Twelve-fold Way

Notes for Math 447

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1 Functions

1.1 General case

The general case is that of functions $f : A \to B$. Here A has k elements and B has n elements. There are n^k such functions.

1.2 Injective case

The injective case is that of injective functions $f : A \to B$. There are $(n)_k$ such functions.

1.3 Surjective case

The surjective case is that of surjective functions $f : A \to B$. There are $n!S(k, n)$ such functions.

2 Multi-indices

2.1 General case

The symmetric group S_k of order $k!$ acts on A. Consider a general function $f : A \to B$. For each element of B there is an inverse image set in A. The stabilizer group of f is the set of permutations within these inverse image sets. The inverse image sets have size $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = k$. The order of the stabilizer group is $k_1! \cdots k_n!$. The orbits correspond to multi-indices. The size of an orbit is thus the multinomial coefficient $k!/(k_1! \cdots k_n!)$. The number of functions is

$$n^k = \sum_{k_1,\ldots,k_n} \frac{k!}{k_1! \cdots k_n!}. \tag{1}$$

2.2 Injective case

Consider an injective function $f : A \to B$. This is the special case when each $k_i \leq 1$, and we require $k \leq n$. The order of the stabilizer subgroup is 1. The orbits correspond to subsets of B of size k. The size of an orbit is $k!$. The total number of injective functions is

$$(n)_k = \binom{n}{k} k!. \tag{2}$$

2.3 Surjective case

Consider a surjective function $f : A \to B$. This is the special case when each $k_i \geq 1$. The order of the stabilizer subgroup is $k_1! \cdots k_n!$. The size of an orbit is the multinomial coefficient $k!/(k_1! \cdots k_n!)$. The number of surjective functions is

$$n!S(k,n) = \sum_{k_1 \geq 1,\ldots,k_n \geq 1} \frac{k!}{k_1! \cdots k_n!}. \tag{3}$$

3 Set Partitions

3.1 General case

The symmetric group S_n of order $n!$ acts on B. Consider a general function $f : A \to B$. Let p_j be the number of points of B that have inverse images with j elements. We have $p_0 + p_2 + p_3 + \cdots = n$. In particular, p_0 is the number of points in B that are not in the range of f. The stabilizer group of f permutes these points and thus has order $p_0!$. The orbits correspond to set partitions of A into at most n blocks. The size of an orbit is thus $n!/p_0!$. If we write i for the number of points in the range of f, then this is $n!/(n-i)! = (n)_i$. The total number of functions is thus

$$n^k = \sum_{i=0}^{n} S(k,i)(n)_i. \tag{4}$$

Every function is given by a set partition of A into some number $i \leq n$ of blocks, together with an injection from the set of blocks to B.

3.2 Injective case

Consider an injective function $f : A \to B$. We must have $k \leq n$. Then $p_1 = k$ and $p_0 = n - k$. The stabilizer group of f has order $p_0! = (n-k)!$. The size of an orbit is thus $n!/(n-k)! = (n)_k$. For $k \leq n$ there is one orbit. Thus for $k \leq n$ we have that the total number of injective functions is $(n)_k$.

3.3 Surjective case

Consider a surjective function $f : A \to B$. Then $p_0 = 0$, so the order of the stabilizer subgroup is $0! = 1$. The orbits correspond to set partitions of A with n blocks. The size of an orbit is $n!$. The number of surjective functions is $S(k,n)n!$.

4 Integer Partitions

4.1 General case

The symmetric group $S_k \times S_n$ of order $k!n!$ acts on $A \times B$. Consider a general function $f : A \to B$. Let p_j be the number of points of B that have inverse images with j elements. In particular, p_0 is the number of points in B that are not in the range of f, so we write $p_0 = n - i$, where i is the number of points in the range. Write a multi-index $k_1, \ldots, k_i$ such that each j occurs p_j times. The stabilizer group of f permutes these points. However it also simultaneously permutes blocks of the same size, together with the image points associated with these blocks. Furthermore, it permutes that points within the blocks. So the stabilizer subgroup has order $k_1! \cdots k_i! \cdot p_0!p_1!p_2!p_3! \cdots$. The orbits correspond to integer partitions with no more than n parts. It follows that the size of an orbit is $k!/(k_1! \cdots k_i!)$ times $n!/(p_0!p_1!p_2!p_3! \cdots)$. The number of functions is thus

$$n^k = \sum_{i=0}^{n} \sum_{p_1,p_2,p_3,\ldots} \frac{k!}{k_1! \cdots k_i!} \frac{1}{p_1!p_2!p_3! \cdots} (n)_i. \tag{5}$$

The sum is over $p_1, p_2, p_3, \ldots$ with sum i. Every function from A to B is given by an integer partition of k into some number i of parts, together with a set partition compatible with this integer partition, together with an injection of the blocks into B.

4.2 Injective case

Consider an injective function $f : A \to B$. We must have $k \leq n$. Then $p_1 = k$ and $p_0 = n - k$. The stabilizer group of f has order $p_0!p_1! = (n-k)!k!$. The size of an orbit is thus $n!/(n-k)! = (n)_k$. For $k \leq n$ there is one orbit. For $k \leq n$ the total number of injective functions is $(n)_k$.

4.3 Surjective case

Consider a surjective function $f : A \to B$. The stabilizer group of f has order $k_1! \cdots k_n! \cdot p_1!p_2!p_3! \cdots$. The orbits correspond to integer partitions with n parts. It follows that the size of an orbit is $k!/(k_1! \cdots k_n!)$ times $n!/(p_1!p_2!p_3! \cdots)$. The number of functions is thus

$$n!S(k,n) = \sum_{p_1,p_2,p_3,\ldots} \frac{k!}{k_1! \cdots k_n!} \frac{1}{p_1!p_2!p_3! \cdots} n!. \tag{6}$$

The sum is over $p_1, p_2, p_3, \ldots$ with sum n. Every function from A to B is given by an integer partition of k into n parts, together with a set partition compatible with this integer partition, together with a bijection of the blocks onto B.