Non-selfadjoint Ornstein-Uhlenbeck semigroups

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1 Introduction

The talk will describe Ornstein-Uhlenbeck semigroups and explore an example in infinite dimensions related to the renormalization group of quantum field theory. An Ornstein-Uhlenbeck semigroup describes a diffusion process with constant diffusion and linear drift. The resulting effect is Gaussian convolution followed by rescaling. The first goal of the talk is to contrast the situation of detailed balance, where the drift coefficient is self-adjoint, with a very different situation where the drift coefficient is skew-adjoint. It is shown that in the latter situation there can be a stationary measure, but only in infinite dimensions and only when the drift coefficient is given by an operator with absolutely continuous spectrum. The second goal of the talk is to show that one version of the renormalization group is given by a particular Ornstein-Uhlenbeck semigroup of this second type and to characterize this semigroup abstractly.

The first part briefly reviews the theory of vector fields, that is, of autonomous systems of ordinary differential equations. Near an isolated zero of the vector field, that is, near a stationary solution of the system of differential equations, the equation is typically (but not always) equivalent to a system given by a linear vector field (a matrix equation).

The second part is a similar review of the theory of stochastic differential equations driven by white noise (the derivative of a Wiener process). For such equations the solutions are random, but there are semigroups of linear operators that describe the evolution of expectations of functions of the process. In some circumstances there are stationary probability densities. One special case is when the equation satisfies the detailed balance condition. Then it is comparatively easy to compute the stationary probability densities. However, the processes described in this talk need not satisfy detailed balance.

The third part introduces the main object of study, the Ornstein-Uhlenbeck processes and their associated semigroups. Such a process is defined by a stochastic differential equation given by a linear vector field and a non-degenerate constant diffusion coefficient matrix. It is the natural object for studying the behavior of a stochastic differential equation near a stable stationary point of the vector field.

The fourth part poses the question of whether an Ornstein-Uhlenbeck semigroup with a vector field leaving a quadratic form constant can have an stationary probability density. The answer is negative in finite dimensional situations. But in infinite dimensions there are many examples. It is shown that in such examples the generator of the linear vector field must have absolutely continuous spectrum.

The fifth part is devoted to an important example of the situation where the vector field leaves a quadratic form invariant. This quadratic form is defined on a space of functions, and it is given by the integral of the square of the gradient of the function. The vector field generates a scaling of the functions that leaves this form invariant. This is the Ornstein-Uhlenbeck semigroup corresponding to the renormalization group of quantum field theory. The action of the semigroup on measures is to first scale wave numbers in the range from 0 to 1 to the range from 0 to e^t. Then the convolution integrates out fluctuations corresponding to wave numbers from 1 to e^t. The result is an effective description of fluctuations at low wave numbers. This example is characterized abstractly by Lie algebra calculations.

2 Vector fields

A vector field defines for each point $\mathbf{x}$ in n dimensional space a corresponding n component vector $\mathbf{b}(\mathbf{x})$. It defines a flow by solving the system of ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) \tag{2.1}$$

with initial condition $\mathbf{x} = \mathbf{x}_0$ at $t = 0$.

The local properties of vector fields are relatively simple. Near each point $\mathbf{x}$ at which the vector field $\mathbf{b}(\mathbf{x}) \neq \mathbf{0}$ there is a change of coordinates that makes the vector field a constant vector field. The situation is more complicated in the neighborhood of a isolated zero $\mathbf{x}_0$, where $\mathbf{b}(\mathbf{x}_0) = 0$. The vector field has a linearization at such a stationary point. This is a matrix given by $-A = (\nabla\mathbf{b})(\mathbf{x}_0)$. (The minus sign is chosen for later convenience.) If there are no non-trivial integer relations among the eigenvalues, then, according to the Sternberg linearization theorem [4], there is a change of coordinates that makes the stationary point the zero vector and makes the vector field a linear vector field $-A\mathbf{x}$.

In the case when the eigenvalues satisfy an integer relation, this can fail. The simplest example is a vector field in the plane whose linearization at zero has eigenvalues $-i$ and i. The linear vector field with these eigenvalues has solutions with constant period 2π. But there also nonlinear vector fields with the same linearization, but with periods that differ from 2π. Since the period is invariant under change of coordinates, such a non-linear vector field is not equivalent to the linear vector field under change of coordinates. Such examples are rather special, and we shall ignore them in the following.

Thus to study local behavior near a zero we look at the special case when $\mathbf{b}(\mathbf{x}) = -A\mathbf{x}$, a linear vector field. It is possible to solve the differential equation explicitly. This differential equation is $d\mathbf{x}/dt = -A\mathbf{x}$ with solution

$$\mathbf{x}(t) = \exp(-tA)\mathbf{x}(0). \tag{2.2}$$

3 Stochastic differential equations

A stochastic differential equation of the form

$$d\mathbf{x} = \mathbf{b}(\mathbf{x})\, dt + \sigma(\mathbf{x})d\mathbf{w}(t) \tag{3.1}$$

with initial condition $\mathbf{x} = \mathbf{x}_0$ at $t = 0$ defines a Markov diffusion process. One can think of this as the motion of a particle with a deterministic flow given by the vector field $\mathbf{b}(\mathbf{x})$ and a random flow determined by the diffusion matrices $\sigma(\mathbf{x})$. Here $\mathbf{w}(t)$ consists of n independent copies of the Wiener process (sometimes also called Brownian motion). The matrices $\sigma(\mathbf{x})$ are each strictly positive symmetric n by n matrices.

Let f be a smooth function, and let $f(\mathbf{x}_0, t) = E_{\mathbf{x}_0}[f(\mathbf{x}(t))]$ be the expected value of $f(\mathbf{x}(t))$ at time t, when the particle is started at $\mathbf{x}_0$ at time zero. Then $f(\mathbf{x}, t)$ satisfies the *backward equation*

$$\frac{\partial f}{\partial t} = \frac{1}{2}\sigma^2(\mathbf{x})\nabla\nabla f + \mathbf{b}(\mathbf{x}) \cdot \nabla f \tag{3.2}$$

with the initial condition $f(\mathbf{x}, 0) = f(\mathbf{x})$. The mapping that sends $f(\mathbf{x})$ to $f(\mathbf{x}, t)$ is called the *semigroup* associated to this process. On the other hand, let $\rho(\mathbf{x})$ be a density at time zero, and define the density $\rho(\mathbf{x}, t)$ at time t by

$$\int \rho(\mathbf{x}, t)f(\mathbf{x})\, d\mathbf{x} = \int \rho(\mathbf{x}_0)f(\mathbf{x}_0, t)\, d\mathbf{x}_0. \tag{3.3}$$

Then $\rho(\mathbf{x}, t)$ satisfies the equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\nabla \cdot \nabla(\sigma^2(\mathbf{x})\rho) - \nabla \cdot (\mathbf{b}(\mathbf{x})\rho) \tag{3.4}$$

with the initial condition $\rho(\mathbf{x}, t) = \rho(\mathbf{x})$. This is the *forward equation* (also called the Kolmogorov equation or the Fokker-Planck equation). The mapping that sends $\rho(\mathbf{x})$ into $\rho(\mathbf{x}, t)$ is the action of the *adjoint semigroup*. The forward equation can also be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\mathbf{b}(\mathbf{x})\rho - \frac{1}{2}\nabla(\sigma^2(\mathbf{x})\rho)] = 0. \tag{3.5}$$

The quantity in square brackets is called the *current*.

The equation for an stationary probability density ρ is obtained by setting the time derivative in the forward equation equal to zero. Suppose there is a solution. Then there is a decomposition of the drift $\mathbf{b}$ into two parts, so that $\mathbf{b} = \mathbf{u} + \mathbf{v}$. The decomposition is

$$\mathbf{u}\rho = \frac{1}{2}\nabla(\sigma^2\rho) \tag{3.6}$$

and

$$\mathbf{v}\rho = \mathbf{b}\rho - \frac{1}{2}\nabla(\sigma^2\rho). \tag{3.7}$$

Thus $\mathbf{v}\rho$ is the current. If we set $\rho = c\exp(-U)$, then the equation for $\mathbf{u}$ is

$$\mathbf{u} = \frac{1}{2}\nabla\sigma^2 - \frac{1}{2}\sigma^2\nabla U. \tag{3.8}$$

This decomposition gives the decomposition of the operator L that occurs in the backward equation $\partial f/\partial t = Lf$ into a self-adjoint part and a skew-adjoint part. Thus

$$L = \frac{1}{2}\sigma^2\nabla\nabla + \mathbf{b}\cdot\nabla = [\frac{1}{2}\sigma^2\nabla\nabla + \mathbf{u}\cdot\nabla] + \mathbf{v}\cdot\nabla. \tag{3.9}$$

The self-adjoint part involving $\mathbf{u}$ describes purely diffusive motion that is even under time reversal. The skew-adjoint part involving $\mathbf{v}$ describes deterministic motion that is odd under time reversal. Thus it is given by a first order differential operator.

The condition of *detailed balance* says that the current $\mathbf{v} = 0$, so that $\mathbf{b} = \mathbf{u}$. (This condition is also called time reversibility or self-adjointness.) Since $\mathbf{b}$ is known, it follows that $\mathbf{u}$ is also known. In that case it is relatively easy to solve the equation for the potential U and hence for the probability density $\rho = e^{-U}$.

4 The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is the special case when the process is of the form $\sigma^2 = Q$, a constant matrix, and $\mathbf{b}(\mathbf{x}) = -A\mathbf{x}$, a linear vector field. It is possible to solve the stochastic differential equation explicitly. This makes it also possible to solve the backward and forward equations more or less explicitly.

The stochastic differential equation is

$$d\mathbf{x} = -A\mathbf{x}\,dt + \sigma\,d\mathbf{w}(t). \tag{4.1}$$

It has solution

$$\mathbf{x}(t) = \exp(-tA)\mathbf{x}(0) + \int_0^t \exp(-(t-s)A)\sigma\mathbf{w}(s)\,ds. \tag{4.2}$$

From this it is easy to see that $\mathbf{x}(t)$ is Gaussian with mean $\exp(-tA)\mathbf{x}_0$ and covariance matrix

$$C_t = \int_0^t \exp(-tA)Q\exp(-tA^*)\,ds. \tag{4.3}$$

Here A^* denotes the transpose of the matrix A. The solution of the backward equation is the Ornstein-Uhlenbeck semigroup given by

$$f(\mathbf{x}_0, t) = \int f(\exp(-tA)\mathbf{x}_0 + \mathbf{y})\, d\mu_t(\mathbf{y}), \tag{4.4}$$

where μ_t is a Gaussian measure with mean zero and covariance C_t. This expression is called the *Mehler formula*.

Assume that the eigenvalues of A have strictly positive real parts. Then the integral

$$C = \int_0^\infty \exp(-tA) Q \exp(-tA^*)\, ds \tag{4.5}$$

converges. The stationary probability measure is the Gaussian measure μ with this covariance. That is, its density is $\rho(\mathbf{x}) = c\exp(-U(\mathbf{x}))$, where $U(\mathbf{x}) = \mathbf{x}^* C^{-1}\mathbf{x}$. The covariance operators C_t have the simple explicit expression in terms of C as

$$C_t = C - \exp(tA) C \exp(-tA^*). \tag{4.6}$$

Furthermore, by differentiation

$$AC + CA^* = Q. \tag{4.7}$$

The problem of computing the covariance C that determines the stationary measure μ reduces to the problem of solving this equation. The solution is given by the integral expression above, but this is not always easy to compute explicitly.

The condition for detailed balance is that the vector field has the representation $-A\mathbf{x} = -(1/2)Q\nabla U(\mathbf{x})$. This happens when $Q^{-1}A$ is a symmetric matrix, that is, $Q^{-1}A = A^* Q^{-1}$. This in turn may be written as $AQ = QA^*$. The solution C is then given by $C = \frac{1}{2}A^{-1}Q = \frac{1}{2}QA^{*-1}$.

Notice that the condition of detailed balance says that A is self-adjoint with respect to Q^{-1}, that is, that $A = QA^*Q^{-1}$. Equivalently, it says that A^* is self-adjoint with respect to the generating covariance Q, that is, that $A^* = Q^{-1}AQ$. In the following we shall be interested in systems that are very far from satisfying detailed balance. In fact, the generator will be skew-adjoint.

5 Invariant quadratic forms

The Ornstein-Uhlenbeck has a generalization to infinite dimensions. The setting is a Banach space E in which $\exp(-tA)$ acts. This is the space on which the Gaussian measures are concentrated. The continuous linear forms on E form a dual Banach space E^*. These forms in E^* correspond to the linear random variables on E. Then $\exp(-tA^*)$ acts in the dual Banach space E^*.

We shall refer to a non-degenerate positive symmetric quadratic form C on E^* as a covariance operator. We may also think of C as an operator from E^* to E. If there is a Gaussian measure on E with mean zero and with covariance C, then the covariance of two Gaussian random variables u, v in E^* are given by C via the formula uCv. The covariance operator determines the Gaussian measure.

Given a covariance C, we can complete the space E^* in the C inner product. This is the natural space $H(C)$ of Gaussian random variables with finite variance. Since $E^* \subset H(C)$ is dense, we are allowed to think of the dual space of this Hilbert space as a Hilbert space $H(C^{-1})$ with inner product C^{-1} contained in the space E. Thus $H(C^{-1}) \subset E$. Notice that C is an isomorphism from $H(C)$ to $H(C^{-1})$.

Later on we shall need to compare two covariances. If C and S are covariances, and if there exists a constant m such that $C \leq mS$, then it follows that $S^{-1} \leq mC^{-1}$ and hence $CS^{-1}C \leq mC \leq m^2 S$. This shows that C is a bounded operator from $H(S)$ to $H(S^{-1})$.

The Ornstein-Uhlenbeck semigroup with generator $-A$ and generating covariance Q is defined as the mapping that sends F into $\exp(-tL)F$ given by

$$(\exp(-tL)F)(\phi) = \int F(\exp(-tA)\phi + \chi)\, d\mu_t(\chi). \tag{5.1}$$

Here μ_t is the Gaussian measure with semigroup covariance C_t given in terms of A and Q by the formula of the previous section. This of course is just the expectation of a Gaussian random variable with mean $\exp(-tA)\phi$ and covariance C_t.

There is also a dual action of the Ornstein-Uhlenbeck semigroup on measures. If the expression for C given above converges, then there is a stationary Gaussian measure with this covariance.

Here is a situation that is rather far from detailed balance. Consider a positive quadratic form S such that $\exp(-tA)S\exp(-tA^*) = S$. This says that the group $\exp(-tA^*)$ leaves the S inner product invariant and therefore is a unitary group on $H(S)$. The equation may also be written as $\exp(-tA^*)S^{-1}\exp(-tA) = S^{-1}$. It then says that the group $\exp(-tA)$ leaves the S^{-1} inner product invariant and therefore is a unitary group of operators on $H(S^{-1})$. These equations have infinitesimal forms $AS + SA^* = 0$ and $A^*S^{-1} + S^{-1}A = 0$. The first condition says that A^* is skew-adjoint with respect to S, that is, that $A^* = -S^{-1}AS$. The other condition says that A is skew-adjoint with respect to S^{-1}, that is, that $A = -SA^*S^{-1}$. Since S is an isomorphism from $H(S)$ to $H(S^{-1})$, it follows in particular that A and $-A^*$ are isomorphic as operators on the two Hilbert spaces.

In such an example there can be a stationary measure only in the infinite dimensional situation. Also, there is no longer a unique stationary Gaussian measure C satisfying $AC + CA^* = Q$ for the Ornstein-Uhlenbeck process. In fact, each Gaussian measure with covariance $C + aS$ with $a \geq 0$ is also stationary.

Theorem 5.1 *Consider an Ornstein-Uhlenbeck semigroup with generator* $-A$ *and generating covariance* Q. *Consider another covariance* S *such that* $Q \leq kS$. *Suppose that* A *is skew-adjoint with respect to* S^{-1}, *so that* $\exp(-tA)$ *leaves* S^{-1}

invariant. Say that the stationary covariance

$$C = \int_0^\infty \exp(-tA) Q \exp(-tA^*)\, dt \tag{5.2}$$

exists and satisfies $C \leq mS$. Then A has absolutely continuous spectrum.

Proof: The operator QS^{-1} is bounded as an operator on $H(S^{-1})$. Furthermore, we can write

$$CS^{-1} = \int_0^\infty \exp(-tA) QS^{-1} \exp(tA)\, dt. \tag{5.3}$$

Thus as operators in this space QS^{-1} is positive and CS^{-1} is bounded. It follows from a standard fact in quantum scattering theory [3] that A has absolutely continuous spectrum. End of proof.

6 The renormalization group

We now want to consider a situation in which there are two covariances and two generators of unitary groups. The first covariance defines a Hilbert space that will be denoted by H. The inner product defined by this covariance is used to identify H with its dual space. Thus the covariance may be represented by the identity operator. Its associated generator is B with $B + B^* = 0$. The operator $B = -B^*$ is skew-adjoint in H.

The second covariance is denoted by S and the generator is A with $AS + SA^* = 0$. Thus there is a Hilbert space $H(S)$ with inner product given by S, and the operator A^* is skew-adjoint in this space. Also there is a dual Hilbert space Hilbert space $H(S^{-1})$ with inner product given by S^{-1}, and the operator A is skew-adjoint in this space. The inner product of H may be used to identify the covariance S with an operator acting in H.

In order to characterize the usual renormalization group make the hypothesis

$$A = B + 1. \tag{6.1}$$

This gives the identity $A + A^* = 2$. This in turn leads to the commutator identity $AS - SA = -2S$.

Proposition 6.1 *The identities $AS + SA^* = 0$ and $A + A^* = 2$ imply that*

$$\exp(-tA) f(S) \exp(-tA^*) = e^{-2t} f(e^{2t} S). \tag{6.2}$$

In infinitesimal form, this is

$$Af(S) + f(S)A^* = -2Sf'(S) + 2f(S). \tag{6.3}$$

Proof: We have

$$\exp(-tA)S\exp(-tA^*) = S. \tag{6.4}$$

Hence

$$\exp(-tA)S\exp(tA) = e^{2t}S, \tag{6.5}$$

and so

$$\exp(-tA)f(S)\exp(tA) = f(e^{2t}S), \tag{6.6}$$

Finally,

$$\exp(-tA)f(S)\exp(-tA^*) = e^{-2t}f(e^{2t}S), \tag{6.7}$$

End of proof.

Corollary 6.2 *Under the same hypothesis, we have*

$$\exp(-tA)f(S^{-1})\exp(-tA^*) = e^{-2t}f(e^{-2t}S^{-1}). \tag{6.8}$$

In infinitesimal form, this is

$$Af(S^{-1}) + f(S^{-1})A^* = 2S^{-1}f'(S^{-1}) + 2f(S^{-1}. \tag{6.9}$$

Say that $Q = g(S^{-1})$. Then we can use the infinitesimal equation to solve for $C = f(S^{-1})$ satisfying $AC + CA^* = Q$. We get the differential equation $2[uf'(u) + f(u)] = g(u)$ with solution determined by $2f(u) = (1/u)G(u)$, where $G'(u) = g(u)$ with $G(0) = 0$. So $C = (1/2)SG(S^{-1})$.

Theorem 6.3 *Consider an Ornstein-Uhlenbeck process with generator $-A$ satisfying $A+A^* = 2$ such that there is an invariant covariance S with $AS+SA^* = 0$. Let the generating covariance defining the Ornstein-Uhlenbeck process be defined by $Q = g(S^{-1})$. Let $G'(u) = g(u)$ with $G(0) = 0$. Then the semigroup covariance is*

$$C_t = \int_0^t \exp(-sA)Q\exp(-sA^*)\,ds = \frac{1}{2}S[G(S^{-1}) - G(e^{-2t}S^{-1})] \tag{6.10}$$

with limit

$$C = \int_0^\infty \exp(-sA)Q\exp(-sA^*)\,ds = \frac{1}{2}SG(S^{-1}). \tag{6.11}$$

Proof: From the previous result

$$C_t = \int_0^t e^{-2s}g(e^{-2s}S^{-1})\,ds = \frac{1}{2}S\int_{e^{-2t}S^{-1}}^{S^{-1}} g(u)\,du. \tag{6.12}$$

End of proof.

It remains to choose the function g with $Q = g(S^{-1})$ and compute the corresponding C_t. One example where it is easy to compute is when $g(u) = e^{-u}$ and $G(u) = 1 - e^{-u}$.

However there is another example that will make the physical interpretation more transparent. This is where $g(u)$ is a smooth approximation to $\delta(u-1)$, that is, a bump concentrated near one. Then $G(u)$ is a smooth approximation to the indicator function of the interval from 1 to infinity. It follows that $G(u) - G(e^{-2t}u)$ is a smooth approximation to the indicator function of the interval from 1 to e^{2t}.

The example in view is when the Hilbert space is $H = L^2(\mathbf{R}^n)$ with $n > 2$. The first covariance is just the usual inner product on this Hilbert space. The skew-adjoint operator B is given by

$$B = -x \cdot \nabla - \frac{n}{2}. \tag{6.13}$$

Then

$$(\exp(-tB)u)(x) = u(e^t x)e^{tn/2}. \tag{6.14}$$

is a unitary group of scaling transformations.

The other covariance in this example is $S = (-\Delta)^{-1}$. The reason for taking $n > 2$ is so that this covariance is defined for nice functions. The corresponding inverse covariance is the negative of the Laplace operator, that is, $S^{-1} = -\Delta$. The generator $-A = -B - 1$ satisfies $AS + SA^* = 0$.

In this application the spectral values of $S^{-1} = -\Delta$ correspond via the Fourier transform to squares $u = |\mathbf{k}|^2$ of wave numbers $\mathbf{k}$. One has an initial measure that describes fluctuations roughly in some range of wave numbers $|\mathbf{k}|$ from 0 to 1. One wants to effectively integrate out the fluctuations with wave numbers $|\mathbf{k}|$ in the interval from e^{-t} to 1, thus getting the effective behavior for small wave numbers.

This is implemented by the Ornstein-Uhlenbeck process in the following way. The first thing is to scale the wave numbers by e^t. This gives a new range of $|\mathbf{k}|$ from 0 to e^t. Then the covariance C_t integrates effectively over the range of wave numbers $|\mathbf{k}|$ from 1 to e^t. This leaves the resulting wave numbers $|\mathbf{k}|$ in the range from 0 to 1. These give an effective description of the behavior of the original problem for very low values of wave number, that is, for very long distances.

There is much more to be said about infinite dimensional Ornstein-Uhlenbeck semigroups and about the renormalization group example. One of the most interesting issues is the non-uniqueness of stationary measures [1]. In particular, there can be non-Gaussian stationary measures. The paper [2] gives the beginning of such an analysis.

References

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