Math 464 - Midterm I   Name:

Show your work and explain all answers!

1. (10 points per part) For each part of this problem, A, B, C are events of a probability space. **Note:** The different parts of this problem are independent of one another.

a) If $P(A) = 0.4, P(B^c) = 0.6$ and $P(A \cup B) = 0.5$, find $P(A \cap B)$.

Since B, B^c is a partition of the sample space, $P(B) = 1 - P(B^c) = 0.4$. By the inclusion-exclusion formula,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.4 - 0.5 = 0.3.$$

b) Assume that $P(A \cap B \cap C) \neq 0$. Suppose further that $P(C|A \cap B) = P(C|B)$. Show that $P(A|B \cap C) = P(A|B)$.

Note first that $P(A \cap B \cap C) \neq 0$ implies that $P(A \cap B) \neq 0$ and $P(B \cap C) \neq 0$ since the triple intersection is contained in each of the two double intersections. Similarly, $P(B) \neq 0$. Then observe that

$$\begin{aligned}
P(C|A \cap B) &= \frac{P(A \cap B \cap C)}{P(A \cap B)} \text{ and} \\
P(C|B) &= \frac{P(B \cap C)}{P(B)}. \text{ Hence, by assumption} \\
\frac{P(A \cap B \cap C)}{P(A \cap B)} &= \frac{P(B \cap C)}{P(B)} \text{ which implies that} \\
\frac{P(A \cap B \cap C)}{P(B \cap C)} &= \frac{P(A \cap B)}{P(B)} \text{ or, equivalently,} \\
P(A|B \cap C) &= P(A|B).
\end{aligned}$$

c) Suppose $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cap B^c)$ if A and B are independent.

Since B, B^c is a partition of the sample space, $P(A) = P(A \cap B) + P(A \cap B^c)$. Hence, by independence of A and B,

$$P(A \cap B^c) = P(A) - P(A \cap B) = 0.4 - (0.4 \times 0.6) = 0.16.$$

2. (10 points) A box contains 10 red balls and 8 blue balls. Seven balls are drawn at random. Find the probability that 4 are red and 3 are blue.

The experiment described here is to draw, successively and without replacement, seven balls at random from a collection of 18 balls, ten of which are red and eight of which are blue. The probability of one such random drawing, corresponding to the event described in the question, is,

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 8 \cdot 7 \cdot 6}{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}.$$

The decreasing sequence in the denominator corresponds to the fact that the number of available balls to choose from decreases by one after each selection. The same multiplication principles applies to the two decreasing sequences in the numerator corresponding to the sequence of red balls chosen, respectively the sequence of blue balls chosen. There are $\binom{7}{4}$ different ways to choose a sequence of four red and three blue balls. Multiplying this number times the basic probability above yields:

$$\frac{7!}{4!3!} \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 8 \cdot 7 \cdot 6}{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12} = \frac{\binom{10}{4} \cdot \binom{8}{3}}{\binom{18}{7}}.$$

3. (10 points per part) Consider an experiment in which a fair coin is tossed successively and independently.

a) What is the probability that exactly four heads will occur in 8 tosses?

For the experiment of eight tosses there are 2^8 possible outcomes. This is the number of different words of length eight that can be consturcted on a two letter (H or T) alphabet. Because the coin is fair, the probability of getting a head (or of getting a tail) on any given toss is 1/2. Because these tosses are independent each outcome has probability 2^{-8}. Hence the event of having four heads (and therefore four tails) in eight tosses is

$$\binom{8}{4} \cdot 2^{-8}.$$

b) What is the probability that the fourth head occurs on the eighth toss? (This question is separate from part (a); i.e., it is not conditional on the event described in part (a)).

The probability space is the same as that described in part (a). However, the event is different because here we require the eighth toss to be a head. Hence, the number of different outcomes in this event is equal to the number of outcomes in the event that there are three heads in seven tosses. Therefore the probability of this event is

$$\binom{7}{3} \cdot 2^{-8}.$$

4. (10 points per part) You have 5 double-headed coins, 3 normal coins, 6 double-tailed coins. Pick one at random, don't look at it, and toss it.

a) What is the probability you will get a head?

We take the sample space here to be the outcome of randomly choosing one of the fourteen coins and tossing it. Since each coin has two sides, there are 28 possible outcomes. Each outcome has equal probability. (Note that if one chooses a double-headed coin, the result of tossing it will always show a head; nevertheless, there are two possible outcomes since the coin has two sides. It is just that each side has the property–i.e., belongs to the event–of showing a head. A similar statement holds for double-tailed coins.) Let H denote the event of the experiment showing a head; let T denote the event of the experiment showing a tail; let DH denote the event of the coin chosen being a double-head; let DT denote the event of the coin chosen being a double-tail; and, finally, let N denote the event of the coin chosen being normal. By assumption, the events DH, N, DT constitute a partition of the sample space. Hence, by the partition theorem,

$$\begin{aligned} P(H) &= P(H|DH)P(DH) + P(H|N)P(N) + P(H|DT)P(DT) \\ &= 1 \cdot \frac{5}{14} + \frac{1}{2} \cdot \frac{3}{14} + 0 \cdot \frac{6}{14} \\ &= \frac{13}{28} \end{aligned}$$

b) You toss the coin and get tails. What is the probability that the coin is normal?

Since the event $T = H^c, P(T) = 1 - P(H) = \frac{15}{28}$. The question asks us to find the probability of N conditional on T. Note that the event

$N \cap T$ contains just three outcomes since each of the three normal coins has just one side that shows tails. Thus we may calculate the desired probability as follows:

$$P(N|T) = \frac{P(N \cap T)}{P(T)} = \frac{3/28}{15/28} = \frac{1}{5}.$$

5. (10 points) Let X be a Poisson random variable. Find $\mathbb{E}[\frac{1}{1+X}]$.

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+X}\right] &= \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \\
&= \frac{e^{-\lambda}}{\lambda} \left[e^{\lambda} - 1\right] \\
&= \frac{1 - e^{-\lambda}}{\lambda}.
\end{aligned}$$

6. (10 points) Let X be a geometric random variable of mean 2. What is the parameter p of the distribution of X?

Recall the derivation of the mean of a geometric random variable that was derived in class and in the text:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} kpq^{k-1} \\
&= p \sum_{k=0}^{\infty} kq^{k-1} \\
&= p \frac{d}{dq} \sum_{k=0}^{\infty} q^k
\end{aligned}$$

$$
\begin{aligned}
&= p\frac{d}{dq}(1-q)^{-1} \\
&= p(1-q)^{-2} \\
&= \frac{1}{p}.
\end{aligned}
$$

Thus, if $\mathbb{E}[X] = 2$, then $p = 1/2$.

Extra Credit Problem (10 points) Suppose that n different balls are distributed in n different boxes. What is the probability that *exactly* one box is empty?

The sample space for this experiment consists of all distinct mappings of the n balls to the n boxes. Hence, the size of the sample space is n^n and each of this outcomes has the same probability. For the event of having *exactly* one box empty, one observes that there are two distinguihed boxes. One is the empty box; the other is a box containing two balls. All other boxes contain exactly one ball each. This is the only type of arrangement that is possible if all balls are to be used and only one box is empty. To count the number of different mappings that will yield such an arrangement, we first choose the two distinct boxes; there are $\binom{n}{2}$ ways to do this. Then we must decide which of the two distinguished boxes is empty and which contains two balls. There are two ways of doing this. Next we must choose two balls to go into the box that is to contain two balls. Again there are $\binom{n}{2}$ ways to do this. Finally there are $(n-2)!$ ways to arrange the remaining $n-2$ balls in the remaining $n-2$ boxes. Putting this all together we have that the desired probability is

$$
\frac{\binom{n}{2} \cdot 2 \cdot \binom{n}{2} \cdot (n-2)!}{n^n} = \binom{n}{2}\frac{n!}{n^n}.
$$