Math 464 - Homework 1

1. Let A, B and C be events.

(a) Write each of the following events in terms of A, B, C using intersections, unions and complements:

(i) None of A, B, C occurs.

(ii) At least one of A, B, C occurs.

(iii) Exactly one of A, B, C occurs.

(iv) Exactly two of A, B, C occur.

(v) Exactly three of A, B, C occur.

(b) Now suppose

$$P(A) = \tfrac{5}{10} \qquad P(B) = \tfrac{7}{10} \qquad P(C) = \tfrac{6}{10}$$
$$P(A \cap B) = \tfrac{3}{10} \qquad P(B \cap C) = \tfrac{4}{10} \qquad P(A \cap C) = \tfrac{2}{10}$$
$$P(A \cap B \cap C) = \tfrac{1}{10}$$

Find the probability of each of the events (i) to (v) in part (a).

2. A point (X, Y) is picked at random (uniformly) from the rectangle whose corners are $(0, 0), (2, 0), (0, 1), (2, 1)$.

(a) What is the probability that $X > Y$?

(b) What is the probability that $X > Y$ given that $X \leq 1$?

3. We roll a six-sided die until we get a 3.

(a) What is the probability it takes exactly 4 rolls?

(b) What is the probability it takes at least 4 rolls?

(c) What is the probability it takes exactly 10 rolls given that it takes at least 4 rolls?

4. If A, B and C are events and P is a probability measure, show that

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

This is a special case of a more general result called the "inclusion-exclusion formula".

5.

(a) Prove that for all events A and B with $P(B) > 0$, we have

$$P(A^c|B) = 1 - P(A|B).$$

You should not use the fact (which we proved in class) that conditional probability is a probability measure. You should give a direct proof using the denition of conditional probability.

(b) Show that $P(A|B^c)$ is not always equal to $1 - P(A|B)$. (It suffices to give a counterexample.)

6. In a hospital 1% of the patients have disease X. There is a test for disease X. If a patient has X, the test will be positive 90% of the time. If a patient does not have X, the test will be positive 5% of the time. (This is called a *false positive.*) If a randomly chosen patient has a positive test, what is the probability he or she has disease X?

7. Four mathematicians are hiding a coin in each of their hands. The coins are all silver or gold. Two of them have one gold and one silver coin. One of them has two gold coins. One of them has two silver coins. We pick a mathematician at random and pick one of his or her hands at random. The coin in that hand is revealed to be gold.

(a) What is the probability the coin in the other hand is also gold?

(b) What is the probability the coin in the other hand is silver?

8. An urn contains 10 red balls and 5 yellow balls. In all parts of this problem we will draw two balls, although in different ways. Let E be the event *both balls are red*, F the event that *both balls are yellow* and G the event that *one is red, one is yellow.*

(a) I draw two balls at once from the urn. What are the probabilities of E, F and G?

(b) Now I draw one ball from the urn, note its color, put it back in the urn and then draw another ball. What are the probabilities of E, F and G? (This is called *sampling with replacement.*)

(c) Now I draw one ball from the urn, note its color, do not put it back in the urn and then draw another ball. What are the probabilities of E, F and G? (This is called *sampling without replacement.*)