_\pm(\theta).$$

Depending on the different combinations of  $M_\pm, m_\pm$ , we have five generic types of attractors for equation (1.3). They are

**(I) Two rank one attractors** ( $m_+, m_- > 0$ ). Under the assumption that  $m_+, m_- > 0$ , the return map  $\mathcal{R} : \Sigma_+ \cup \Sigma_- \rightarrow \Sigma_+ \cup \Sigma_-$  is decomposed into two separate maps defined on  $\Sigma_+$  and  $\Sigma_-$ , which we denote as  $\mathcal{R}_+$  and  $\mathcal{R}_-$ , respectively. We observe that in this case the stable manifold and the unstable manifold of the solution  $(0, 0, \omega t)$  in the extended phase space *do not* intersect. See Fig. 1(a). This is a case in which expansions from (b) is in full play for large  $\omega$  but that from (a) is missing in action. For  $\sigma \in \{+, -\}$ , let  $(\theta_1^\sigma, \mathbb{X}_1^\sigma) = \mathcal{R}_\sigma(\theta^\sigma, \mathbb{X}^\sigma)$ . We have from (3.6),

$$(4.5) \quad \begin{aligned} \theta_1^\sigma &= \theta^\sigma + \mathbf{a}_\sigma - \frac{\omega}{\beta} \ln \mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu) + \mathcal{O}_{\theta^\sigma, \mathbb{X}^\sigma, p}(\mu) \\ \mathbb{X}_1^\sigma &= \mathbf{b}_\sigma [\mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu)]^{\frac{\alpha}{\beta}} \end{aligned}$$

where

$$\mathbb{F}_\sigma(\theta^\sigma, \mathbb{X}^\sigma, \mu) = \mathcal{W}_\sigma(\theta^\sigma) + \mathbf{k}_\sigma \mathbb{X}^\sigma + \mathbb{E}_\sigma(\theta^\sigma, \mu) + \mathcal{O}_{\theta^\sigma, \mathbb{X}^\sigma, p}(\mu).$$

(4.5) are in the form of (2.1) and we are in the case discussed in Sect. 2.2. In particular, Propositions 2.4-2.6 and Corollary 2.2 all apply. In this case  $\Lambda$  consists of the one saddle in the middle and two other disjoint components we denote as  $\Lambda_+$  and  $\Lambda_-$ .  $\Lambda_+$  is around  $\ell_+$  and  $\Lambda_-$  is around  $\ell_-$ . For  $|\omega|$  sufficiently small,  $\Lambda_+$  and  $\Lambda_-$  are attractive invariant circles inside of  $\Sigma_+$  and  $\Sigma_-$  respectively. For  $|\omega|$  large, there exists a set of positive measure for  $\mu$  such that the corresponding attractors  $\Lambda_\pm$  are rank one attractors with SRB measures and full stochastic behavior. Through Proposition 2.5, 2.6 and Corollary 2.2, the Newhouse theory, the theory of SRB measures, the theory of Hénon-like and rank one attractors are all rigorously applied to the attractor  $\Lambda$  of equation (1.3).

**(II) One rank one attractor** ( $M_+, M_- < 0$ ;  $M_+ < 0, m_- > 0$ ;  $M_- < 0, m_+ > 0$ ). We start with the first of the three listed combinations for  $m_\pm, M_\pm$ . Under the assumption that  $M_+, M_- < 0$  we have  $\tilde{\sigma} = -\sigma$  in (3.6) for  $\mathcal{R}$ . All solutions of (1.2) initiated from  $\Sigma_+$  first follows  $\ell_+$  to reach  $\Sigma_-$ , then follows  $\ell_-$  to get back to  $\Sigma_+$ . See Fig. 1(b). In this case, expansions from (b) is in full play for large  $\omega$  and that from (a) is again missing in action.  $\mathcal{R}$  is a rank one map but it is now defined not on one but on the union of two annulus. To study the dynamics of  $\mathcal{R}$  we can proceed in two ways: one is to study  $\mathcal{R}^2$  restricting on  $\Sigma_+$ , a family of rank one maps well-defined from  $\Sigma_+$  to  $\Sigma_+$ . This is a direct approach but we would not be able to take advantage

of the rigorous results presented in Sect. 2.2 because  $\mathcal{R}^2$  restricted on  $\Sigma_+$  assumes a different form from (2.1). The second approach is through the observation that the theory developed in [WY1]-[WY3] and the results of Sect. 2.2 are indifferent if we change the domain of the maps from one annulus to the union of two, and the results presented in Sect. 2.2, including Propositions 2.4-2.6 and Corollary 2.2 all apply to  $\mathcal{R}$ . The overall dynamics in this case is similar to that of type (I) except that, instead of two separated rank one attractors staying close to  $\ell_+$  and  $\ell_-$  respectively, we have in this case one that goes around  $\ell_+$  and  $\ell_-$  alternatively.

The other two cases ( $M_- < 0, m_+ > 0$  and  $M_+ < 0, m_- > 0$ ) are simpler: For  $M_- < 0, m_+ > 0$ , nothing stays in the neighborhood of  $\ell_-$  but there is an attractor, represented by a family of 2D maps in the form of (2.1), that stays close to  $\ell_+$ . The case for  $M_+ < 0, m_- > 0$  is similar.

**(III) One tangle and one rank one attractor** ( $m_+ < 0 < M_+, m_- > 0$  or  $m_- < 0 < M_-, m_+ > 0$ ). Let us assume that  $m_+ < 0 < M_+, m_- > 0$ . In this case the perturbed stable and unstable branch from  $\ell_+$  intersect each other, forming a homoclinic tangle but the two from  $\ell_-$  are pulled apart. See Fig. 1(c). The expansions created by (a) is now in action on  $\Sigma_+$  but it does not effect the dynamics of  $\Lambda$  on  $\Sigma_-$ .

Restricted on  $\Sigma_-$ ,  $\mathcal{R}$  is  $\mathcal{R}_\sigma$  in (4.5) with  $\sigma = -$ .  $\mathcal{R}_-$  is in the form of (2.1). It induces an attractor on  $\Sigma_-$ , which we denote as  $\Lambda_-$ . All results presented in Sect. 2.2, including Propositions 2.4-2.6 and Corollary 2.2, apply to  $\Lambda_-$ .

Restricted on  $\Sigma_+$ ,  $\mathcal{R}$  is again in the form of (2.1). The equation

$$\mathbb{F}_+ = \mathcal{W}_+(\theta^+) + \mathbf{k}_+ \mathbb{X}^+ + \mathbb{E}_+(\theta^+, \mu) + \mathcal{O}_{\theta^+, \mathbb{X}^+, p}(\mu) = 0$$

divides  $\Sigma_+$  into two collections of vertical strips, one we denote as  $V = \cup V_i$  and the other we denote as  $U = \cup U_i$ .  $\mathbb{F}_+ > 0$  on  $V$  and  $\mathbb{F}_+ < 0$  on  $U$ .  $U$  is mapped by  $\mathcal{R}$  to  $\Sigma_-$  and are trapped in  $\Sigma_-$ , never coming back to  $\Sigma_+$ . The dynamics of  $\mathcal{R}$  on  $V$  is the same as the ones discussed in Sect. 2.1. We have an attractor, which we denote as  $\Lambda_+$ , that stays close to  $\ell_+$ , to which the results presented in Sect. 2.1, in particular, Propositions 2.1-2.3 and Corollary 2.1 all apply.

In short, the attractor  $\Lambda$  for equation (1.3) in this case again contain three disjoint components, one is the saddle in the middle, and the other two we denote as  $\Lambda_-$  and  $\Lambda_+$ .  $\Lambda_-$  is around  $\ell_-$ , it is defined by a family of rank one maps, to which the results of [WY1]-[WY3] and [WO], summarized in Sect. 2.2, fully apply.  $\Lambda_+$  is a homoclinic tangle, to which the results of [WOk], summarized in Sect. 2.1, fully apply.

**(IV) Mixture of two tangles** ( $m_+, m_- < 0 < M_+, M_-$ ) Assume that  $m_+, m_- < 0 < M_+, M_-$ , both  $\ell_+$  and  $\ell_-$  are broken into two intersecting curves, forming two homoclinic tangles that interact to each other. See Fig. 1(d). Here the singular expansions from (a) is in full play.

In this case,  $\mathbb{F}_\sigma = 0$ ,  $\sigma = \{+, -\}$ , divide  $\Sigma_\sigma$  into collections of vertical strips, which we denote as  $U_\sigma$  where  $\mathbb{F}_\sigma < 0$  and  $V_\sigma$  where  $\mathbb{F}_\sigma > 0$ .  $U_\sigma$  and  $V_\sigma$ ,  $\sigma \in \{+, -\}$ , are all non-empty. The action of  $\mathcal{R}$  on each vertical strip in these collections are as follows: (1) first  $\mathcal{R}$  compresses it in the vertical direction and stretches it in the horizontal

direction, making the image infinitely long towards both ends; (2)  $\mathcal{R}$  then folds the images, copying the shape of the graph of  $\theta + \omega\beta^{-1} \ln |\mathcal{W}_\sigma(\theta)|$  on the corresponding  $\theta$ -interval; and (3) this image is then put in and wrapped around  $\Sigma_{\tilde{\sigma}}$  infinitely many times where  $\tilde{\sigma} = \sigma$  if  $\mathbb{F}_\sigma > 0$  and  $\tilde{\sigma} = -\sigma$  if  $\mathbb{F}_\sigma < 0$ .

We can again think of the map (3.6) for  $\mathcal{R}$  in the Main Theorem as a family of 2D maps defined on two annulus, and regard it as unfolded from a 1D singular limit defined from  $S^1 \cup S^1$  to  $S^1 \cup S^1$ . To these maps, unfortunately, the existing theory on rank one maps in [WY1]-[WY3] do not generalize directly and the corresponding conclusions presented in Sect. 2.2 do not apply. The main obstacle here is the singularity introduced by  $\ln |\mathbb{F}_\sigma|$  in (3.6) on the vertical curves defined by  $\mathbb{F}_\sigma = 0$ . With the irregular expansions of (a) in full action, the regularity assumed for the rank one maps in [WY1]-[WY3] is violated. Though we see nothing *a priori* against the conclusions of Proposition 2.5 in this case, there should be at least a substantial overhaul, due to the exceedingly complicated nature of the theory of [WY1]-[WY2], in order to for us to claim such correspondence for  $\mathcal{R}$ .

Let us now turn to what we can rigorously prove through (3.6) for  $\mathcal{R}$ . We observe that Propositions 2.1 and 2.2 and Corollary 2.1 remain valid for  $\mathcal{R}$ . By deleting vertical strips around the critical points of the 1D limit  $f : S^1 \cup S^1 \rightarrow S^1 \cup S^1$  in (3.13) and the curves defined by  $\mathbb{F}_\pm = 0$ , we obtain a collection of vertical strips from  $\Sigma = \Sigma_+ \cup \Sigma_-$ . The invariant subset defined by these vertical strips through the iterations of  $\mathcal{R}$  is uniformly hyperbolic, which we refer to as *the* Smale horseshoe. It is again *the* complicated structure that is created by transversal intersections of the stable and unstable manifolds depicted in Fig. 1(d). The traditional horseshoes constructed by Smale in [S] and others are part of this structure. Second, let  $q$  be a periodic saddle in the Smale horseshoe, then the unstable manifold and the stable manifold of  $q$  form non-degenerate, transversal homoclinic tangency for infinitely many values of  $\mathbf{a} := \omega\beta^{-1} \ln \mu^{-1}$  as  $\mathbf{a} \rightarrow \infty$ . The proof of this statement is the same as that of Proposition 2.2 in Sect. 2.1. Corollary 2.1 then follows from Proposition 2.2, and the Newhouse theory and the theory of Hénon-like attractors are again applied rigorously to  $\Lambda$  in this case.

**(V) One tangle mixed with one rank one map** ( $m_+ < 0 < M_+, M_- < 0$ ;  $m_- < 0 < M_-, M_+ < 0$ ) Assume  $m_+ < 0 < M_+, M_- < 0$ . Similar to (III), in this case  $\ell_+$  is broken into two intersecting curves but the stable and unstable manifold of  $(0, 0, \omega t)$  from  $\ell_-$  are pulled apart. However, with  $M_- < 0$ , the unstable branch from  $\ell_-$  will come back to trace  $\ell_+$ . See Fig. 1(e).

In this case,  $\mathbb{F}_+ = 0$  again divide  $\Sigma_+$  into two collections of vertical strips  $V_+ = \cup V_i$  where  $\mathbb{F}_+ > 0$  and  $U_+ = \cup U_i$  where  $\mathbb{F}_+ < 0$ . Under  $\mathcal{R}$  in (3.6), the image of  $V_i$  wraps around  $\Sigma_+$  and the image of  $U_i$  wraps around  $\Sigma_-$ . On  $\Sigma_-$ ,  $\mathcal{R}$  is completely well-defined.  $\Sigma_-$  is compressed in the vertical direction and the horizontal shape of the image copies the shape of the graph of the 1D limit  $f_-(\theta) = \theta + \mathbf{a} + \ln |\mathcal{W}_-(\theta)|$ . This image is then put into  $\Sigma_+$ .

To investigate the structure of  $\Lambda$  we decompose  $\Lambda$  into three disjoint parts. The first part, which we denote as  $\Lambda(V_+)$ , consists of all solutions in  $\Lambda$  that stay close to

$\ell_+$  for all time.  $\Lambda(V_+)$  is exactly the one homoclinic tangle studied in [WOk]. All results presented in Sect. 2.1, in particular, Propositions 2.1-2.3 and Corollary 2.1 applies to  $\Lambda(V_+)$ .

The second part of  $\Lambda$ , which we denote as  $\Lambda(U_+)$ , consists all orbits of  $\mathcal{R}$  that return to  $\Sigma_+$  in  $U_+$  without ever hitting  $V_+$ . Again by using (3.6) we can construct a uniformly hyperbolic invariant subset for all  $\mu$  in  $\Lambda(U_+)$  using the singular expansion (a). We can then further obtain infinitely many values of  $\mu$ , for which the stable and the unstable manifold of a periodic saddle inside of this uniformly hyperbolic set form transversal non-degenerate homoclinic tangency, leading again to the existence of Newhouse sinks and Hénon-like attractors in  $\Lambda(U_+)$ . We note that all solutions inside of  $\Lambda(U_+)$  go around  $\ell_+$  and  $\ell_-$  alternatively.

The third part of  $\Lambda$ , which we denote as  $\tilde{\Lambda}$ , is a much larger set than  $\Lambda(V_+) \cup \Lambda(U_+)$ . It consists all orbits that switches back and forth between  $V_+$  and  $U_+$  under the iterations of  $\mathcal{R}$ . From (3.6) for  $\mathcal{R}$ , we again obtain a uniformly hyperbolic invariant set by using the singular expansion of (a). In fact for any give bi-infinite sequences of positive integers, there is an orbit of  $\mathcal{R}$  hitting  $V_+$  and  $U_+$  alternatively with the assigned number of hits according to the given sequence. These solutions will stay close and go around  $\ell_+$  for the arbitrarily assigned number of times, then go around  $\ell_+$  and  $\ell_-$  alternatively as many times as the next number in the given sequence, and so on. Again, let  $q$  be a periodic saddle in  $\tilde{\Lambda}$ , the stable manifold and the unstable manifold of  $q$  form transversal, non-degenerate homoclinic tangency for infinitely many  $\mu$  regarding  $\ln \mu^{-1}$  as a parameter, leading again, correspondingly, to Newhouse sinks and Hénon-like attractors in  $\tilde{\Lambda}$ .

When  $\omega$  is large, this case is similar to that of (IV) above in the sense that  $\mathcal{R}$  is a family of rank one maps with singularity on the set defined by  $\mathbb{F}_+ = 0$  on  $\Sigma_+$ . Though we expect  $\mathcal{F}_a$  admits SRB measures and full stochastic behavior on  $\Lambda$ , we can not apply the theory of rank one maps developed in [WY1]-[WY2].

## 5. ANALYSIS OF A PERIODICALLY PERTURBED DUFFING EQUATION

In this section we use a periodically perturbed Duffing equation to illustrate how to apply the Main Theorem and the conclusions of Section 4 to the analysis of a given equation. We start with the Duffing equation

$$(5.1) \quad \frac{d^2q}{dt^2} - q + q^3 = 0.$$

Let  $p = \frac{dq}{dt}$ , (5.1) has two homoclinic solution to  $(q, p) = (0, 0)$  and they are  $\ell_{\pm} = (q_{\pm}(t), p_{\pm}(t))$  where

$$(5.2) \quad q_{\pm}(t) = \pm \frac{2\sqrt{2}e^t}{1 + e^{2t}}, \quad p_{\pm}(t) = \pm \frac{2\sqrt{2}(e^t - e^{3t})}{(1 + e^{2t})^2}.$$

We add a non-linear damping term to (5.1) to form a new autonomous equation

$$(5.3) \quad \frac{d^2q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^3 = 0$$

where  $\lambda, \gamma > 0$  are parameters. Denote  $p = \frac{dq}{dt}$ .  $(q, p) = (0, 0)$  is now a dissipative saddle.

**Proposition 5.1.** *There exists  $\varepsilon_0 > 0$  sufficiently small, such that for  $\lambda \in [0, \varepsilon_0)$ , there exists a  $\gamma_\lambda, |\gamma_\lambda| \leq K\lambda$  such that for  $\gamma = \gamma_\lambda$ , equation (5.3) has two homoclinic solution for  $(q, p) = (0, 0)$ .*

**Proof:** Re-write equation (5.3) as

$$\frac{d^2q}{dt^2} - q + q^3 = -\lambda(1 - \gamma\lambda^{-1}q^2)\frac{dq}{dt}.$$

Let  $\tau := \gamma\lambda^{-1}$ . Regard  $\lambda$  as the magnitude of an autonomous perturbation. The standard Melnikov function for  $\ell_+ = (q_+(t), p_+(t))$  is

$$\mathcal{W} = \int_{-\infty}^{\infty} (1 - \tau q_+^2(s)) p_+^2(s) ds.$$

We obtain a unique  $\tau$  for a homoclinic solution for (5.3) provided that

$$\int_{-\infty}^{\infty} q_+^2(s) p_+^2(s) ds \neq 0,$$

which is clearly the case here. Denote this homoclinic solution as  $\ell_{+,\lambda} = (q_{+,\lambda}(t), p_{+,\lambda}(t))$  and observe that  $\ell_{-,\lambda} = (-q_{+,\lambda}(t), -p_{+,\lambda}(t))$  is also a solution for (5.3).  $\square$

Let  $0 < \lambda_0 < \varepsilon_0$  be fixed and  $\gamma_0 = \gamma_{\lambda_0}$  be as in Proposition 5.1.  $(q, p) = (0, 0)$  is a dissipative saddle and the eigenvalues are  $-\alpha$  and  $\beta$  where

$$(5.4) \quad \alpha = \frac{1}{2}(\sqrt{\lambda_0^2 + 4} + \lambda_0), \quad \beta = \frac{1}{2}(\sqrt{\lambda_0^2 + 4} - \lambda_0).$$

We fix a  $\lambda_0$  such that  $\alpha, \beta$  are non-resonant up to order  $r$  for some  $r > 3$ .  $(q, p) = (0, 0)$  is a dissipative saddle because  $-\alpha + \beta = -\lambda_0 < 0$ . We add two more perturbations, one autonomous as an additional damping term and the other time periodic, to equation (5.3) to obtain a new equation

$$(5.5) \quad \frac{d^2q}{dt^2} + (\lambda_0 - \gamma q^2 - \hat{\gamma} q^3) \frac{dq}{dt} - q + q^3 = \mu q^2 \sin 2\pi\omega t$$

where  $\mu$  is a parameter representing the magnitude of the forcing and  $\omega$  is the forcing frequency. The term  $\hat{\gamma} q^3$  is introduced to allow various dynamical scenarios of Sect. 4.2.

To apply the main theorem to (5.5), first we introduce two new parameters  $\rho, \hat{\rho}$  to replace  $\gamma$  and  $\hat{\gamma}$  as follows. We fix a pair  $(\lambda_0, \gamma_0)$  from Proposition 5.1 and write

$$\gamma = \gamma_0 - \mu\rho, \quad \hat{\gamma} = -\mu\hat{\rho}.$$

We re-write equation (5.5) as

$$(5.6) \quad \frac{d^2q}{dt^2} + (\lambda_0 - \gamma_0 q^2) \frac{dq}{dt} - q + q^3 = \mu(-\rho q^2 \frac{dq}{dt} - \hat{\rho} q^3 \frac{dq}{dt} + q^2 \sin 2\pi\omega t).$$

In what follows,  $\lambda_0, \gamma_0$  are fixed and the parameters we vary are  $\rho, \hat{\rho}, \omega$  and  $\mu$ . We assume that  $|\rho|, |\hat{\rho}|, \omega \ll \mu^{-1}$ . To write (5.6) in the form of (1.3), we introduce new phase variables  $(x, y)$  by letting

$$(5.7) \quad x = q + p, \quad y = -\alpha q - \beta p.$$

In reverse,

$$(5.8) \quad q = \frac{1}{\alpha + \beta}(\beta x - y), \quad p = \frac{1}{\alpha + \beta}(\alpha x + y)$$

and the equation in  $(x, y)$  are in the form of

$$(5.9) \quad \frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y, t), \quad \frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y, t),$$

where  $\alpha, \beta$  are as in (5.4),

$$(5.10) \quad \begin{aligned} f(x, y) &= -\frac{1}{(\alpha + \beta)^3}(\beta x - y)^2[\gamma_0(\alpha x + y) + (\beta x - y)] \\ g(x, y) &= -\frac{\beta}{(\alpha + \beta)^3}(\beta x - y)^2[\gamma_0(\alpha x + y) + (\beta x - y)], \end{aligned}$$

and

$$(5.11) \quad \begin{aligned} P(x, y, t) &= \frac{-\rho}{(\alpha + \beta)^3}(\beta x - y)^2(\alpha x + y) + \frac{-\hat{\rho}}{(\alpha + \beta)^3}(\beta x - y)^3 \\ &\quad + \frac{1}{(\alpha + \beta)^2}(\beta x - y)^2 \sin 2\pi\omega t \\ Q(x, y, t) &= \frac{-\beta\rho}{(\alpha + \beta)^3}(\beta x - y)^2(\alpha x + y) + \frac{-\beta\hat{\rho}}{(\alpha + \beta)^3}(\beta x - y)^2 \\ &\quad + \frac{\beta}{(\alpha + \beta)^2}(\beta x - y)^2 \sin 2\pi\omega t. \end{aligned}$$

By Proposition 5.1, the unperturbed equation, obtained by setting  $\mu = 0$  in (5.9), has two homoclinic solutions to  $(x, y) = (0, 0)$ , which we denote as

$$\ell_{\pm, \lambda_0} = \{(x, y) = (a_{\pm, \lambda_0}(t), b_{\pm, \lambda_0}(t))\}.$$

In comparison to the notations in Section 3, the subscript  $\lambda_0$  is added to mark the dependency on  $\lambda_0$ .

We are now ready to apply the Main Theorem. From (5.9)-(5.11), (5.4) and Proposition 5.1, the return map  $\mathcal{R}$  assumes the form of (3.6) and the Melnikov functions  $\mathcal{W}_{\pm}(\theta)$  are

$$(5.12) \quad \begin{aligned} \mathcal{W}_+(\theta) &= A_{\lambda_0}\rho + B_{\lambda_0}\hat{\rho} + \sqrt{J_{c, \lambda_0}^2(\omega) + J_{s, \lambda_0}^2(\omega)} \sin \theta \\ \mathcal{W}_-(\theta) &= A_{\lambda_0}\rho - B_{\lambda_0}\hat{\rho} + \sqrt{J_{c, \lambda_0}^2(\omega) + J_{s, \lambda_0}^2(\omega)} \sin \theta. \end{aligned}$$

(5.12) is directly from (3.2) for  $\mathcal{W}_{\pm}(\theta)$ , using (5.10) for  $f(x, y), g(x, y)$  and (5.11) for  $P(x, y, t)$  and  $Q(x, y, t)$ .  $A_{\lambda_0}, B_{\lambda_0}, J_{s, \lambda_0}, J_{c, \lambda_0}$  are constants derived directly by using (3.2) for  $\mathcal{W}_{\pm}(\theta)$ .  $A_{\lambda_0}, B_{\lambda_0}$  are independent of  $\omega$  but  $J_{s, \lambda_0}, J_{c, \lambda_0}$  are dependents of

$\omega$ . Here we save the trouble of writing in explicit terms the long formula for these constants, which we will not need for the rest of this study except in the case of  $\lambda_0 = 0$ .  $\ell_{-, \lambda_0}(t) = -\ell_{+, \lambda}(t)$  is used for  $\mathcal{W}_-(\theta)$  in (5.12).

We also have

**Proposition 5.2.** *Let  $\omega > 0$  be fixed. Then there exists a  $\delta(\omega) > 0$  so that for all  $0 < \lambda_0 < \delta(\omega)$ , (i)  $A_{\lambda_0}, B_{\lambda_0} > 0$  and (ii)  $J_{c, \lambda_0}^2 + J_{s, \lambda_0}^2 \neq 0$ .*

**Proof:** Though  $A_{\lambda_0}, B_{\lambda_0}, J_{c, \lambda_0}(\omega), J_{s, \lambda_0}(\omega)$  are not explicitly computable for  $\lambda_0 > 0$ , their respective values at  $\lambda_0 = 0$  are attainable through direct computations. The estimates of this Proposition are then proved through the continuous dependency of the respective quantities on  $\lambda_0$ .

In the case of  $\lambda_0 = 0$ , the unperturbed equation is the Duffing equation, which we write in Hamiltonian form as

$$\frac{dq}{dt} = \partial_p H(q, p), \quad \frac{dp}{dt} = -\partial_q H(q, p)$$

where

$$H(q, p) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + \frac{1}{4}q^4.$$

Using the second remark in Sect. 3D, we have for  $\lambda_0 = 0$ ,

$$\mathcal{W}_+(\theta) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} p(s)(\rho q^2(s)p(s) + \hat{\rho} q^3(s)p(s) + q^2(s) \sin(2\pi\omega s + 2\pi\theta)) ds,$$

from which we obtain

$$A_0 = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} q^2(s)p^2(s) ds \quad B_0 = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} q^3(s)p^2(s) ds;$$

$$J_{c,0} = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} p(s)q^2(s) \cos(2\pi\omega s) ds \quad J_{s,0} = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} p(s)q^2(s) \sin(2\pi\omega s) ds$$

where  $(q(t), p(t)) = (q_+(t), p_+(t))$  is as in (5.3). From these explicit formula we obtain both (i) and (ii) of this proposition for  $\lambda_0 = 0$ .  $\square$

To apply the result of Section 4, we start with

**Proposition 5.3.** *Recall that  $M_{\pm} = \max_{\theta \in S^1} \mathcal{W}_{\pm}(\theta)$ ,  $m_{\pm} = \min_{\theta \in S^1} \mathcal{W}_{\pm}$ , then*

$$(5.13) \quad \begin{aligned} M_{\pm} &= A_{\lambda_0} \rho \pm B_{\lambda_0} \hat{\rho} + J \\ m_{\pm} &= A_{\lambda_0} \rho \pm B_{\lambda_0} \hat{\rho} - J \end{aligned}$$

where

$$J = \sqrt{J_{c, \lambda_0}^2(\omega) + J_{s, \lambda_0}^2(\omega)}.$$

**Proof:** This follows directly from (5.12).  $\square$

We finally have

**Theorem 2.** *The four straight lines defined by*

$$M_{\pm} = A_{\lambda_0}\rho \pm B_{\lambda_0}\hat{\rho} + J = 0, \quad m_{\pm} = A_{\lambda_0}\rho \pm B_{\lambda_0}\hat{\rho} - J = 0,$$

*divide the  $(\hat{\rho}, \rho)$ -plane into nine parameter regions, and on each of these parameter regions the dynamics of the attractor  $\Lambda$  for equation (5.6) is as one of the five types listed in Sect. 4.2. See Fig. 6 for the specific type for each of the nine parameter regions.*

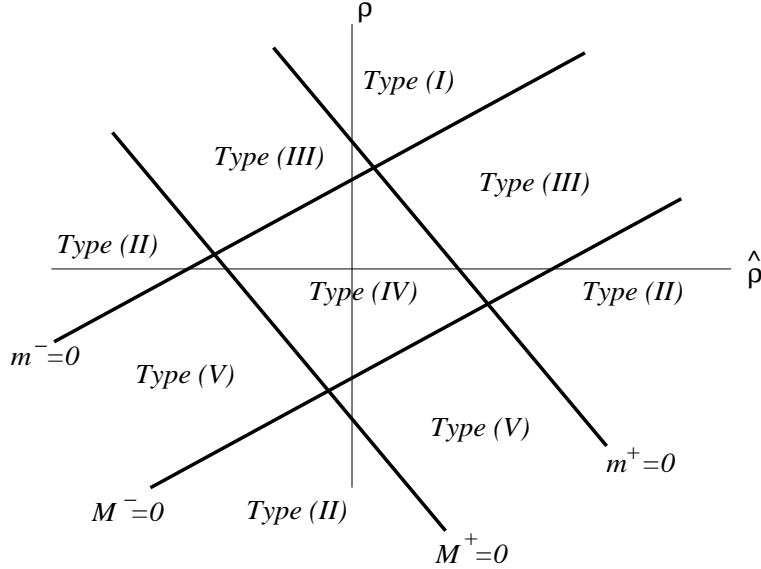


Fig. 6 Dynamic scenarios on  $(\hat{\rho}, \rho)$ -plane

**Proof:** The classifications of dynamical types of the attractors  $\Lambda$  illustrated in Fig. 6 follow directly from the conditions for each of dynamical scenarios in term of different combinations of  $M_{\pm}, m_{\pm}$  presented in Sect. 4.2 and Proposition 5.3. There is, however, one potentially serious obstacle in applying the conclusions of Sect. 4.2 to the analysis of a given equation such as (5.6). In Sect. 4.2, all dynamical scenarios are derived by using (3.6) for  $\mathcal{R}$ , in which we first fix  $\mathcal{W}_{\pm}(\theta)$ , then vary  $\omega$  independently without changing  $\mathcal{W}_{\pm}(\theta)$ . In the current analysis,  $\mathcal{W}_{\pm}(\theta)$  do depend on  $\omega$ , and with  $J_{c,\lambda_0}(\omega)$  and  $J_{s,\lambda_0}(\omega)$  decaying exponentially with respect to  $\omega$ , it is not clear if the expansion introduced by large  $\omega$  in Sect. 4.2 could materialize. This is in fact not a problem for equation (5.6) because  $\rho, \hat{\rho}$  are parameters independent of  $\omega$ . From (5.12) we re-write  $\mathcal{W}_{\pm}(\theta)$  as

$$\mathcal{W}_{\pm}(\theta) = J (A_{\lambda_0}\rho J^{-1} \pm B_{\lambda_0}\hat{\rho} J^{-1} + \sin 2\pi\theta)$$

where  $J = \sqrt{J_{c,\lambda_0}^2(\omega) + J_{s,\lambda_0}^2(\omega)}$ . As far as the expansions in map (3.6) is concerned, the factor  $J$  in front is not relevant and the rest of the  $\mathcal{W}_{\pm}(\theta)$  are made independent of  $\omega$  by our fixing the values of  $\rho J^{-1}$  and  $\hat{\rho} J^{-1}$ .  $\square$

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