Math 464 - Homework 3

1. A probability distribution has values $0, 1, 2, 3, \ldots$ with probabilities $\pi_k = ce^{-\lambda}\lambda^{2k}/k!$ for a certain c. What is c?

2. We toss a fair coin until we get two heads in a row or two tails in a row. Let N be the total number of tosses and let $X = N - 1$. It always takes at least two tosses, so the image of X is $1, 2, 3, \cdots$. Show that X is a geometric random variable and find the value of the parameter p.

3. The probability mass function of a discrete random variable X is given in the table below

x	-3	-1	0	1	2	3	5	8
f(x)	0.1	0.2	0.15	0.2	0.1	0.15	0.05	0.05

Compute the following

a. the probability X is even.

b. the probability that $1 \leq X \leq 8$.

c. the probability that X is -3 given that $X \leq 0$.

d. the probability that $X \geq 3$ given that $X > 0$.

4. Let X be the number of eggs laid by an insect. We suppose that X is a Poisson random variable with parameter λ. Each egg produces an insect with probability p. The eggs are independent of each other. Let Y be the number of insects that hatch from the X eggs. Show that Y is also a Poisson random variable and find its parameter (in terms of λ and p). *Hint*: Use the partition theorem:

$$P(Y = k) = \sum_{n=k}^{\infty} P(Y = k|X = n)P(X = n)$$

and argue that if we are given that $X = n$ then Y is a binomial random variable.

5. Let X be a Poisson random variable with parameter λ. a) Find the mean and variance of X. b) Let $Y = X+1$. Find the expectation $\mathbb{E}(Y)$ and variance

var(Y). c) Let $Y = 2X$. Find $\mathbb{E}(Y)$ and var(Y). d) Let $Y = -2X + 1$. Find $\mathbb{E}(Y)$ and var(Y).

6. Toss a fair coin three times. For toss $j = 1, 2, 3$; let $X_j = 1$ if the toss is heads; -1 if the toss is tails. a) Let $Y = X_1 + X_2 + X_3$. Find the mass function of $Y, \mathbb{E}(Y)$, and var(Y). b) Same question for $Y = X_1X_2X_3$.

7. I have a box with three red balls and two black balls. I draw a ball at random, replace it, and add one of the same color (so now there are six balls). I repeat this step. Let X be the number of red balls and Y the number of black balls now in the box. a) Find the distribution of X. b) Find $\mathbb{E}(X)$. c) Find the mass function of Y . d) Find $\mathbb{E}(Y)$.

8. An unfair coin has probability p of heads. I toss it until I get heads, then I toss it some more until I get tails. Let X be the total number of tosses. Here are some possible outcomes:

$HT : X = 2$
$THT : X = 3$
$HHHHT : X = 5$
$TTHHHHT : X = 7$

Find the mean and variance of X . **Hint**: write X as the sum of two random variables.

9. Suppose that the probability a child is male is p and the probability the child is female is $1 - p$. We choose four children at random. Let A be the event that there is at most one girl and B the event that there are children of both sexes. Show that there is a value of p for which A and B are independent events.

10. Let X be a discrete random variable with the geometric distribution.

a. Compute $P(X > n)$.

b. Show that $P(X > n + k|X > n) = P(X > k)$.

c. The geometric distribution may be realized by tossing a biased coin until we get heads. (X is the total number of tosses needed.) Explain why the equation in part (b) is "obvious" in this experiment.