Math 464 - Midterm II Name:

Show your work and explain all answers!

1. (7 points per part) Let X and Y be independent random variables each having the standard normal distribution, $N(0,1)$.

a) Find the joint density of $U = aX+bY$ and $V = bX-aY$, where $a^2+b^2>0$.

$$\begin{aligned}
\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\
|J| &= \left|\det\left[\frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right]\right| = \frac{1}{a^2+b^2} \\
f_{XY}(x,y)dxdy &= \frac{1}{2\pi}e^{-\frac{1}{2}\left(x^2+y^2\right)}dxdy \\
&= \frac{1}{2\pi\left(a^2+b^2\right)}e^{-\frac{1}{2\left(a^2+b^2\right)}\left(u^2+v^2\right)}dudv = f_{UV}(u,v)dudv
\end{aligned}$$

b) What are the densities of U and V respectively? Are U and V independent?

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{UV}(u,v)dv \\
&= \frac{1}{\sqrt{2\pi\left(a^2+b^2\right)}}e^{-\frac{1}{2\left(a^2+b^2\right)}u^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\left(a^2+b^2\right)}}e^{-\frac{1}{2\left(a^2+b^2\right)}v^2}dv \\
&= \frac{1}{\sqrt{2\pi\left(a^2+b^2\right)}}e^{-\frac{1}{2\left(a^2+b^2\right)}u^2} \text{ since the integrand is the density for } N(0,a^2+b^2) \\
f_V(v) &= \frac{1}{\sqrt{2\pi\left(a^2+b^2\right)}}e^{-\frac{1}{2\left(a^2+b^2\right)}v^2} \text{ by a symmetric argument.}
\end{aligned}$$

It is then immediate that $f_{UV}(u,v) = f_U(u)f_V(v)$ and therefore U and V are independent.

2. (9 points) Let X be uniform on $[0,1]$. Let $Y = -\ln X$; find the density function f_Y.

$$\begin{aligned}
P(Y \le y) &= P(-\ln X \le y) = P(\ln X \ge -y) \text{ for } 0 \le y < \infty \text{ (0 otherwise)} \\
&= 1 - P(\ln X < -y) = 1 - P(\ln X \le -y) \text{ since } P(\ln X = -y) = 0; \\
&= 1 - P(X \le e^{-y}) = 1 - e^{-y} \text{ for } y \ge 0 \text{ (0 otherwise) since } X \text{ is uniform.} \\
f_Y(y) &= \frac{d}{dy}P(Y \le y) = e^{-y} \text{ for } y > 0 \text{ (0 otherwise).}
\end{aligned}$$

Hence Y has an exponential distribution with parameter 1.

3. (8 points per part) Two random variables X and Y have a joint density of the form

$$f(x,y) \quad = \quad \begin{cases} \frac{c}{(x)^3} & \text{if } 1 < x < \infty \text{ and } 1 < y < x \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of c that makes $f(x;y)$ into a density. If you are unable to find this constant, just keep calling it c and go on.

$$\begin{aligned} 1 &= c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = c\int_{1}^{\infty} dx \int_{1}^{x} \frac{1}{x^3}dy = c\int_{1}^{\infty}\left(\frac{1}{x^2}-\frac{1}{x^3}\right)dx \\ &= c\left(\frac{-1}{x}+\frac{1}{2x^2}\right)|_1^{\infty} = c\,(1-1/2) = c/2 \end{aligned}$$

Hence $c = 2$.

b) Find the density of X.

$$f_X(x) \quad = \quad \int_{-\infty}^{\infty} f(x,y)dy = \int_{1}^{x}\frac{2}{x^3}dy = \frac{2}{x^2}-\frac{2}{x^3} \text{ for } x > 1 (0 \text{ otherwise})$$

c) Verify that your formula in b) actually satisfies the requirements of a density.

For $x > 1, \frac{1}{x^2} > \frac{1}{x^3}$; hence, $f_X(x) \geq 0$. Also, by the calculation on the second line of (a), the total integral of $f_X(x)$ equals 1.

d) Find the conditional expectation $\mathbb{E}(Y|X)$.

$$\begin{aligned} \mathbb{E}(Y|X) &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_{-\infty}^{\infty} y\frac{f(x,y)}{f_X(x)}dy \\ &= \int_{-\infty}^{\infty} y\frac{2/x^3\mathbb{I}_{1<y<x<\infty}}{2\,(1/x^2-1/x^3)\,\mathbb{I}_{1<x<\infty}}dy \\ &= \int_{1}^{x}\frac{y}{x-1}dy \quad \big(\text{N.B., } f_{Y|X} \text{ is the uniform density on } (1,x)\big) \\ &= \frac{1}{x-1}\frac{y^2}{2}|_1^x = \frac{1}{x-1}\frac{x^2-1}{2} = \frac{x+1}{2}. \end{aligned}$$

4. (7 points per part) Let $X_1, X_2, \ldots, X_n$ be independent random variables each taking values in $\{0, 1, 2, \ldots\}$.

a) Consider the random variable $T = X_1 + \cdots + X_n$. Use generating functions to calculate $\mathbb{E}(T)$.

By independence,

$$\begin{aligned} G_T(s) &= G_{X_1}(s) \cdots G_{X_n}(s) \\ \mathbb{E}(T) &= G'_T(1) \\ &= \sum_{i=1}^{n} G_{X_1}(1) \cdots G_{X_{i-1}}(1) G'_{X_i}(1) G_{X_{i-1}}(1) \cdots G_{X_n}(1) \\ &= \sum_{i=1}^{n} G'_{X_i}(1) = \sum_{i=1}^{n} \mathbb{E}(X_i) \end{aligned}$$

b) Now suppose that the X_i all have the same mass function. Use generating functions to calculate $\mathbb{E}(T^2)$.

By independence and equality of mass functions,

$$\begin{aligned} G_T(s) &= G_X(s)^n \\ G'_T(s) &= nG_X(s)^{n-1} G'_X(s) \\ G''_T(s) &= n(n-1)G_X(s)^{n-2} \left(G'_X(s)\right)^2 + nG_X(s)^{n-1} G''_X(s) \\ \mathbb{E}(T^2) = G''_T(1) + G'_T(1) &= n(n-1)G'_X(1)^2 + nG''_X(1) + nG'_X(1) \\ &= n(n-1)\mathbb{E}(X)^2 + n\mathbb{E}(X^2) \end{aligned}$$

c) Still assuming that the X_i all have the same mass function, calculate the variance of T.

$$\begin{aligned} \mathrm{Var}(T) &= \mathbb{E}(T^2) - \mathbb{E}(T)^2 \\ &= n(n-1)\mathbb{E}(X)^2 + n\mathbb{E}(X^2) - (n\mathbb{E}(X))^2 \\ &= -n\mathbb{E}(X)^2 + n\mathbb{E}(X^2) \\ &= n\mathrm{Var}(X) \end{aligned}$$

5. (8 points per part) Let X and Y be independent random variables each having a geometric distribution with parameter p. Set $Z = Y - X$ and $M = \min(X, Y)$.

a) Show that for integer z and integer $m \geq 0$

$$P(M = m, Z = z) \quad = \quad \begin{cases} P(X = m - z)P(Y = m), & z < 0 \\ P(X = m)P(Y = m + z) & z \geq 0 \end{cases}$$

$$\begin{aligned}
Z < 0 \quad &\Longrightarrow \quad X > Y \Longrightarrow \min(X, Y) = Y \text{ and} \\
Z = z \quad &\Longleftrightarrow \quad X = Y - z \\
&\Longrightarrow \quad P(M = m, Z = z) = P(Y = m, X = Y - z) = P(Y = m, X = m - z) \\
&= \quad P(X = m - z)P(Y = m) \text{ by independence of } X \text{ and } Y; \\
Z \geq 0 \quad &\Longrightarrow \quad X \leq Y \Longrightarrow \min(X, Y) = X \text{ and} \\
Z = z \quad &\Longleftrightarrow \quad Y = X + z \\
&\Longrightarrow \quad P(M = m, Z = z) = P(X = m, Y = X + z) = P(X = m, Y = m + z) \\
&= \quad P(X = m)P(Y = m + z) \text{ by independence of } X \text{ and } Y.
\end{aligned}$$

b) Show that part a) implies that for integer z and integer $m \geq 0$,

$$P(M = m, Z = z) \quad = \quad p^2(1 - p)^{2m}(1 - p)^{|z|}.$$

By (a) the proof of this identity reduces to two cases

$$\begin{aligned}
z < 0 : P(M = m, Z = z) \quad &= \quad P(X = m - z)P(Y = m) \\
&= \quad p(1 - p)^{m - z}p(1 - p)^m = p^2(1 - p)^{2m + |z|} \\
z \geq 0 : P(M = m, Z = z) \quad &= \quad P(X = m)P(Y = m + z) \\
&= \quad p(1 - p)^m p(1 - p)^{m + z} = p^2(1 - p)^{2m + |z|}
\end{aligned}$$

Note that the probability weights used here for the geometric distribution, $P(X = k) = pq^k$, differ from those we have been using based on the text; however, if I had taken $m \geq 1$ with our usual convention $P(X = k) = pq^{k-1}$ the result would have come out to be the same.

c) Are M and Z independent? Provide a proof or counter-example for your answer.

Since one can write $P(M = m, Z = z) = f(m)g(z)$ where $f(m) = p(1 - p)^{2m}$ and $g(z) = p(1 - p)^{|z|}$, by a theorem from the text, M and Z are independent. One can in fact directly calculate that $P(M = m) = (1 + q)pq^{2m}$ and $P(Z = z) = p(1 - p)^{|z|}/(1 + q)$.

Extra Credit Problem (10 points) If X and Y are continuous random variables having joint density function $f(x,y)$, find the joint density function of the random variables $W = X^2$ and $Z = Y^2$.

On the domain $x, y > 0$, one has $x = \sqrt{w}, y = \sqrt{z}$ for $z, w > 0$. Hence,

$$|J| = \left|\det\begin{pmatrix} 1/2\sqrt{w} & 0 \\ 0 & 1/2\sqrt{z} \end{pmatrix}\right| = \frac{1}{4\sqrt{wz}}$$

By the change of variables formula,

$$f_{WZ}(w,z) = \frac{1}{4\sqrt{wz}} f_{XY}(\sqrt{w}, \sqrt{z})$$

Alternatively one may observe that for $w, z > 0$

$$\begin{aligned}
P(W \le w, Z \le z) &= P(-\sqrt{w} \le X \le \sqrt{w}, -\sqrt{z} \le Y \le \sqrt{z}) \\
&= P(X \le \sqrt{w}, Y \le \sqrt{z}) - P(X \le \sqrt{w}, Y \le -\sqrt{z}) \\
&- P(X \le -\sqrt{w}, Y \le \sqrt{z}) + P(X \le -\sqrt{w}, Y \le -\sqrt{z}) \\
f_{WZ}(w,z) &= \frac{\partial^2}{\partial w \partial z} P(W \le w, Z \le z) \\
&= \frac{1}{4\sqrt{wz}} \big(f_{XY}(\sqrt{w}, \sqrt{z}) + f_{XY}(\sqrt{w}, -\sqrt{z}) + f_{XY}(-\sqrt{w}, \sqrt{z}) + f_{XY}(-\sqrt{w}, -\sqrt{z})\big)
\end{aligned}$$