Math 464 Hw 4

1.

a) $f_X(j) = P(X=j) = P(X=j, Y=n-j)$

$$= \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} = \binom{n}{j} p^j (1-p)^{n-j} \qquad (j=0,1,2,\cdots,n)$$

$f_Y(k) = P(Y=k) = P(X=n-k, Y=k)$

$$= \frac{n!}{(n-k)!\,k!} p^{n-k} (1-p)^k = \binom{n}{k} p^{n-k} (1-p)^k \qquad (k=0,1,2,\cdots,n)$$

$X \sim$ Binomial (n,p) $\quad$ $Y \sim$ ~~Bion~~ Binomial $(n,(1-p))$

b) X and Y are not independent since $f_{X,Y}(j,k)$ does not always equal to $f_X(j) \cdot f_Y(k)$

~~Fore~~ For example, $f_X(0) = (1-p)^n$. $f_Y(0) = p^n$ $\quad$ $f_{X,Y}(0,0) = 0 \neq (1-p)^n \cdot p^n$.

c) $f_X(j) = P(X=j) = \sum_{k=0}^{n-j} P(X=j, Y=k)$

$$= \sum_{k=0}^{n-j} \frac{n!}{j!\,k!\,(n-j-k)!} p^j q^k (1-p-q)^{n-j-k}$$

$$= \frac{n!}{j!(n-j)!} p^j \sum_{k=0}^{n-j} \frac{(n-j)!}{k! \cdot (n-j-k)!} q^k (1-p-q)^{n-j-k}$$

$$= \binom{n}{j} \cdot p^j \cdot (q+1-p-q)^{n-j}$$

$$= \binom{n}{j} \cdot p^j (1-p)^{n-j}$$

which is binomial with parameters n and p. Similarly, the marginal for Y is binomial with parameters n and $1-p$.

Hence the marginals for this joint mass function are the same as ~~for~~ those for the joint mass function in Part (a).

2. Exercise 4 in Chapter 3.

$$P(X \geq x, Y \geq y) = P\left(\bigcup_{z \geq x} \{X = z\} \cap \bigcup_{w \geq y} \{Y = w\} \right)$$

$$= P\left(\bigcup_{z \geq x, w \geq y} [\{X = z\} \cap \{Y = w\}] \right)$$

(the distributively of intersection over union)

$$= \sum_{z \geq x} \sum_{w \geq y} P(\{X = z\} \cap \{Y = w\})$$

(by countable additively since the union in the previous line is over a disjoint family)

$$= \sum_{z \geq x} \sum_{w \geq y} P(\{X = z\}) P(\{Y = w\})$$

(since X, Y are assumed to be independent RVs)

$$= \sum_{z \geq x} P(\{X = z\}) \cdot \sum_{w \geq y} P(\{Y = w\})$$

(by the distributively of multiplication over addition)

$$= P\left(\bigcup_{z \geq x} \{X = z\} \right) \cdot P\left(\bigcup_{w \geq y} \{Y = w\} \right)$$

(by countable additivity again)

$$= P(X \geq x) \cdot P(Y \geq y)$$

3. Problem 4 in Chapter 3

$$P(X_i = k) = 1/N \quad \text{for } k = 1, 2, \cdots, N.$$

$$P(X_i \le k) = \frac{k}{N}$$

$$\therefore \; P(U_n \le k)$$

$$= P(\min\{X_1, X_2 \cdots, X_n\} \le k)$$

$$\cancel{= P \quad X_1}$$

$$= 1 - P(\min\{X_1, X_2 \cdots, X_n\} > k)$$

$$= 1 - \quad P(X_1 > k) \cdot P(X_2 > k) \cdots P(X_n > k)$$

$$= 1 - \quad [1 - P(X_1 \le k)][1 - P(X_2 \le k)] \cdots [1 - P(X_n \le k)]$$

$$= 1 - \cancel{\pi}\left(1 - \frac{k}{N}\right)^n$$

$$P(U_n \le k-1) = 1 - \left(1 - \frac{k-1}{N}\right)^n$$

$$\therefore \; P(U_n = k) = P(U_n \le k) - P(U_n \le k-1) = \left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n.$$

Similarly.

$$P(V_n \le k) = P(\max\{X_1, X_2, \cdots, X_n\} \overset{\le}{\cancel{\ge}} k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdots P(X_n \le k)$$

$$= \left(\frac{k}{N}\right)^n$$

$$P(V_n \le k-1) = \left(\frac{k-1}{N}\right)^n$$

$$\therefore \; P(V_n = k) = P(V_n \le k) - P(V_n \le k-1) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

4. Exercise 3 in Chapter 4

$$G_Y(s) = \sum_{t=0}^{\infty} s^t \, p(Y=t)$$

$$= \sum_{t=0}^{\infty} s^t \, p(kX=t) \qquad (Y=kX)$$

$$= \sum_{t=0}^{\infty} s^t \, p(X=\tfrac{t}{k})$$

$$= \sum_{m=0}^{\infty} s^{km} \cdot p(X=m) \qquad (m=\tfrac{t}{k})$$

$$= \sum_{m=0}^{\infty} (s^k)^m \, p(X=m) = G_X(s^k)$$

$$G_Z(s) = \sum_{t=0}^{\infty} s^t \, p(Z=t)$$

$$= \sum_{t=0}^{\infty} s^t \, p(X=t-k) \qquad (Z=X+k)$$

$$= \sum_{m=-k}^{\infty} s^{m+k} \, p(X=m) \qquad (m=t-k)$$

$$= s^k \sum_{m=-k}^{\infty} s^m \, p(X=m)$$

$$= s^k \sum_{m=0}^{\infty} s^m \, p(X=m) \qquad (P(X<0)=0)$$

$$= s^k \, G_X(s)$$

5. Exercise 4 in Chapter 4

$$G_X(s) = \sum_{k=0}^{a} s^k \, p(X=k)$$

$$= \sum_{k=0}^{a} s^k \cdot \frac{1}{a+1}$$

$$= \frac{1}{a+1} \cdot (1+s+s^2+\cdots+s^a)$$

$$= \frac{1}{a+1} \cdot \frac{1-s^{a+1}}{1-s}$$

6. Exercise 6 in Chapter 4

For Negative binomial Distribution.

$$G_X(s) = \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n q^{k-n} s^k$$

$$= \left(\frac{ps}{1-qs}\right)^n \qquad \text{if } |s| < q^{-1}$$

$$E(X) = G_X'(1)$$

$$= n\left(\frac{ps}{1-qs}\right)^{n-1} \cdot \frac{p}{(1-qs)^2} \Big|_{s=1}$$

$$= \frac{np^n}{(1-q)^{n+1}} = \frac{n}{p}$$

$$E(X^2) = G_X''(1) + G_X'(1)$$

$$= \left\{ n(n-1) \cdot \left(\frac{ps}{1-qs}\right)^{n-2} \cdot \left[\frac{p}{(1-qs)^2}\right]^2 + n\left(\frac{ps}{1-qs}\right)^{n-1} \cdot 2pq\,(1-qs)^{-3} \right\}\Big|_{s=1} + \frac{n}{p}$$

$$= \frac{nq+n^2}{p^2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{nq}{p^2}$$

7. Exercise 7 in Chapter 4

$G_X(s) = e^{\lambda(s-1)}$

$G_Y(s) = e^{\mu(s-1)}$ According to Theorem 4C

$G_{X+Y}(s) = G_X(s) \cdot G_Y(s)$

$= e^{\lambda(s-1)} \cdot e^{\mu(s-1)}$

$= e^{(\lambda+\mu)(s-1)}$

which shows the sum of two independent random variables, having the Poisson distribution with parameters λ and μ respectively, has the Poisson distribution also, with parameter $\lambda+\mu$.