Math 464 Hw 5.

1. Exercise 15 in Chapter 5

$X \sim N(0,1)$.

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\qquad Y = X^2$.

According to Example 34 in Page 66.

$f_Y(y) = 0$ if $y < 0$.

$f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$ $\qquad$ if $y \geq 0$.

$$= \frac{1}{2\sqrt{y}} \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \right)$$

$$= \frac{1}{2^{\frac{1}{2}}\sqrt{\pi}} y^{-\frac{1}{2}} \cdot e^{-\frac{y}{2}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left(\tfrac{1}{2}\right)^{+\frac{1}{2}} \cdot y^{-\frac{1}{2}} e^{-\frac{y}{2}}$$

$$= \frac{1}{2\Gamma(\frac{1}{2})} \cdot \left(\tfrac{1}{2}\right)^{-\frac{1}{2}} y^{-\frac{1}{2}} \cdot e^{-\frac{y}{2}} \qquad (\Gamma(\tfrac{1}{2}) = \sqrt{\pi}).$$

$$= \frac{1}{2\Gamma(\frac{1}{2})} \cdot \left(\frac{y}{2}\right)^{-\frac{1}{2}} e^{-\frac{y}{2}}$$

Therefore, $Y = X^2 \sim \chi_1^2$

2. Problem 1 in Chapter 5

$$f(x) = \frac{1}{2} c e^{-c|x|} \qquad \text{for } x \in R$$

$$EX = \int_{-\infty}^{\infty} x \cdot \frac{1}{2} c e^{-c|x|} \, dx$$

$$= 0 \qquad \left(\text{since } x \cdot \frac{1}{2} c e^{-c|x|} \text{ is an odd function}\right)$$

$$EX^2 = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} c e^{-c|x|} \, dx$$

$$= \int_0^{\infty} x^2 \cdot c \cdot e^{-cx} \, dx$$

$$= \int_0^{\infty} -x^2 \cdot d e^{-cx}$$

$$= -x^2 \cdot e^{-cx} \Big|_0^{\infty} + \int_0^{\infty} e^{-cx} \cdot 2x \, dx$$

$$= 0 + 2 \int_0^{\infty} e^{-cx} \cdot x \, dx$$

$$= -\frac{2}{c} \int_0^{\infty} x \, de^{-cx}$$

$$= -\frac{2}{c} \cdot \left[x \cdot e^{-cx} \Big|_0^{\infty} - \frac{1}{-c} \int_0^{\infty} e^{-cx} \, d(-cx) \right]$$

$$= -\frac{2}{c} \cdot \left[0 + \frac{1}{c} \left(e^{-cx}\right)\Big|_0^{\infty} \right]$$

$$= -\frac{2}{c} \cdot \left(-\frac{1}{c}\right) = 2c^{-2}$$

$$\text{Var } X = EX^2 - (EX)^2$$

$$= 2c^{-2}$$

3.

$X \sim$ Exponential.

$$F(X) = \begin{cases} 0 & \text{if } X \le 0 \\ 1 - e^{-\lambda x} & \text{if } X > 0 \end{cases}$$

$$P(X \ge 0.01) = 1 - P(X < 0.01)$$

$$= 1 - F(0.01)$$

$$= 1 - \left(1 - e^{-\lambda \cdot 0.01}\right) = e^{-0.01 \cdot \lambda} = \frac{1}{2} \Rightarrow \lambda = 100 \log 2$$

$$P(X \ge t) = 1 - P(X < t)$$

$$= 1 - F(t)$$

$$= e^{-\lambda t} = e^{-100 \log 2 \cdot t} = 0.9$$

$$\therefore -100 \log 2 \cdot t = \log(0.9)$$

$$\therefore t = -\frac{\log(0.9)}{100 \log 2} = 0.00152$$

4. $\{y_i\} \sim$ uniform $[0,1]$ $\quad (i=1,2,3,4)$

Base on the formula of order statistics

$$f_{Y_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_Y(x) \left[F_Y(x)\right]^{j-1} \left[1-F_Y(x)\right]^{n-j}$$

We know

$f_Y(y) = 12y^2(1-y)$ $\quad$ (y is the second largest)

$\therefore P(Y > 1/2)$

$$= \int_{\frac{1}{2}}^{1} 12y^2(1-y)\,dy$$

$$= 12 \cdot \left(\frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Big|_{\frac{1}{2}}^{1}$$

$$= 12 \cdot \left\{\frac{1}{12} - \left(\frac{1}{3}\cdot\left(\frac{1}{2}\right)^3 - \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4\right)\right\}$$

$$= 1 - \frac{1}{2} + 3 \cdot \frac{1}{16} = \frac{11}{16}$$

There is another way to derive $f_Y(y)$. By countable additivity,

$F_Y(y) = P(X_1, X_2, X_3, X_4 \le y) + P($ three variables are $\le y$ and the other is $> y)$

$$= y^4 + C_4^3 y^3(1-y)$$

$$= y^4 + 4y^3(1-y) = 4y^3 - 3y^4$$

It follows that $f_Y(y) = \frac{d}{dy} F_Y(y) = 12y^2(1-y)$

5. $X \sim$ uniform $(0,1)$

$f_X(x) = 1.$ $\quad F_X(x) = \int_0^x 1 \cdot dx = x$ $\qquad Y = -X$

$$P(Y \le y) = P(-X \le y) = P(X \ge -y)$$

$$= 1 - P(X \le -y)$$

$$= 1 - \int_0^{-y} 1\,dx = 1 - (-y) = 1+y$$

$\therefore F_Y(y) = 1+y$

$\therefore f_Y(y) = (1+y)' = 1$ $\qquad$ Hence $f_Y(y) = \begin{cases} 1 & -1 < y < 0 \\ 0 & \text{else} \end{cases}$

6. X is a random variable with density $f_X(x)$

$Y = |X|$

$$P(Y \le y) = P(|X| \le y) = \begin{cases} 0 & y < 0 \\ P(-y \le X \le y) & y \ge 0 \end{cases}$$

If $y \ge 0$. $\quad P(-y \le X \le y) = \int_{-y}^{y} f_X(x)\,dx = F_Y(y)$

Hence $f_Y(y) = \begin{cases} 0 & y < 0 \\ f_X(y) + f_X(-y) & y \ge 0 \end{cases}$

8. $X_1, X_2 \cdots X_n$ iid $\sim f_X(x) = 1$, $F_X(x) = x$

Base on the formula of order statistics

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x) [F_X(x)]^{j-1} [1-F_X(x)]^{n-j}$$

We know

$$f_Y(y) = n \cdot y^{n-1}$$

$$f_Z(z) = n(n-1) z^{n-2} \cdot (1-z)$$

$$EY = \int_0^1 y \cdot n y^{n-1} dy = n \cdot \int_0^1 y^n dy = \frac{n}{n+1}$$

$$EY^2 = \int_0^1 y^2 \cdot n y^{n-1} dy = n \int_0^1 y^{n+1} dy = \frac{n}{n+2}$$

$$Var\ Y = EY^2 - (EY)^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n}{(n+1)^2(n+2)}$$

$$EZ = \int_0^1 z\ n(n-1) z^{n-2} \cdot (1-z)\, dz$$

$$= n(n-1) \int_0^1 z^{n-1} (1-z)\, dz = \frac{n-1}{n+1}$$

$$EZ^2 = \int_0^1 z^2\ n(n-1) z^{n-2} (1-z)\, dz$$

$$= n(n-1) \int_0^1 z^n (1-z)\, dz = \frac{n(n-1)}{(n+1)(n+2)}$$

$$Var\ Z = EZ^2 - (EZ)^2 = \frac{n(n-1)}{(n+1)(n+2)} - \frac{(n-1)^2}{(n+1)^2} = \frac{2n-2}{(n+1)^2(n+2)}$$

There is another way to derive $f_Y(y)$ and $f_Z(z)$

By countable additivity,

$$F_Y(y) = P(X_1, X_2, X_3 \cdots, X_n \leq y)$$

$$= y^n$$

$F_Z(z) = P(X_1, X_2, X_3, \cdots, X_n \leq z) + P($n-1 variables are $\leq z$ and the other is $> z)$

$$= z^n + C_n^{n-1} z^{n-1} (1-z)$$

$$= z^n + n z^{n-1} (1-z)$$

$$= n z^{n-1} - (n-1) z^n$$

It follows that

$$f_Y(y) = \frac{d}{dy} F_Y(y) = n y^{n-1}$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = n(n-1) z^{n-2} - n(n-1) z^{n-1}$$

$$= n(n-1) z^{n-2} (1-z)$$