Math 464 Hw 3

1. Since $\sum_{k=0}^{\infty} \pi_k = \sum_{k=0}^{\infty} ce^{-\lambda} \cdot \lambda^{2k}/k!$

$$= ce^{-\lambda} \sum_{k=0}^{\infty} (\lambda^2)^k/k!$$

$$= ce^{-\lambda} \cdot e^{\lambda^2} = ce^{\lambda^2-\lambda} = 1$$

$$c = \frac{1}{e^{\lambda^2-\lambda}} = e^{\lambda-\lambda^2}$$

2.

X	1	2	3	....	$2n+1$
	HH	THH	HTHH	....	$\underbrace{HT \cdots HT}_{2n}$ HH
	TT	HTT	THTT	....	$\underbrace{TH \cdots TH}_{2n}$ TT

$$P(X=N-1) = \frac{1}{2^N} \cdot 2 = \frac{1}{2^{N-1}}$$

$k = N-1 = 1, 2, 3, \cdots$

$$P(X=k) = \frac{1}{2^k} = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{k-1} = p \cdot q^{k-1} \quad \text{where } p = \frac{1}{2}$$

3. a. $P(X \text{ is even}) = P(X=0) + P(X=2) + P(X=8)$

$$= 0.15 + 0.1 + 0.05 = 0.3$$

b $P(1 \leq X \leq 8) = P(X=1) + P(X=2) + P(X=3) + P(X=5) + P(X=8)$

$$= 0.2 + 0.1 + 0.15 + 0.05 + 0.05$$

$$= 0.55$$

c. $P(X=-3 \mid X \le 0)$

$= P\{(X=-3) \cap (X \le 0)\} / P(X \le 0)$

$$= \frac{P(X=-3)}{P(X \le 0)} = \frac{P(X=-3)}{P(X=-3)+P(X=-1)+P(X=0)} = \frac{0.1}{0.1+0.2+0.15} = 0.222$$

d $P(X \ge 3 \mid X > 0)$

$$= \frac{P\{(X \ge 3) \cap (X > 0)\}}{P(X>0)}$$

$$= \frac{P(X \ge 3)}{P(X>0)} = \frac{P(X=3)+P(X=5)+P(X=8)}{P(X=1)+P(X=2)+P(X=3)+P(X=5)+P(X=8)}$$

$$= \frac{0.15+0.05+0.05}{0.2+0.1+0.15+0.05+0.05} = \frac{0.25}{0.55} = \frac{5}{11}$$

4. X : # of eggs laid by an insect (Poisson with parameter λ)

Each egg produces an insect with probability p (eggs are independent of each of them)

Y: # of insects that hatch from the X eggs.

$$P(Y=k \mid X=n) = \binom{n}{k} p^k q^{n-k}$$

$$P(Y=k) = \sum_{n=k}^{\infty} P(Y=k \mid X=n) \cdot P(X=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k \cdot q^{n-k} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{n!}{k!\,(n-k)!} \cdot p^k \cdot q^{n-k} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$= \frac{e^{-\lambda} \cdot p^k}{k!} \sum_{n=k}^{\infty} \frac{q^{n-k} \cdot \lambda^n}{(n-k)!}$$

$$= \frac{e^{-\lambda} \cdot p^k}{k!} \sum_{n=0}^{\infty} \frac{q^n \lambda^{n+k}}{n!}$$

$$= \frac{e^{-\lambda} \cdot p^k \cdot \lambda^k}{k!} \sum_{n=0}^{\infty} \frac{q^n \cdot \lambda^n}{n!}$$

$$= \frac{e^{-\lambda} \cdot p^k \cdot \lambda^k}{k!} \sum_{n=0}^{\infty} \frac{(q\lambda)^n}{n!}$$

$$= \frac{e^{-\lambda} \cdot p^k \cdot \lambda^k}{k!} \cdot e^{q\lambda}$$

$$= \frac{e^{q\lambda-\lambda} \cdot (\cancel{\lambda q}\,\lambda p)^k}{k!} \overset{q=1-p}{=\!=} e^{-\lambda p} \frac{(\lambda p)^k}{k!}$$

Therefore, Y is Poisson with parameter λp.

5 $X \sim \text{Poisson}(\lambda) \qquad \lambda > 0$

$$f(x) = \frac{\lambda^x}{x!} \exp(-\lambda) \qquad x = \{0, 1, \cdots\}$$

(a) $$EX = \sum_{x=0}^{\infty} x f(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x}{x!} \exp(-\lambda)$$

$$= \exp(-\lambda) \cdot \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \overset{s=x-1}{=\!=} \exp(-\lambda) \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{s!} = \exp(-\lambda) \cdot \lambda \cdot \sum_{s=0}^{\infty}$$

$$\overset{\text{Taylor Series}}{=\!=\!=} \exp(-\lambda) \cdot \lambda \cdot \exp(\lambda) = \lambda$$

$$EX^2 = \sum_{x=0}^{\infty} x^2 f(x)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot \frac{\lambda^x}{x!} \exp(-\lambda)$$

$$= \exp(-\lambda) \cdot \sum_{x=1}^{\infty} x^2 \cdot \frac{\lambda^x}{x!}$$

$$= \exp(-\lambda) \cdot \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x}{(x-1)!}$$

$$= \exp(-\lambda) \cdot \sum_{x=1}^{\infty} \left[(x-1) \cdot \frac{\lambda^x}{(x-1)!} + \frac{\lambda^x}{(x-1)!} \right]$$

$$= \exp(-\lambda) \cdot \left[\sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} + \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} \right]$$

From the computation of EX, we know that $\sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \lambda \cdot \exp(\lambda)$

And $$\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} \overset{t=x-2}{=\!=} \sum_{t=0}^{\infty} \frac{\lambda^{t+2}}{t!} = \lambda^2 \cdot \sum_{t=0}^{\infty} \frac{\lambda^t}{t!} = \lambda^2 \cdot \exp(\lambda)$$

Therefore $$EX^2 = \exp(-\lambda) \cdot \left[\lambda \exp(\lambda) + \lambda^2 \exp(\lambda) \right]$$

$$= \lambda + \lambda^2$$

$$\mathrm{Var}\,X = EX^2 - (EX)^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

b) $Y = X+1$

$$EY = E(X+1) = EX+1 = \lambda+1$$

$$EY^2 = E[(X+1)^2] = E(X^2+2X+1)$$

$$= EX^2 + 2EX + 1 = \lambda + \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 + \lambda$$

$$\mathrm{Var}\,Y = EY^2 - (EY)^2$$

$$= (\lambda+1)^2 + \lambda - (\lambda+1)^2 = \lambda$$

c) $Y = 2X$

$$EY = E(2X) = 2EX = 2\lambda$$

$$EY^2 = E[(2X)^2] = E(4X^2) = 4EX^2 = 4\cdot(\lambda+\lambda^2)$$

$$\mathrm{Var}\,Y = EY^2 - (EY)^2 = 4(\lambda+\lambda^2) - 4\lambda^2 = 4\lambda$$

d) $Y = -2X+1$

$$EY = E(-2X+1) = -2EX+1 = -2\lambda+1$$

$$EY^2 = E[(-2X+1)^2] = E(4X^2-4X+1) = 4EX^2 - 4EX + 1$$

$$= 4(\lambda+\lambda^2) - 4(\overset{\lambda}{\cancel{-2\lambda+1}}) + 1$$

$$= 4\lambda^2 + 1$$

$$\therefore \mathrm{Var}\,Y = EY^2 - (EY)^2 = 4\lambda^2 + 1 - (-2\lambda+1)^2 = 4\lambda$$

6. (a)

Y	-3	-1	1	3
$f(y)$	$C_3^0(\frac{1}{2})^0(\frac{1}{2})^3$	$C_3^1(\frac{1}{2})^1(\frac{1}{2})^2$	$C_3^2(\frac{1}{2})^2\cdot\frac{1}{2}$	$C_3^3\cdot(\frac{1}{2})^3\cdot(\frac{1}{2})^0$
	$\parallel$	$\parallel$	$\parallel$	$\parallel$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$EY = \sum y f(y)$$

$$= -3\times\frac{1}{8} + (-1)\times\frac{3}{8} + 1\times\frac{3}{8} + 3\times\frac{1}{8} = 0$$

$$EY^2 = \sum y^2 f(y)$$

$$= (-3)^2\times\frac{1}{8} + (-1)^2\cdot\frac{3}{8} + 1^2\times\frac{3}{8} + 3^2\cdot\frac{1}{8}$$

$$= \frac{9}{8} + \frac{3}{8} + \frac{3}{8} + \frac{9}{8} = 3$$

$$\mathrm{Var}\,Y = EY^2 - (EY)^2 = 3 - 0^2 = 3$$

(b)

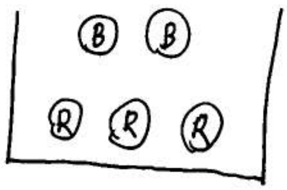
Y	-1	1
$f(y)$	$C_3^0\cdot(\frac{1}{2})^0(\frac{1}{2})^3 + C_3^2(\frac{1}{2})^2\cdot\frac{1}{2}$	$C_3^1(\frac{1}{2})(\frac{1}{2})^2 + C_3^2(\frac{1}{2})^3\cdot(\frac{1}{2})^0$
	$\parallel$	$\parallel$
	$\frac{1}{2}$	$\frac{1}{2}$

$$EY = \sum y f(y) = (-1)\times\frac{1}{2} + 1\times\frac{1}{2} = 0$$

$$EY^2 = \sum y^2 f(y) = (-1)^2\times\frac{1}{2} + 1^2\times\frac{1}{2} = 1$$

$$\mathrm{Var}\,Y = EY^2 - (EY)^2 = 1$$

7.



After the two steps, the box has 7 balls.

Image $(X) = 3, 4, 5$

Image $(Y) = 2, 3, 4$

a) $p(X=3) = \frac{2}{5} \cdot \frac{3}{6} = \frac{1}{5}$

$p(X=4) = \frac{3}{5} \cdot \frac{2}{6} + \frac{2}{5} \cdot \frac{3}{6} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

$P(X=5) = \frac{3}{5} \cdot \frac{4}{6} = \frac{2}{5}$

b) $EX = 3 \cdot \frac{1}{5} + 4 \cdot \frac{2}{5} + 5 \cdot \frac{2}{5} = \frac{21}{5}$

c) $p(Y=2) = \frac{3}{5} \cdot \frac{4}{6} = \frac{2}{5}$

$p(Y=3) = \frac{2}{5} \cdot \frac{3}{6} + \frac{3}{5} \cdot \frac{2}{6} = \frac{2}{5}$

$P(Y=4) = \frac{2}{5} \cdot \frac{3}{6} = \frac{1}{5}$

d) $EY = 2 \cdot \frac{2}{5} + 3 \cdot \frac{2}{5} + 4 \cdot \frac{1}{5} = \frac{14}{5}$

9. $P(A) = p^4 + \binom{4}{1} p^3 \cdot (1-p)$

$= p^4 + 4p^3(1-p)$

$= 4p^3 - 3p^4$

$P(B) = 1 - p^4 - (1-p)^4$

$= 1 - p^4 - (1 - 4p + 6p^2 - 4p^3 + p^4)$

$= 4p - 6p^2 + 4p^3 - 2p^4$

$P(A \cap B) = \binom{4}{1} p^3 \cdot (1-p) = 4p^3 - 4p^4$

A and B are independent

$\Leftrightarrow \quad P(A \cap B) = P(A) \cdot P(B)$

$\Leftrightarrow \quad 4p^3 - 4p^4 = (4p^3 - 3p^4)(4p - 6p^2 + 4p^3 - 2p^4)$

$\Leftrightarrow \quad 3p^5 - 10p^4 + 17p^3 - 18p^2 + 10p - 2 = 0$ Suppose $f(p) = 3p^5 - 10p^4 + 17p^3 - 18p^2 + 10p - 2$

Since ~~pt~~ $f(0) = -2 < 0$ $\quad f(1) = 3 - 10 + 17 - 18 + 10 - 2 = 0$

$f(\frac{1}{2}) = 0.09375 > 0$

So there must be exists $p \in (0, \frac{1}{2})$ such ~~st~~ that $f(p) = 0$ according to the ~~Mean~~ Intermediate Value Theorem.

Hence, there is a value for p for which A and B are independent events.

Another way: Solve the polynomial

$3p^5 - 10p^4 + 17p^3 - 18p^2 + 10p - 2 = 0 \quad \Rightarrow p = 0.411332$

[$p \in (0,1)$ & p is a real]

10. $X \sim$ geometric distribution

$$f(x) = (1-p)^{x-1}p \qquad (x = 1, 2, 3 \cdots)$$

(a) $P(X>n)$

$$= \sum_{x=n+1}^{\infty} (1-p)^{x-1}p = \frac{p(1-p)^n}{1-(1-p)} = (1-p)^n$$

(b) $P(X>n+k \mid X>n)$

$$= \frac{P((X>n+k) \cap (X>n))}{P(X>n)} = \frac{P(X>n+k)}{P(X>n)} = \frac{(1-p)^{n+k}}{(1-p)^n}$$

$$= (1-p)^k = P(X>k)$$

c) Since the coin one tosses does not have a "memory" of these failures, the conditional probability distribution of the number of additional trials does not depend on how many failures have been observed. In fact, the geometric distribution is ~~in fact~~ the only memoryless discrete distribution