Math 464 Hw 7

1. Exercise 3 in Chapter 7

(i) X is uniformly distributed on (a,b).

$\therefore f(x) = \frac{1}{b-a} \qquad x \in (a,b)$

$$E(X^k) = \int_a^b x^k \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \cdot \frac{x^{k+1}}{k+1}\Big|_a^b = \frac{b^{k+1} - a^{k+1}}{(b-a)(k+1)} \qquad (k=1,2,\cdots)$$

(ii) $X \sim \text{gamma}(w, \lambda)$

$$\therefore f(x) = \begin{cases} \frac{1}{\Gamma(w)} \lambda^w \cdot x^{w-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$E(X^k) = \int_0^\infty x^k \cdot \frac{1}{\Gamma(w)} \lambda^w x^{w-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^w}{\Gamma(w)} \int_0^\infty x^{k+w-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^w}{\Gamma(w)} \cdot \frac{\Gamma(k+w)}{\lambda^{k+w}} = \frac{\Gamma(w+k)}{\lambda^k \Gamma(w)} \qquad (k=1,2,\cdots)$$

(iii) $X \sim \chi^2(n)$

$$\therefore f(x) = \begin{cases} \frac{1}{2\Gamma(\frac{1}{2}n)} \cdot (\frac{1}{2}x)^{\frac{1}{2}n-1} e^{-\frac{1}{2}x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$E(X^k) = \int_0^\infty x^k \cdot \frac{1}{2\Gamma(\frac{1}{2}n)} \cdot \left(\tfrac{1}{2}x\right)^{\frac{1}{2}n-1} e^{-\frac{1}{2}x}\, dx$$

$$= \frac{2^k}{2\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty \left(\tfrac{1}{2}x\right)^{(\frac{1}{2}n+k)-1} e^{-\frac{1}{2}x}\, dx$$

$$= \frac{\cancel{2k}\; 2^k}{2\Gamma(\frac{1}{2}n)} \cdot 2\Gamma\left(\tfrac{1}{2}n+k\right)$$

$$= 2^k \cdot \frac{\Gamma(k+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)} \qquad (k=1,2,\cdots)$$

2. Exercise 4 in Chapter 7

$$g(x,y) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho^2}}\ \exp\left(-\tfrac{1}{2}Q(x,y)\right)$$

where

$$Q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$$

Let $\begin{cases} u = \dfrac{x-\mu_1}{\sigma_1} \\ v = \dfrac{y-\mu_2}{\sigma_2} \end{cases} \Rightarrow \begin{cases} x = \sigma_1 u + \mu_1 \\ y = \sigma_2 v + \mu_2 \end{cases}$

$$|J| = \left|\,\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\,\right| = \left|\,\begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix}\,\right| = \sigma_1\sigma_2$$

$\therefore$ The joint density of u and v. is

$$f(u,v) = g(\sigma_1 u+\mu_1,\ \sigma_2 v+\mu_2)\ |J|$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\ \exp\left[-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\left(u^2 - 2\rho uv + v^2\right)\right]$$

which is a standard bivariate normal distribution.

From Page 95 of Text book,

we know $\rho = E(UV) - E(U)E(V)$

$$= E\left(\frac{X-\mu_1}{\sigma_1}\cdot\frac{Y-\mu_2}{\sigma_2}\right) - E\left(\frac{X-\mu_1}{\sigma_1}\right)E\left(\frac{Y-\mu_2}{\sigma_2}\right)$$

$$= \frac{1}{\sigma_1\sigma_2}\left(E(XY) - \mu_1 E(Y) - \mu_2 E(X) + \mu_1\mu_2\right) - \frac{1}{\sigma_1\sigma_2}$$

$$\left(E(X)\cdot E(Y) - \mu_1 E(Y) - \mu_2 E(X) + \mu_1\mu_2\right)$$

$$= \frac{1}{\sigma_1 \sigma_2} \left(EXY - EX \cdot EY \right) = \frac{1}{\sigma_1 \sigma_2} \operatorname{cov}(X,Y)$$

$$\therefore \operatorname{cov}(X,Y) = \rho \sigma_1 \sigma_2$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{\rho \sigma_1 \sigma_2}{\sqrt{\sigma_1^2 \cdot \sigma_2^2}} = \rho$$

3. Exercise 5 in Chapter 7.

$S_n = X_1 + X_2 + \cdots + X_n.$

$S_m = X_1 + X_2 + \cdots + X_n + \cdots + X_m$

$$\mathrm{cov}(S_n, S_m) = \sum_{i=1}^{n} \mathrm{cov}(X_i, X_i) + \sum_{\substack{j \neq k \\ 1 \le j \le n \\ 1 \le k \le m}} \mathrm{cov}(X_j, X_k)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

(The second term equals to 0 because $X_1, X_2, \cdots$ is a sequence of uncorrelated random variables)

$$= n\sigma^2$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1 + X_2 + \cdots + X_n)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{\substack{j \neq k \\ 1 \le j \le n \\ 1 \le k \le n}} \mathrm{cov}(X_j, X_k)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

$$= n\sigma^2$$

Hence $\mathrm{cov}(S_n, S_m) = \mathrm{Var}(S_n) = n\sigma^2$ if $n < m$.

4. Exercise 6 in Chapter 7

$$f(x,y) = \begin{cases} \frac{1}{y} e^{-y - x/y} & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(XY) = \int_0^\infty \int_0^\infty xy \cdot \frac{1}{y} \cdot e^{-y - x/y} \, dx\, dy$$

$$= \int_0^\infty \int_0^\infty x e^{-y} \cdot e^{-x/y} \, dx\, dy$$

$$= \int_0^\infty e^{-y} dy \int_0^\infty -xy \, d e^{-x/y}$$

$$= \int_0^\infty e^{-y} \cdot y^2 \, dy$$

$$= 2$$ (Integrade by parts)

$$EX = \int_0^\infty x f_X(x) \, dx$$

$$= \int_0^\infty x \int_0^\infty \frac{1}{y} e^{-y - x/y} \, dy \, dx$$

$$= \int_0^\infty \int_0^\infty \frac{x}{y} e^{-y} \cdot e^{-x/y} \, dx\, dy$$

$$= \int_0^\infty y e^{-y} \, dy = 1$$

$$EY = \int_0^\infty y f_Y(y) \, dy$$

$$= \int_0^\infty y \int_0^\infty \frac{1}{y} e^{-y - x/y} \, dx \, dy$$

$$= \int_0^\infty e^{-y}\,dy \int_0^\infty e^{-x/y}\,dx$$

$$= \int_0^\infty y e^{-y}\,dy = 1$$

$\therefore$ $\operatorname{cov}(X,Y) = E(XY) - EX \cdot EY = 1.$

5. Exercise 7 in Chapter 7

(i) $X \sim \text{gamma}(w, \lambda)$

$$\therefore f(x) = \begin{cases} \frac{1}{\Gamma(w)} \lambda^{w} \cdot x^{w-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$M_X(t) = \int_0^{\infty} e^{tx} \cdot \frac{1}{\Gamma(w)} \lambda^{w} \cdot x^{w-1} e^{-\lambda x} \, dx$$

$$= \frac{\lambda^{w}}{\Gamma(w)} \int_0^{\infty} x^{w-1} e^{-(-t+\lambda) x} \, dx$$

$$= \frac{\lambda^{w}}{\Gamma(w)} \cdot \frac{\Gamma(w)}{(-t+\lambda)^{w}} = \left(\frac{1}{-\frac{t}{\lambda} + 1} \right)^{w} = \left(\frac{1}{1 - \frac{t}{\lambda}} \right)^{w}$$

(ii) $X \sim \text{Poisson}(\lambda)$

$$P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \qquad x = 0, 1, 2 \cdots$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \cdot \frac{(e^{t} \lambda)^{x}}{x!}$$

$$= \cancel{\sum_{x=0}^{\infty} \cdot e^{-\lambda}}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{t} \lambda)^{x}}{x!}$$

$$= e^{-\lambda} \cdot e^{e^{t} \lambda} = \exp[\lambda(e^{t} - 1)]$$

6. Exercise 8 in Chapter 7

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let $Y = X - \mu$. $\quad X = Y + \mu$. $\quad \frac{dx}{dy} = 1$

$$\therefore f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

$$\therefore E(Y^3) = \int_{-\infty}^{\infty} y^3 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$(y^3 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$ is an odd function).

$$= 0$$

$$\therefore E\left[(X-\mu)^3\right]$$

$$= E\left(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3\right)$$

$$= EX^3 - 3\mu EX^2 + 3\mu^2 EX - \mu^3$$

$(EX = \mu, \quad EX^2 = (EX)^2 + \mathrm{Var}\, X = \mu^2 + \sigma^2)$

$$= EX^3 - 3\mu \cdot (\mu^2 + \sigma^2) + 3\mu^2 \cdot \mu - \mu^3$$

$$= 0$$

$$\Rightarrow EX^3 = \mu^3 + 3\mu\sigma^2$$

7. Exercise 1 in Chapter 8

$$E\left(\left(Z_n - 0\right)^2\right) = \left(n^{\alpha} - 0\right)^2 \cdot \frac{1}{n} + (0-0)^2 \cdot \left(1 - \frac{1}{n}\right)$$

$$= n^{2\alpha - 1} \to 0 \quad \text{as} \quad n \to \infty$$

if and only if $2\alpha - 1 < 0$, that is $\alpha < \frac{1}{2}$

Hence Z_n converges to 0 in mean square if and only if $\alpha < \frac{1}{2}$

8. Exercise 3 in Chapter 8

$N_n = X_1 + X_2 + \cdots + X_n$.

$$X_i = \begin{cases} 1 & p = 1/3 \\ 0 & p = 2/3 \end{cases}$$

$\therefore EX_i = 1/3$

Therefore, using Theorem 8A $\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \to 1/3$ in mean square as $n \to \infty$.

In other words, as $n \to \infty$, $\frac{N_n}{N} \to \frac{1}{3}$ in mean square.

9. Exercise 6 in Chapter 8

We know the number of sixes X in n throws of a fair die is distributed as Binomial $(n, \frac{1}{6})$.

$\therefore EX = \frac{1}{6}n \quad Var X = \frac{5}{36}n$.

$$P\left(\frac{1}{6}n - \sqrt{n} < X < \frac{1}{6}n + \sqrt{n}\right) = P\left(\left|X - \frac{1}{6}n\right| < \sqrt{n}\right) = 1 - P\left(\left|X - \frac{1}{6}n\right| \geq \sqrt{n}\right)$$