Vector fields and differential forms

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Part I

Analysis on a manifold

Chapter 1

Vectors

1.1 Linear algebra

The algebraic setting is an n-dimensional real vector space V with real scalars. However we shall usually emphasize the cases $n = 2$ (where it is easy to draw pictures) and $n = 3$ (where it is possible to draw pictures).

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V, and if a, b are real scalars, then the linear combination $a\mathbf{u} + v\mathbf{v}$ is also a vector in V. In general, linear combinations are formed via scalar multiplication and vector addition. Each *vector* in this space is represented as an arrow from the origin to some point in the plane. A scalar multiple $c\mathbf{u}$ of a vector $\mathbf{u}$ scales it and possibly reverses its direction. The sum $\mathbf{u} + \mathbf{v}$ of two vectors $\mathbf{u}, \mathbf{v}$ is defined by the parallelogram law Figure **??**. The original two arrows determine a parallelogram, and the sum arrow runs from the origin to the opposite corner of the parallelogram.

1.2 Manifolds

An n-dimensional manifold is a set whose points are characterized by the values of n coordinate functions. Mostly we shall deal with examples when n is 1, 2, or 3.

We say that an open subset U of $\mathbf{R}^n$ is a nice region if it is diffeomorphic to an open ball in $\mathbf{R}^n$. Each such nice region U is diffeomorphic to each other nice region V. For the moment the only kind of manifold that we consider is a manifold described by coordinates that put the manifold in correspondence with a nice region. Such a manifold can be part of a larger manifold, as we shall see later.

Thus this preliminary concept of manifold is given by a set M such that there are coordinates $x_1, \ldots, x_n$ that map M one-to-one onto some nice region U in $\mathbf{R}^n$. There are many other coordinate systems $u_1, \ldots, u_n$ defined on M. What is required is that they are all related by diffeomorphisms, that is, if one goes from U to M to V by taking the coordinate values of $x_1, \ldots, x_n$ back to the corresponding points in M and then to the corresponding values of $u_1, \ldots, u_n$, then this is always a diffeomorphism.

A function z on M is said to be smooth if it can be expressed as a smooth function $z = f(x_1, \ldots, x_n)$ of some coordinates $x_1, \ldots, x_n$. Of course then it can also be expressed as a smooth function $z = g(u_1, \ldots, u_n)$ of any other coordinates $u_1, \ldots, u_n$.

1.3 Local and global

There is a more general concept of a manifold. The idea is that near each point the manifold looks like an open ball in $\mathbf{R}^n$, but on a large scale it may have a different geometry. An example where $n = 1$ is a circle. Near every point one can pick a smooth coordinate, the angle measured from that point. But there is no way of picking a single smooth coordinate for the entire circle.

Two important examples when $n = 2$ are a sphere and a torus. The usual spherical polar coordinates for a sphere are not smooth near the north and south poles, and they also have the same problem as the circle does as one goes around a circle of constant latitude. A torus is a product of two circles, so it has the same problems as a circle.

In general, when we look at a manifold near a point, we are taking the *local* point of view. Most of what we do in these notes is local. On the other hand, when we look at the manifold as a whole, we are taking the *global* point of view. Globally a sphere does not look like an open disk, since there is no way of representing a sphere by a map that has the form of an open disk.

1.4 A gas system

The most familiar manifold is n-dimensional Euclidean space, but this example can be highly misleading, since it has so many special properties. A more typical example is the example of an $n - 1$ dimensional surface defined by an equation. For example, we could consider the circle $x^2 + y^2 = 1$. This has a coordinate near every point, but no global coordinate.

However an more typical example is one that has no connection whatever to Euclidean space. Consider a system consisting of a gas in a box of volume V held at a pressure P. Then the states of this system form a two-dimensional manifold with coordinates P, V. According to thermodynamics, the temperature T is some (perhaps complicated) function of P and V. However P may be also a function of V and T, or V may be a function of P and T. So there are various coordinate systems that may be used to describe this manifold. One could use P, V, or V, T, or V, T. The essential property of a manifold is that one can locally describe it by coordinates, but there is no one preferred system.

1.5 Vector fields

A *vector field* $\mathbf{v}$ may be identified with a linear partial differential operator of the form

$$\mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}. \tag{1.1}$$

Here the $x_1, \ldots, x_n$ are coordinates on a manifold M. Each coefficient v_i is a smooth function on M. Of course it may always be expressed as a function of the values $x_1, \ldots, x_n$, but we do not always need to do this explicitly.

The picture of a vector field is that at each point of the manifold there is a vector space. The origin of each vector in this vector space is the corresponding point in the manifold. So a vector field assigns a vector, represented by an arrow, to each point of the manifold. In practice one only draws the arrows corresponding to a sampling of points, in such a way to give an overall picture of how the vector field behaves Figure **??**.

The notion of a vector field as a differential operator may seem unusual, but in some ways it is very natural. If z is a smooth function on M, then the directional derivative of z along the vector $\mathbf{v}$ is

$$dz \cdot \mathbf{v} = \mathbf{v}z = \sum_{i=1}^{n} v_i \frac{\partial z}{\partial x_i}. \tag{1.2}$$

The directional derivative of z along the vector field $\mathbf{v}$ is the differential operator $\mathbf{v}$ acting on z.

In the two dimensional case a vector field might be of the form

$$L_{\mathbf{v}} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}. \tag{1.3}$$

Here x, y are coordinates on the manifold, and $a = f(x,y)$ and $b = g(x,y)$ are smooth functions on the manifold.

If z is a smooth function, then

$$dz \cdot \mathbf{v} = a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y}. \tag{1.4}$$

If the same vector field is expressed in terms of coordinates u, v, then by the chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}. \tag{1.5}$$

Also

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}. \tag{1.6}$$

So

$$dz \cdot \mathbf{v} = \left(\frac{\partial u}{\partial x}a + \frac{\partial u}{\partial y}b\right)\frac{\partial z}{\partial u} + \left(\frac{\partial v}{\partial x}a + \frac{\partial v}{\partial y}b\right)\frac{\partial z}{\partial v}. \tag{1.7}$$

The original linear partial differential operator is seen to be

$$\mathbf{v} = \left(\frac{\partial u}{\partial x}a + \frac{\partial u}{\partial y}b\right)\frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial x}a + \frac{\partial v}{\partial y}b\right)\frac{\partial}{\partial v}. \tag{1.8}$$

This calculation shows that the when the vector field is expressed in new coordinates, then these new coordinates are related to the old coordinates by a linear transformation.

One point deserves a warning. Say that u is a smooth function on a manifold. Then the expression $\frac{\partial}{\partial u}$ is in general not well-defined. If, however, we know that the manifold is 2-dimensional, and that u, v form a coordinate system, the $\frac{\partial}{\partial u}$ is a well-defined object, representing differentiation along a curve with v held constant along that curve.

This warning is particularly important when one compares coordinate systems. A particular nasty case is when u, v is one coordinate system, and u, w is another coordinate system. Then it is correct that

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial u} + \frac{\partial v}{\partial u}\frac{\partial}{\partial v}. \tag{1.9}$$

This is totally confusing, unless one somehow explicitly indicates the coordinate system as part of the notation. For instance, one could explicitly indicate which coordinate is being held fixed. Thus the last equation would read

$$\left.\frac{\partial}{\partial u}\right|_w = \left.\frac{\partial}{\partial u}\right|_v + \left.\frac{\partial v}{\partial u}\right|_w \left.\frac{\partial}{\partial v}\right|_u. \tag{1.10}$$

In this notation, the general chain rule for converting from u, v derivatives to x, y derivatives is

$$\left.\frac{\partial}{\partial x}\right|_y = \left.\frac{\partial u}{\partial x}\right|_y \left.\frac{\partial}{\partial u}\right|_v + \left.\frac{\partial v}{\partial x}\right|_y \left.\frac{\partial}{\partial v}\right|_u, \tag{1.11}$$

$$\left.\frac{\partial}{\partial y}\right|_x = \left.\frac{\partial u}{\partial y}\right|_x \left.\frac{\partial}{\partial u}\right|_v + \left.\frac{\partial v}{\partial y}\right|_x \left.\frac{\partial}{\partial v}\right|_u. \tag{1.12}$$

1.6 Systems of ordinary differential equations

A vector field is closely related to a system of ordinary differential equations. In the two dimensional case such a system might be expressed by

$$\frac{dx}{dt} = f(x, y) \tag{1.13}$$

$$\frac{dy}{dt} = g(x, y). \tag{1.14}$$

The intuitive meaning of such an equation is that the point on the manifold is a function of time t, and its coordinates are changing according to the system of ordinary differential equations.

If we have a solution of such an equation, and if z is some smooth function on the manifold, then the rate of change of z is give by the chain rule by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{1.15}$$

According to the equation this is

$$\frac{dz}{dt} = f(x,y)\frac{\partial z}{\partial x} + g(x,y)\frac{\partial z}{\partial x} = \mathbf{v}z. \tag{1.16}$$

This shows that the system of ordinary differential equations and the vector field are effectively the same thing.

1.7 The straightening out theorem

Theorem 1 (Straightening out theorem) . *If*

$$\mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \neq 0 \tag{1.17}$$

is a vector field that is non-zero near some point, then near that point there is another coordinate system $u_1, \ldots, u_n$ *in which it has the form*

$$\mathbf{v} = \frac{\partial}{\partial u_j}. \tag{1.18}$$

Here is the idea of the proof of the straightening out theorem. Say that $v_j \neq 0$. Solve the system of differential equations

$$\frac{dx_i}{dt} = v_i \tag{1.19}$$

with initial condition 0 on the surface $x_j = 0$. This can be done locally, by the existence theorem for systems of ordinary differential equations with smooth coefficients. The result is that x_i is a function of the coordinates x_i for $i \neq j$ restricted to the surface $x_j = 0$ and of the time parameter t. Furthermore, since $dx_j/dt \neq 0$, the condition $t = 0$ corresponds to the surface $x_j = 0$. So if $x_1, \ldots, x_n$ corresponds to a point in M near the given point, we can define for $i \neq j$ the coordinates u_i to be the initial value of x_i on the $x_j = 0$, and we can define $u_j = t$. In these coordinates the differential equation becomes

$$\frac{du_i}{dt} = 0, i \neq j, \tag{1.20}$$

$$\frac{du_j}{dt} = 1. \tag{1.21}$$

Example. Consider the vector field

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{1.22}$$

away from the origin. The corresponding system is

$$\frac{dx}{dt} = -y \tag{1.23}$$

$$\frac{dy}{dt} = x. \tag{1.24}$$

Take the point to be $y = 0$, with $x > 0$. Take the initial condition to be $x = r$ and $y = 0$. Then $x = r\cos(t)$ and $y = r\sin(t)$. So the coordinates in which the straightening out takes place are polar coordinates r, t. Thus if we write $x = r\cos(\phi)$ and $y = r\sin(\phi)$, we have

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial \phi}, \tag{1.25}$$

where the partial derivative with respect to ϕ is taken with r held fixed.

Example. Consider the Euler vector field

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}, \tag{1.26}$$

where the partial derivative with respect to r is taken with fixed ϕ. We need to stay away from the zero at the origin. If we let $t = \ln(r)$, then this is

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r} = \frac{\partial}{\partial t}, \tag{1.27}$$

where the t derivative is taken with ϕ fixed.

1.8 Linearization at a zero

Say that a vector field defining a system of ordinary differential equations has an isolated zero. Thus the coefficients satisfy $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$ at the point with coordinate values x^*, y^*. At a zero of the vector field the solution of the system of ordinary differential equations has a fixed point.

Write $\tilde{x} = x - x^*$ and $\tilde{y} = y - y^*$. Then the differential equation takes the form

$$\frac{d\tilde{x}}{dt} = f(x^* + \tilde{x}, y^* + \tilde{y}) \tag{1.28}$$

$$\frac{d\tilde{y}}{dt} = g(x^* + \tilde{x}, y^* + \tilde{y}). \tag{1.29}$$

Expand in a Taylor series about x^*, y^*. The result is

$$\frac{d\tilde{x}}{dt} = \frac{\partial f(x^*, y^*)}{\partial x}\tilde{x} + \frac{\partial f(x^*, y^*)}{\partial y}\tilde{y} + \cdots \tag{1.30}$$

$$\frac{d\tilde{y}}{dt} = \frac{\partial g(x^*, y^*)}{\partial x}\tilde{x} + \frac{\partial g(x^*, y^*)}{\partial y}\tilde{y} + \cdots. \tag{1.31}$$

The linearization is the differential equation where one neglects the higher order terms. It may be written

$$\frac{d\tilde{x}}{dt} = \frac{\partial f(x^*, y^*)}{\partial x}\tilde{x} + \frac{\partial f(x^*, y^*)}{\partial y}\tilde{y} \tag{1.32}$$

$$\frac{d\tilde{y}}{dt} = \frac{\partial g(x^*, y^*)}{\partial x}\tilde{x} + \frac{\partial g(x^*, y^*)}{\partial y}\tilde{y}. \tag{1.33}$$

It is of the form

$$\frac{d\tilde{x}}{dt} = a\tilde{x} + b\tilde{y} \tag{1.34}$$

$$\frac{d\tilde{y}}{dt} = c\tilde{x} + c\tilde{y}, \tag{1.35}$$

where the coefficients are given by the values of the partial derivatives at the point where the vector field vanishes. It can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}. \tag{1.36}$$

The behavior of the linearization is determined by the eigenvalues of the matrix. Here are some common cases.

Stable node Real eigenvalues with $\lambda_1 < 0, \lambda_2 < 0$.

Unstable node Real eigenvalues with $\lambda_1 > 0, \lambda_2 > 0$.

Hyperbolic fixed point (saddle) Real eigenvalues with $\lambda_1 < 0 < \lambda_2$.

Stable spiral Nonreal eigenvalues with $\lambda = \mu \pm i\omega$, $\mu < 0$.

Unstable spiral Nonreal eigenvalues with $\lambda = \mu \pm i\omega$, $\mu > 0$.

Elliptic fixed point (center) Nonreal eigenvalues $\lambda = \pm i$.

There are yet other cases when one of the eigenvalues is zero.

Example. A classic example is the pendulum

$$\frac{dq}{dt} = \frac{1}{m}p \tag{1.37}$$

$$\frac{dp}{dt} = -mg\sin(\frac{1}{a}q). \tag{1.38}$$

Here $q = a\theta$ represents displacement, and p represents momentum. The zeros are at $\theta = n\pi$. When n is even this is the pendulum at rest in a stable position; when n is odd this is the pendulum at rest upside down, in a very unstable position. The linearization at a zero is

$$\frac{dq}{dt} = \frac{1}{m}p \tag{1.39}$$

$$\frac{dp}{dt} = -\frac{mg}{a}(-1)^n q. \tag{1.40}$$

In matrix form this is

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -(-1)^n \frac{mg}{a} & 0 \end{pmatrix}\begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix}. \tag{1.41}$$

The eigenvalues λ are given by $\lambda^2 = -(-1)^n \frac{g}{a}$. When n is even we get an elliptic fixed point, while when n is odd we get a hyperbolic fixed point.

The following question is natural. Suppose that a vector field has an isolated zero. At that zero it has a linearization. When is it possible to choose coordinates so that the vector field is given in those new coordinates by its linearization? It turns out that the answer to this question is negative in general [13].

1.9 Problems

1. Straightening out. A vector field that is non-zero at a point can be transformed into a constant vector field near that point by a change of coordinate system. Pick a point away from the origin, and find coordinates u, v so that

$$-\frac{y}{x^2+y^2}\frac{\partial}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial}{\partial y} = \frac{\partial}{\partial u}. \tag{1.42}$$

2. Linearization. Consider the vector field

$$\mathbf{u} = x(4-x-y)\frac{\partial}{\partial x} + (x-2)y\frac{\partial}{\partial y}. \tag{1.43}$$

   Find its zeros. At each zero, find its linearization. For each linearization, find its eigenvalues. Use this information to sketch the vector field.

3. Nonlinearity. Consider the vector field

$$\mathbf{v} = (1+x^2+y^2)y\frac{\partial}{\partial x} - (1+x^2+y^2)x\frac{\partial}{\partial y}. \tag{1.44}$$

   Find its linearization at zero. Show that there is no coordinate system near 0 in which the vector field is expressed by its linearization. Hint: Solve the associated system of ordinary differential equations, both for $\mathbf{v}$ and for its linearization. Find the period of a solution is both cases.

4. Nonlinear instability. Here is an example of a fixed point where the linear stability analysis gives an elliptic fixed point, but changing to polar coordinates shows the unstable nature of the fixed point:

$$\frac{dx}{dt} = -y + x(x^2+y^2) \tag{1.45}$$

$$\frac{dy}{dt} = x + y(x^2+y^2). \tag{1.46}$$

   Change the vector field to the polar coordinate representation, and solve the corresponding system of ordinary differential equations.

Chapter 2

Forms

2.1 The dual space

The objects that are dual to vectors are 1-forms. A *1-form* is a linear transformation from the n-dimensional vector space V to the real numbers. The 1-forms also form a vector space V^* of dimension n, often called the dual space of the original space V of vectors. If α is a 1-form, then the value of α on a vector $\mathbf{v}$ could be written as $\alpha(\mathbf{v})$, but instead of this we shall mainly use $\alpha \cdot \mathbf{v}$. The condition of being linear says that

$$\alpha \cdot (a\mathbf{u} + b\mathbf{v}) = a\alpha \cdot \mathbf{u} + b\alpha \cdot \mathbf{v}. \tag{2.1}$$

The vector space of all 1-forms is called V^*. Sometimes it is called the *dual space* of V.

It is important to note that the use of the dot in this context is not meant to say that this is the inner product (scalar product) of two vectors. In Part III of this book we shall see how to associate a form $\mathbf{gu}$ to a vector $\mathbf{u}$, and the inner product of $\mathbf{u}$ with $\mathbf{w}$ will then be $\mathbf{gu} \cdot \mathbf{w}$.

There is a useful way to picture vectors and 1-forms. A vector is pictured as an arrow with its tail at the origin of the vector space V. A 1-form is pictured by its contour lines (in two dimensions) or its contour planes (in three dimensions) Figure **??**. These are parallel lines or parallel planes that represent when the values of the 1-form are multiples of some fixed small number $\delta > 0$. Sometimes it is helpful to indicate which direction is the direction of increase. The value $\alpha \cdot \mathbf{v}$ of a 1-form α on a vector $\mathbf{v}$ is the value associated with the contour that passes through the head of the arrow.

Each contour line is labelled by a numerical value. In practice one only draws contour lines corresponding to multiples of some fixed small numerical value. Since this numerical value is somewhat arbitrary, it is customary to just draw the contour lines and indicate the direction of increase. The contour line passing through the origin has value zero. A more precise specification of the 1-form would give the numerical value associated with at least one other contour line.

A scalar multiple $c\alpha$ of a 1-form α has contour lines with increased or decreased spacing, and possibly with reversed direction of increase. The sum $\alpha + \beta$ of two 1-forms α, β is defined by adding their values. The sum of two 1-forms may also be indicated graphically by a parallelogram law. The two forms define an array of parallelograms. The contour lines of the sum of the two forms are lines through two (appropriately chosen) corners of the parallelograms Figure **??**.

2.2 Differential 1-forms

A *differential form* is a linear transformation from the vector fields to the reals given by

$$\alpha = \sum_{i=1}^{n} a_i dx_i. \tag{2.2}$$

The value of α on the vector field $\mathbf{v}$ is

$$\alpha \cdot \mathbf{v} = \sum_{i=1}^{n} a_i v_i. \tag{2.3}$$

If z is a scalar function on M, then it has a *differential* given by

$$dz = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} dx_i. \tag{2.4}$$

This is a special kind of differential form. In general, a differential form that is the differential of a scalar is called an *exact* differential form.

If z is a smooth function on M, and $\mathbf{v}$ is a vector field, then the *directional derivative* of z along $\mathbf{v}$ is

$$dz \cdot \mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial z}{\partial x_i}. \tag{2.5}$$

It is another smooth function on M.

Theorem 2 (Necessary condition for exactness) *If $\alpha = \sum_{i=1}^{n} a_i dx_i$ is an exact differential form, then its coefficients satisfy the integrability conditions*

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}. \tag{2.6}$$

When the integrability condition is satisfied, then the differential form is said to be *closed*. Thus the theorem says that every exact form is closed.

In two dimensions an exact differential form is of the form

$$dh(x,y) = \frac{\partial h(x,y)}{\partial x}\, dx + \frac{\partial h(x,y)}{\partial y}\, dy. \tag{2.7}$$

If $z = h(x,y)$ this can be written in a shorter notation as

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy. \tag{2.8}$$

It is easy to picture an exact differential form in this two-dimensional case. Just picture contour curves of the function $z = h(x, y)$. These are curves defined by $z = h(x, y) = c$, where the values of c are spaced by some small $\delta > 0$. Notice that adding a constant to z is does not change the differential of z. It also does not change the contour curves of z. For determination of the differential form what is important is not the value of the function, since this has an arbitrary constant. Rather it is the spacing between the contour curves that is essential.

In this picture the exact differential form should be thought of a closeup view, so that on this scale the contour curves look very much like contour lines. So the differential form at a point depends only on the contour lines very near this point.

In two dimensions a general differential form is of the form

$$\alpha = f(x, y)\,dx + g(x, y)\,dy. \tag{2.9}$$

The condition for a closed form is

$$\frac{\partial g(x, y)}{\partial x} = \frac{\partial f(x, y)}{\partial y}. \tag{2.10}$$

If the form is not closed, then it is not exact. The typical differential form is not closed.

We could also write this as

$$\alpha = p\,dx + q\,dy. \tag{2.11}$$

The condition for a closed form is

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}. \tag{2.12}$$

It somewhat harder to picture a differential 1-form that is not exact. The idea is to draw contour lines near each point that somehow join to form contour curves. However the problem is that these contour curves now must have end points, in order to keep the density of lines near each point to be consistent with the definition of the differential form.

Example. A typical example of a differential form that is not exact is $y\,dx$. The contour lines are all vertical. They are increasing to the right in the upper half plane, and they are increasing to the left in the lower half plane. However the density of these contour lines must diminish near the x axis, so that some of the lines will have end points at their lower ends (in the upper half plane) or at their upper ends (in the lower half plane).

A differential form may be expressed in various coordinate systems. Say, for instance, that

$$\alpha = p\,dx + q\,dy. \tag{2.13}$$

We may write

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv, \tag{2.14}$$

$$dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv. \tag{2.15}$$

Inserting this in the expression for the 1-form α, we obtain

$$\alpha = \left(\frac{\partial x}{\partial u} p + \frac{\partial y}{\partial u} q\right) du + \left(\frac{\partial x}{\partial v} p + \frac{\partial y}{\partial v}\right) dv. \tag{2.16}$$

Contrast this with the corresponding equation 1.8 for vector fields; the coefficients do not transform the same way.

The condition that a differential form is closed or exact does not depend on the coordinate system. Notice that the theory of differential forms is extraordinarily different from the theory of vector fields. A nonzero vector field may always be straightened out locally. For differential forms this is only possible if the form is exact.

A final observation may help in making the comparison between forms and vector fields. If u is a smooth function on the manifold, then du is a well-defined 1-form. There is no need to for u to be part of a coordinate system. On the other hand, for the vector field $\frac{\partial}{\partial u}$ to be well-defined, it is necessary to specify what other variables are being held constant. For instance, we could specify that the coordinate system under consideration is u, w, or even explicitly indicate by writing $\frac{\partial}{\partial u}\big|_w$ that the variable w is to be held fixed.

2.3 Ordinary differential equations in two dimensions

A classic application of these ideas is ordinary differential equations in the plane. Such an equation is often written in the form

$$p\,dx + q\,dy = 0. \tag{2.17}$$

Here $p = f(x, y)$ and $q = g(x, y)$ are functions of x, y. The equation is determined by the differential form $p\,dx + q\,dy$, but two different forms may determine equivalent equations. For example, if $\mu = h(x, y)$ is a non-zero scalar, then the form $\mu p\,dx + \mu q\,dy$ is a quite different form, but it determines an equivalent differential equation.

If $p\,dx + q\,dy$ is exact, then $p\,dx + q\,dy = dz$, for some scalar z depending on x and y. The solution of the differential equation is then given implicitly by $z = c$, where c is constant of integration.

If $p\,dx + q\,dy$ is not exact, then one looks for an integrating factor μ such that

$$\mu(p\,dx + q\,dy) = dz \tag{2.18}$$

is exact. Once this is done, again the solution of the differential equation is then given implicitly by $z = c$, where c is constant of integration.

Theorem 3 *Suppose that $\alpha = p\,dx + q\,dy$ is a differential form in two dimensions that is non-zero near some point. Then α has a non-zero integrating factor μ near the point, so $\mu\alpha = ds$ for some scalar.*

This theorem follows from the theory of solutions of ordinary differential equations. Finding the integrating factor may not be an easy matter. However, there is a strategy that may be helpful.

Recall that if a differential form is exact, then it is closed. So if μ is an integrating factor, then

$$\frac{\partial \mu p}{\partial y} - \frac{\partial \mu q}{\partial x} = 0. \tag{2.19}$$

This condition may be written in the form

$$p\frac{\partial \mu}{\partial y} - q\frac{\partial \mu}{\partial x} + \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)\mu = 0. \tag{2.20}$$

Say that by good fortune there is an integrating factor μ that depends only on x. Then this gives a linear ordinary differential equation for μ that may be solved by integration.

Example. Consider the standard problem of solving the linear differential equation

$$\frac{dy}{dx} = -ay + b, \tag{2.21}$$

where a, b are functions of x. Consider the differential form $(ay-b)\,dx+dy$. Look for an integrating factor μ that depends only on x. The differential equation for μ is $-d\mu/dx = a\mu$. This has solution $\mu = e^A$, where A is a function of x with $dA/dx = a$. Thus

$$e^A(ay-b)\,dx + e^A\,dy = d(e^A y - S), \tag{2.22}$$

where S is a function of x with $dS/dx = e^A b$. So the solution of the equation is $y = e^{-A}(S+c)$.

Theorem 4 *Consider a differential form $\alpha = p\,dx + q\,dy$ in two dimensions. Suppose that near some point α is not zero. Suppose also that α is not closed near this point. Then near this point there is a new coordinate system u, v with $\alpha = u\,dv$.*

The proof is to note that if $\alpha = p\,dx + q\,dy$ is not zero, then it has a non-zero integrating factor with $\mu\alpha = dv$. So we can write $\alpha = u\,dv$, where $u = 1/\mu$. Since $u\,dv = p\,dx + q\,dy$, we have $u\partial v/\partial x = p$ and $u\partial v/\partial y = q$. It follows that $\partial q/\partial x - \partial p/\partial y = \partial u/\partial x \partial v/\partial y - \partial u/\partial y \partial v \partial x$. Since this is non-zero, the inverse function theorem shows that this is a legitimate change of coordinates.

The situation is already considerably more complicated in three dimensions, the canonical form is relatively complicated. The differential equations book by Ince [9] treats this situation.

2.4 The Hessian matrix and the second derivative test

Say that M is a manifold and z is a smooth function on M. The *first derivative test* says that if z has a local minimum or a local maximum at a point of M, then $dz = 0$ at that point. Consider, for instance, the 2-dimensional case. If $z = h(x, y)$, then at a local maximum or local minimum point $x = x_0, y = y_0$, we have

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = 0. \tag{2.23}$$

This gives two equations $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$ to solve to find the numbers x_0 and y_0.

Suppose again that $z = h(x, y)$, and look at a point $x = x^*, y = y^*$ where the differential $dz = d\,h(x, y)$ is zero. At that point

$$h(x, y) = h(x^*, y^*) + \frac{1}{2}\left(\frac{\partial^2 h(x^*, y^*)}{\partial x^2}\tilde{x}^2 + 2\frac{\partial^2 h(x^*, y^*)}{\partial x \partial y}\tilde{x}\tilde{y} + \frac{\partial^2 h(x^*, y^*)}{\partial y^2}\tilde{y}^2\right) + \cdots. \tag{2.24}$$

This suggests that behavior near x^*, y^* should be compared to that of the quadratic function

$$q(\tilde{x}, \tilde{y}) = \left(\frac{\partial^2 h(x^*, y^*)}{\partial x^2}\tilde{x}^2 + 2\frac{\partial^2 h(x^*, y^*)}{\partial x \partial y}\tilde{x}\tilde{y} + \frac{\partial^2 h(x^*, y^*)}{\partial y^2}\tilde{y}^2\right). \tag{2.25}$$

Write this quadratic form as

$$q(\tilde{x}, \tilde{y}) = \left(a\tilde{x}^2 + 2b\tilde{x}\tilde{y} + d\tilde{y}^2\right). \tag{2.26}$$

This can be written in matrix notation as

$$q(\tilde{x}, \tilde{y}) = \begin{pmatrix} \tilde{x} & \tilde{y} \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}. \tag{2.27}$$

The matrix of second partial derivatives evaluated at the point where the first partial derivatives vanish is called the *Hessian matrix*.

This leads to a *second derivative test*. Suppose that $z = h(x, y)$ is a smooth function. Consider a point where the first derivative test applies, that is, the differential $dz = d\,h(x, y)$ is zero. Consider the case when the Hessian is non-degenerate, that is, has determinant not equal to zero. Suppose first that the determinant of the Hessian matrix is strictly positive. Then the function has either a local minimum or a local maximum, depending on whether the trace is positive or negative. Alternatively, suppose that the determinant of the Hessian matrix is strictly negative. Then the function has a saddle point.

Theorem 5 (Morse lemma) *Let z be a function on a 2-dimensional manifold such that dz vanishes at a certain point. Suppose that the Hessian is non-degenerate at this point. Then there is a coordinate system u, v near the point with*

$$z = z_0 + \epsilon_1 u^2 + \epsilon_2 v^2, \tag{2.28}$$

where ϵ_1 and ϵ_2 are constants that each have the value ± 1.

For the Morse lemma, see J.Milnor, Morse Theory, Princeton University Press, Princeton, NJ, 1969 [11]. The theory of the symmetric Hessian matrix and the Morse lemma have a natural generalization to manifolds of dimension n.

2.5 Lagrange multipliers

There is an interesting version of the first derivative test that applies to the special situation when the manifold is defined implicitly by the solution of certain equations. Consider as an illustration the 3-dimensional case. Then one equation would define a 2-dimensional manifold, and two equations would define a 1-dimensional manifold.

Say that we have a function $v = g(x, y, z)$ such that $v = c$ defines a 2-dimensional manifold. Suppose that at each point of this manifold $dv \neq 0$. Then the tangent space to the manifold at each point consists of all vectors $\mathbf{z}$ at this point such that $dv \cdot \mathbf{z} = 0$. This is a 2-dimensional vector space.

Now suppose that there is a function $u = f(x, y, z)$ that we want to maximize or minimize subject to the constraint $v = g(x, y, z) = c$. Consider a point where the local maximum or local minimum exists. According to the first derivative test, we have $du \cdot \mathbf{z} = 0$ for every $\mathbf{z}$ tangent to the manifold at this point. Pick two linearly independent vectors $\mathbf{z}$ tangent to the manifold at this point. Then the equation $du \cdot \mathbf{z}$ for du gives two equations in three unknowns. Thus the solution is given as a multiple of the nonzero dv. So we have

$$du = \lambda\, dv. \tag{2.29}$$

The λ coefficient is known as the *Lagrange multiplier*.

This equation has a simple interpretation. It says that the only way to increase the function u is to relax the constraint $v = c$. In other words, the change du in u at the point must be completely due to the change dv that moves off the manifold. More precisely, the contour surfaces of u must be tangent to the contour surfaces of v at the point.

The Lagrange multiplier itself has a nice interpretation. Say that one is interested in how the maximum or minimum value depends on the constant c defining the manifold. We see that

$$\frac{du}{dc} = \lambda \frac{dv}{dc} = \lambda. \tag{2.30}$$

So the Lagrange multiplier describes the effect on the value of changing the constant defining the manifold.

Example. Say that we want to maximize or minimize $u = x+y+2z$ subject to $v = x^2 + y^2 + z^2 = 1$. The manifold in this case is the unit sphere. The Lagrange multiplier condition says that

$$du = dx + dy + 2\,dz = \lambda dv = \lambda(2x\,dx + 2y\,dy + 2z\,dz). \tag{2.31}$$

Thus $1 = 2\lambda x$, $1 = 2\lambda y$, and $2 = 2\lambda z$. Insert these in the constraint equation $x^2 + y^2 + z^2 = 1$. This gives $(1/4) + (1/4) + 1 = \lambda^2$, or $\lambda = \pm\sqrt{3/2}$. So $x = \pm\sqrt{2/3}/2$, $y = \pm\sqrt{2/3}/2$, $z = \pm\sqrt{2/3}$.

Say instead that we have two function $v = g(x, y, z)$ and $w = h(x, y, z)$ such that $v = a$ and $w = b$ defines a 1-dimensional manifold. Suppose that at each point of this manifold the differentials dv and dw are linearly idependent (neither is a multiple of the other). Then the tangent space to the manifold at each point consists of all vectors $\mathbf{z}$ at this point such that $dv \cdot \mathbf{z} = 0$ and $dw \cdot \mathbf{z} = 0$. This is a 1-dimensional vector space.

Now suppose that there is a function $u = f(x, y, z)$ that we want to maximize or minimize subject to the constraints $v = g(x, y, z) = a$ and $w = h(x, y, z) = b$. Consider a point where the local maximum or local minimum exists. According to the first derivative test, we have $du \cdot \mathbf{z} = 0$ for every $\mathbf{z}$ tangent to the manifold at this point. Pick a non-zero vector $\mathbf{z}$ tangent to the manifold at this point. Then the equation $du \cdot \mathbf{z}$ for du gives one equation in three unknowns. Thus the solution is given as a linear combination of the basis forms dv and dw at the point. So we have

$$du = \lambda\, dv + \mu\, dw \tag{2.32}$$

Thus we have two Lagrange multipliers when there are two constraints.

Example. Say that we want to maximize or minimize $u = x - 4y + 3z + z^2$ subject to $v = x - y = 0$ and $w = y - z = 0$. The manifold in this case is just a line through the origin. The Lagrange multiplier condition says that

$$dx - 4\, dy + (3 - 2z)\, dz = \lambda(dx - dy) + \mu(dy - dz). \tag{2.33}$$

Thus $1 = \lambda$, $-4 = -\lambda + \mu$, and $(3 - 2z) = -\mu$. When we solve we get $\mu = -3$ and so $z = 0$.

Of course we could also solve this example without Lagrange multipliers. Since the manifold is $x = y = z$, the function to be maximized or minimized is $u = z^2$, and this has its minimum at $z = 0$. The utility of the Lagrange multiplier technique in more complicated problems is that it is not necessary to do such a preliminary elimination before solving the problem.

Example. Here is a simple example to emphasize the point that the Lagrange multiplier technique is coordinate independent. Say that one wants to maximize or minimize z subject to $x^2 + y^2 + z^2 = 1$. The Lagrange multiplier method says to write $dz = \lambda(2x\, dx + 2y\, dy + 2z\, dz)$. This says that $x = y = 0$, and so $z = \pm 1$. In spherical polar coordinates this would be the problem of maximizing $r \cos(\theta)$ subject to $r^2 = 1$. This would give $dr \cos(\theta) - r \sin(\theta)\, d\theta = \lambda 2r\, dr$. Thus $\sin(\theta) = 0$, and the solution is $\theta = 0$ or $\theta = \pi$.

2.6 Covariant and contravariant

There is a terminology that helps clarify the relation between vector fields and differential forms. A scalar quantity z or a differential form $p\, du + q\, dv$ are both quantities that are associated with functions on the manifold. Such quantities are traditionally called *covariant*.

Once this terminology was established, it was natural to call the dual objects *contravariant*. These objects include points of the manifold and vector fields. Often a covariant object may be paired with a contravariant object to give a number or a scalar quantity. Thus, for example, consider the covariant quantity $z = h(u, v)$ and the point $u = a, v = b$, where a, b are real numbers. The corresponding number is $h(a, b)$. For another example, consider the vector field $\mathbf{v} = a\partial/\partial u + b\partial/\partial v$ and the differential form $p\,du + q\,dv$. These give the scalar quantity $ap + bv$.

The covariant-contravariant distinction is a central idea in mathematics. It tends to be lost, however, in certain special contexts. Part III of this book is an exploration of a situation when it is permitted to confuse covariant and contravariant quantities. However, usually it is illuminating to be alert to the difference.

2.7 Problems

1. Exact differentials. Is $(x^2 + y^2)\, dx + 2xy\, dy$ an exact differential form? If so, write it as the differential of a scalar.

2. Exact differentials. Is $(1 + e^x)\, dy + e^x(y - x)\, dx$ an exact differential? If so, write it as the differential of a scalar.

3. Exact differentials. Is $e^y\, dx + x(e^y + 1)\, dy$ an exact differential? If so, write it as the differential of a scalar.

4. Constant differential forms. A differential form usually cannot be transformed into a constant differential form, but there are special circumstances when that can occur. Is it possible to find coordinates u and v near a given point (not the origin) such that

$$-y\, dx + x\, dy = du? \tag{2.34}$$

5. Constant differential forms. A differential form usually cannot be transformed into a constant differential form, but there are special circumstances when that can occur. Is it possible to find coordinates u and v near a given point (not the origin) such that

$$-\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy = du? \tag{2.35}$$

6. Ordinary differential equations. Solve the differential equation $(xy^2 + y)\, dx - x\, dy = 0$ by finding an integrating factor that depends only on y.

7. Hessian matrix. Consider the function $z = x^3 y^2 (6 - x - y)$. Find the point in the x, y plane with $x > 0$, $y > 0$ where $dz = 0$. Find the Hessian matrix at this point. Use this to describe what type of local extremum exists at this point.

8. Lagrange multipliers. Use Lagrange multipliers to maximize $x^2 + y^2 + z^2$ subject to the restriction that $2x^2 + y^2 + 3z^2 = 1$.

Chapter 3

The exterior derivative

3.1 The exterior product

Let $V \times V$ be the set of ordered pairs $\mathbf{u}, \mathbf{v}$ of vectors in V. A *2-form* σ is an anti-symmetric bilinear transformation $\sigma : V \times V \to \mathbf{R}$. Thus for each fixed $\mathbf{v}$ the function $\mathbf{u} \mapsto \sigma(\mathbf{u}, \mathbf{v})$ is linear, and for each fixed $\mathbf{u}$ the function $\mathbf{v} \mapsto (\mathbf{u}, \mathbf{v})$ is linear. Furthermore, $\sigma(\mathbf{u}, \mathbf{v}) = -\sigma(\mathbf{v}, \mathbf{u})$. The vector space of all 2-forms is denoted $\Lambda^2 V^*$. It is a vector space of dimension $n(n-1)/2$.

A 2-form has a geometric interpretation. First consider the situation in the plane. Given two planar 2-forms, at least one of them is a multiple of the other. So the space of planar 2-forms is one-dimensional. However we should not think of such a 2-form as a number, but rather as a grid of closely spaced points. The idea is that the value of the 2-form is proportional to the number of points inside the parallelogram spanned by the two vectors. The actual way the points are arranged is not important; all that counts is the (relative) density of points. Actually, to specify the 2-form one needs to specify not only the points but also an orientation, which is just a way of saying that the sign of the answer needs to be determined.

In three-dimensional space one can think of parallel lines instead of points. The space of 2-forms in three-dimensional space has dimension 3, because these line can have various directions as well as different spacing. The value of the 2-form on a pair of vectors is proportional to the number of lines passing through the parallelogram spanned by the two vectors. Again, there is an orientation associated with the line, which means that one can perhaps think of each line as a thin coil wound in a certain sense.

The sum of two 2-forms may be given by a geometrical construction that somewhat resembles vector addition.

The *exterior product* (or wedge product) $\alpha \wedge \beta$ of two 1-forms is a 2-form. This is defined by

$$(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) = \det \begin{bmatrix} \alpha \cdot \mathbf{u} & \alpha \cdot \mathbf{v} \\ \beta \cdot \mathbf{u} & \beta \cdot \mathbf{v} \end{bmatrix} = (\alpha \cdot \mathbf{u})(\beta \cdot \mathbf{v}) - (\beta \cdot \mathbf{u})(\alpha \cdot \mathbf{v}). \tag{3.1}$$

Notice that $\alpha \wedge \beta = -\beta \wedge \alpha$. In particular $\alpha \wedge \alpha = 0$.

The exterior product of two 1-forms has a nice geometrical interpretation. On two dimensions each of the two 1-forms is given by a family of parallel lines. The corresponding 2-form consists of the points at the intersection of these lines.

In three dimensions each of the two 1-forms is given by a collection of parallel planes. The corresponding 2-form consists of the lines that are the intersections of these planes.

In a similar way, one can define a 3-form τ as an alternating trilinear function from ordered triples of vectors to the reals. In three dimensions a 3-form is pictured by a density of dots.

One way of getting a 3-form is by taking the exterior product of three 1-forms. The formula for this is

$$(\alpha \wedge \beta \wedge \gamma)(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{bmatrix} \alpha \cdot \mathbf{u} & \alpha \cdot \mathbf{v} & \alpha \cdot \mathbf{w} \\ \beta \cdot \mathbf{u} & \beta \cdot \mathbf{v} & \beta \cdot \mathbf{w} \\ \gamma \cdot \mathbf{u} & \gamma \cdot \mathbf{v} & \gamma \cdot \mathbf{w} \end{bmatrix} \tag{3.2}$$

In a similar way one can define r-forms on an n dimensional vector space V. The space of such r-forms is denoted $\Lambda^r V^*$, and it has dimension given by the binomial coefficient $\binom{n}{r}$. It is also possible to take the exterior product of r 1-forms and get an r-form. The formula for this multiple exterior product is again given by a determinant.

The algebra of differential forms is simple. The sum of two r-forms is an r form. The product of an r-form and an s-form is an $r+s$-form. This multiplication satisfies the associative law. It also satisfies the law

$$\beta \wedge \alpha = (-1)^{rs} \alpha \wedge \beta, \tag{3.3}$$

where α is an r-form and β is an s-form. For instance, if $r = s = 1$, then $\alpha \wedge \beta = -\beta \wedge \alpha$. On the other hand, if $r = 1, s = 2$, then $\alpha\beta = \beta\alpha$.

3.2 Differential r-forms

One can also have differential r-forms on a manifold. For instance, on three dimensions one might have a differential 2-form such as

$$\sigma = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy. \tag{3.4}$$

Here x, y, z are arbitrary coordinates, and a, b, c are smooth functions of x, y, z. Similarly, in three dimensions a typical 3-form might have the form

$$\tau = s\, dx \wedge dy \wedge dz. \tag{3.5}$$

Notice that these forms are created as linear combinations of exterior products of 1-forms.

Since these expressions are so common, it is customary in many contexts to omit the explicit symbol for the exterior product. Thus the forms might be written

$$\sigma = a\, dy\, dz + b\, dz\, dx + c\, dx\, dy \tag{3.6}$$

and

$$\tau = s\,dx\,dy\,dz. \tag{3.7}$$

The geometric interpretation of such forms is quite natural. For instance, in the three dimensional situation of these examples, a 1-form is represented by a family of surfaces, possibly ending in curves. Near each point of the manifold the family of surfaces looks like a family of parallel contour planes. A 2-form is represented by a family of curves, possibly ending in points. Near each point of the manifold they look like a family of parallel lines. Similarly, a 3-form is represented by a cloud of points. While the density of points near a given point of the manifold is constant, at distant points of the manifold the densities may differ.

3.3 Properties of the exterior derivative

The exterior derivative of an r-form α is an $r+1$-form $d\alpha$. It is defined by taking the differentials of the coefficients of the r-form. For instance, for the 1-form

$$\alpha = p\,dx + q\,dy + r\,dz \tag{3.8}$$

the differential is

$$d\alpha = dp\,dx + dq\,dy + dr\,dz. \tag{3.9}$$

This can be simplified as follows. First, note that

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz. \tag{3.10}$$

Therefore

$$dp\,dx = \frac{\partial p}{\partial y}\,dy\,dx + \frac{\partial p}{\partial z}\,dz\,dx = -\frac{\partial p}{\partial y}\,dx\,dy + \frac{\partial p}{\partial z}\,dz\,dx. \tag{3.11}$$

Therefore, the final answer is

$$d\alpha = d(p\,dx+q\,dy+r\,dz) = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\,dy\,dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\,dz\,dx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dx\,dy. \tag{3.12}$$

Similarly, suppose that we have a 2-form

$$\sigma = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{3.13}$$

Then

$$d\sigma = da\,dy\,dz + db\,dz\,dx + dc\,dx\,dy = \frac{\partial a}{\partial x}\,dx\,dy\,dz + \frac{\partial b}{\partial y}\,dy\,dz\,dx + \frac{\partial c}{\partial z}\,dz\,dx\,dy. \tag{3.14}$$

This simplifies to

$$d\sigma = d(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy) = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)\,dx\,dy\,dz. \tag{3.15}$$

The geometrical interpretation of the exterior derivative is natural. Consider first the case of two dimension. If α is a 1-form, then it is given by a family of curves, possibly with end points. The derivative $d\alpha$ corresponds to these end points. They have an orientation depending on which end of the curve they are at.

In three dimensions, if α is a 1-form, then it is given by contour surfaces, possibly ending in curves. The 2-form $d\alpha$ is given by the curves. Also, if σ is a 2-form, then it is given by curves that may terminate. Then $d\sigma$ is a 3-form represented by the termination points.

The exterior derivative satisfies various general properties. The exterior derivative of an r-form is an $r+1$ form. There is a product rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta, \tag{3.16}$$

where α is an r-form and β is an s-form. The reason for the $(-1)^r$ is that the d has to be moved past the r form, and this picks up r factors of -1. Another important property is that applying the exterior derivative twice always gives zero, that is, for an arbitrary s-form β we have

$$dd\beta = 0. \tag{3.17}$$

3.4 The integrability condition

This last property has a geometrical interpretation. Take for example a scalar s. Its differential is $\alpha = ds$, which is an exact differential. Therefore ds is represented by curves without end points (two dimensions) or by surfaces without ending curves (three dimensions). This explains why $d\alpha = dds = 0$.

Similarly, consider a 1-form α in three dimensions. Its differential is a 2-form $\sigma = d\alpha$. The 1-form α is represented by surfaces, which may terminate in closed curves. These closed curves represent the 2 form $d\alpha$. Since they have no end points, we see that $d\sigma = dd\alpha = 0$.

In general, if $d\beta = 0$, then we say that β is a closed form. If $\beta = d\alpha$, we say that β is an exact form. The general fact is that if β is exact, then β is closed. The condition that $d\beta = 0$ is called the integrability condition, since it is necessary for the possibility that β can be integrated to get α with $\beta = d\alpha$.

Example. Consider the 2-form $y\,dx$. This is represented by vertical lines that terminate at points in the plane. The density of these lines is greater as one gets farther from the x axis. The increase is to the right above the x axis, and it is to the left below the y axis. The differential of $y\,dx$ is $dy\,dx = -dx\,dy$. This 2-form represents the cloud of terminating points, which has a uniform density. The usual convention that the positive orientation is counterclockwise. So the orientations of these source points are clockwise. This is consistent with the direction of increase along the contours lines.

3.5 Gradient, curl, divergence

Consider the case of three dimensions. Anyone familiar with vector analysis will notice that if s is a scalar, then the formula for ds resembles the formula for the gradient in cartesian coordinates. Similarly, if α is a 1-form, then the formula for $d\alpha$ resembles the formula for the curl in cartesian coordinates. The formula $dds = 0$ then corresponds to the formula $\mathrm{curl}\,\mathrm{grad}s = 0$.

In a similar way, if σ is a 2-form, then the formula for $d\sigma$ resembles the formula for the divergence in cartesian coordinates. The formula $dd\alpha = 0$ then corresponds to the formula $\mathrm{div}\,\mathrm{curl}\mathbf{v} = 0$.

There are, however, important distinctions. First, the differential form formulas take the same form in arbitrary coordinate systems. This is not true for the formulas for the divergence, curl, and divergence. The reason is that the usual definitions of divergence, curl, and divergence are as operations on vector fields, not on differential forms. This leads to a much more complicated theory, except for the very special case of cartesian coordinates on Euclidean space. We shall examine this issue in detail in the third part of this book.

Second, the differential form formulas have natural formulations for manifolds of arbitrary dimension. While the gradient and divergence may also be formulated in arbitrary dimensions, the curl only works in three dimensions.

This does not mean that notions such as gradient of a scalar (a vector field) or divergence of a vector field (a scalar) are not useful and important. Indeed, in some situations they play an essential role. However one should recognize that these are relatively complicated objects. Their nature will be explored in the second part of this book (for the divergence) and in the third part of this book (for the gradient and curl).

The same considerations apply to the purely algebraic operations, at least in three dimensions. The exterior product of two 1-forms resembles in some way the cross product of vectors, while the exterior product of a 1-form and a 2-form resembles a scalar product of vectors. Thus the wedge product of three 1-forms resembles the triple scalar product of vector analysis. Again these are not quite the same thing, and the relation will be explored in the third part of this book.

3.6 Problems

1. Say that the differential 1-form $\alpha = p\,dx + q\,dy + r\,dz$ has an integrating factor $\mu \neq 0$ such that $\mu\alpha = ds$. Prove that $\alpha \wedge d\alpha = 0$. Also, express this condition as a condition on p, q, r and their partial derivatives.

2. Show that $\alpha = dz - y\,dx - dy$ has no integrating factor.

3. Show that the differential 1-form $\alpha = yz\,dx + xz\,dy + dz$ passes the test for an integrating factor.

4. In the previous problem it might be difficult to guess the integrating factor. Show that $\mu = e^{xy}$ is an integrating factor, and find s with $\mu\alpha = ds$.

5. The differential 2-form $\omega = (2xy - x^2)\,dx\,dy$ is of the form $\omega = d\alpha$, where α is a 1-form. Find such an α. Hint: This is too easy; there are many solutions.

6. The differential 3-form $\sigma = (yz + x^2z^2 + 3xy^2z)\,dx\,dy\,dz$ is of the form $\sigma = d\omega$, where ω is a 2-form. Find such an ω. Hint: Many solutions.

7. Let $\sigma = xy^2z\,dy\,dz - y^3z\,dz\,dx + (x^2y + y^2z^2)\,dx\,dy$. Show that this 2-form σ satisfies $d\sigma = 0$.

8. The previous problem gives hope that $\sigma = d\alpha$ for some 1-form α. Find such an α. Hint: This may require some experimentation. Try α of the form $\alpha = p\,dx + q\,dy$, where p, q are functions of x, y, z. With luck, this may work. Remember that when integrating with respect to z the constant of integration is allowed to depend on x, y.

Chapter 4

Integration and Stokes's theorem

4.1 One-dimensional integrals

A one-dimensional manifold C is described by a single coordinate t. Consider an interval on the manifold bounded by $t = a$ and $t = b$. There are two possible orientations of this manifold, from $t = a$ to $t = b$, or from $t = b$ to $t = a$. Suppose for the sake of definiteness that the manifold has the first orientation. Then the differential form $f(t)\,dt$ has the integral

$$\int_C f(t)\,dt = \int_{t=a}^{t=b} f(t)\,dt. \tag{4.1}$$

If s is another coordinate, then t is related to s by $t = g(s)$. Furthermore, there are numbers p, q such that $a = g(p)$ and $b = g(q)$. The differential form is thus $f(t)\,dt = f(g(s))g'(s)\,ds$. The end points of the manifold are $s = p$ and $s = q$. Thus

$$\int_C f(t)\,dt = \int_{s=p}^{s=q} f(g(s))g'(s)\,ds. \tag{4.2}$$

The value of the integral thus does not depend on which coordinate is used.

Notice that this calculation depends on the fact that $dt/ds = g'(s)$ is nonzero. However we could also consider a smooth function u on the manifold that is not a coordinate. Several points on the manifold could give the same value of u, and du/ds could be zero at various places. However we can express $u = h(s)$ and $du/ds = h'(s)$ and define an integral

$$\int_C f(u)\,du = \int_{s=p}^{s=q} f(h(s))h'(s)\,ds. \tag{4.3}$$

Thus the differential form $f(u)\,du$ also has a well-defined integral on the manifold, even though u is not a coordinate.

4.2 Integration on manifolds

Next look at the two dimensional case. Say that we have a coordinate system x, y in a two-dimensional oriented manifold. Consider a region R bounded by curves $x = a$, $x = b$, and by $y = c$, $y = d$. Suppose that the orientation is such that one goes around the region in the order a, b then c, d then b, a then d, c. Then the differential form $f(x, y)\, dx\, dy$ has integral

$$\int_R f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\, dy\right],\ dx. \quad (4.4)$$

The limits are taken by going around the region in the order given by the orientation, first a, b then c, d. We could also have taken first b, a then d, c and obtained the same result.

Notice, by the way, that we could also define an integral with $dy\, dx$ in place of $dx\, dy$. This would be

$$\int_R f(x,y)\, dy\, dx = \int_b^a \left[\int_c^d f(x,y)\, dy\right] dx = \int_c^d \left[\int_b^a f(x,y)\, dx\right],\ dy. \quad (4.5)$$

The limits are taken by going around the region in the order given by the orientation, first c, d then b, a. We could also have taken d, c then a, b and obtained the same result. This result is precisely the negative of the previous result. This is consistent with the fact that $dy\, dx = -dx\, dy$.

These formula have generalizations. Say that the region is given by letting x go from a to b and y from $h(x)$ to $k(x)$. Alternatively, it might be given by letting y go from c to d and x from $p(y)$ to $q(y)$. This is a more general region than a rectangle, but the same kind of formula applies:

$$\int_R f(x,y)\, dx\, dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\, dx\right] dy = \int_a^b \left[\int_{h(x)}^{k(x)} f(x,y)\, dy\right],\ dx. \quad (4.6)$$

There is yet one more generalization, to the case where the differential form is $f(u, v)\, du\, dv$, but u, v do not form a coordinate system. Thus, for instance, the 1-form du might be a multiple of dv at a certain point, so that $du\, dv$ would be zero at that point. However we can define the integral by using the customary change of variable formula:

$$\int_R f(u,v)\, du\, dv = \int_R f(u,v) \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) dx\, dy. \quad (4.7)$$

In fact, since $du = \partial u/\partial x\, dx + \partial u/\partial y\, dy$ and $dv = \partial v/\partial x\, dx + \partial v/\partial y\, dy$, this is just saying that the same differential form has the same integral.

In fact, we could interpret this integral directly as a limit of sums involving only the u, v increments. Partition the manifold by curves of constant x and constant y. This divides the manifold into small regions that look something like parallelograms. Then we could write this sum as

$$\int_R f(u,v)\,du\,dv \approx \sum f(u,v)\left(\Delta u_x\,\Delta v_y - \Delta v_x\,\Delta u_y\right). \tag{4.8}$$

Here the sum is over the parallelograms. The quantity Δu_x is the increment in u from x to $x+\Delta x$, keeping y fixed, along one side of the parallelogram. The quantity Δv_y is the increment in v from y to $y+\Delta y$, keeping y fixed, along one side of the parallelogram. The other quantities are defined similarly. The u, v value is evaluated somewhere inside the parallelogram. The minus sign seems a bit surprising, until one realizes that going around the oriented boundary of the parallelogram the proper orientation makes a change from x to $x+\Delta x$ followed by a change from y to $y+\Delta y$, or a change from y to $y+\Delta y$ followed by a change from $x+\Delta x$ to x. So both terms have the form $\Delta u\,\Delta v$, where the changes are now taken along two sides in the proper orientation, first the change in u, then the change in v.

4.3 The fundamental theorem

The fundamental theorem of calculus says that for every scalar function s we have

$$\int_C ds = s(Q) - s(P). \tag{4.9}$$

Here C is an oriented path from point P to point Q. Notice that the result does not depend on the choice of path. This is because ds is an exact form.

As an example, we can take a path in space. Then $ds = \partial s/\partial x\,dx + \partial s/\partial y\,dy + \partial s/\partial z\,dz$. So

$$\int_C ds = \int_C \frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial y}\,dy + \frac{\partial z}{\partial z}\,dz = \int_C \left(\frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial z}\frac{dz}{dt}\right)dt. \tag{4.10}$$

By the chain rule this is just

$$\int_C ds = \int_C \frac{ds}{dt}\,dt = s(Q) - s(P). \tag{4.11}$$

4.4 Green's theorem

The next integral theorem is Green's theorem. It says that

$$\int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy = \int_{\partial R} p\,dx + q\,dy. \tag{4.12}$$

Here R is an oriented region in two dimensional space, and ∂R is the curve that is its oriented boundary. Notice that this theorem may be stated in the succinct form

$$\int_R d\alpha = \int_{\partial R} \alpha. \tag{4.13}$$

The proof of Green's theorem just amounts to applying the fundamental theorem of calculus to each term. Thus for the second term one applies the fundamental theorem of calculus in the x variable for fixed y.

$$\int_R \frac{\partial q}{\partial x}\,dx\,dy = \int_c^d \left[\int_{C_y} q\,dx\right]\,dy = \int_c^d \left[q(C_y^+) - q(C_y^-)\right]\,dy. \tag{4.14}$$

This is

$$\int_c^d q(C_y^+)\,dy + \int_d^c q(C_y^-)\,dy = \int_{\partial R} q\,dy. \tag{4.15}$$

The other term is handled similarly, except that the fundamental theorem of calculus is applied with respect to the x variable for fixed y. Then such regions can be pieced together to give the general Green's theorem.

4.5 Stokes's theorem

The most common version of Stokes's theorem says that for a oriented two dimensional surface S in a three dimensional manifold with oriented boundary curve ∂S we have

$$\int_S \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\,dy\,dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\,dz\,dx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dx\,dy = \int_{\partial S} (p\,dx + q\,dy + r\,dz). \tag{4.16}$$

Again this has the simple form

$$\int_S d\alpha = \int_{\partial S} \alpha. \tag{4.17}$$

This theorem reduces to Green's theorem. The idea is to take coordinates u, v on the surface S and apply Green's theorem in the u, v coordinates. In the theorem the left hand side is obtained by taking the form $p\,dx + q\,dy + r\,dz$ and applying d to it. The key observation is that when the result of this is expressed in the u, v coordinates, it is the same as if the form $p\,dx + q\,dy + r\,dz$ were first expressed in the u, v coordinates and then d were applied to it. In this latter form Green's theorem applies directly.

Here is the calculation. To make it simple, consider only the $p\,dx$ term. Then taking d gives

$$d(p\,dx) = \left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right)\,dx = \frac{\partial p}{\partial z}\,dz\,dx - \frac{\partial p}{\partial y}\,dx\,dy. \tag{4.18}$$

In u, v coordinates this is

$$d(p\,dx) = \left[\frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) - \frac{\partial p}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\right]\,du\,dv. \tag{4.19}$$

There are four terms in all.

Now we do it in the other order. In u, v coordinates we have

$$p\,dx = p\frac{\partial x}{\partial u}\,du + p\frac{\partial x}{\partial v}\,dv. \tag{4.20}$$

Taking d of this gives

$$d\left(p\frac{\partial x}{\partial u}\,du + p\frac{\partial x}{\partial v}\,dv\right) = \left[\frac{\partial}{\partial u}\left(p\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial x}{\partial u}\right)\right]\,du\,dv. \tag{4.21}$$

The miracle is that the second partial derivatives cancel. So in this version

$$d\left(p\frac{\partial x}{\partial u}\,du + p\frac{\partial x}{\partial v}\,dv\right) = \left[\frac{\partial p}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial x}{\partial u}\right]\,du\,dv. \tag{4.22}$$

Now we can express $\partial p/\partial u$ and $\partial p/\partial v$ by the chain rule. This gives at total of six terms. But two of them cancel, so we get the same result as before.

4.6 Gauss's theorem

Let W be an oriented three dimensional region, and let ∂W be the oriented surface that forms its boundary. Then Gauss's theorem states that

$$\int_W \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)\,dx\,dy\,dz = \int_{\partial W} a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{4.23}$$

Again this has the form

$$\int_W d\sigma = \int_{\partial W} \sigma, \tag{4.24}$$

where now σ is a 2-form. The proof of Gauss's theorem is similar to the proof of Green's theorem.

4.7 The generalized Stokes's theorem

The generalized Stoke's theorem says that

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega. \tag{4.25}$$

Here ω is a $(k-1)$-form, and $d\omega$ is a k-form. Furthermore, Ω is a k dimensional region, and $\partial\Omega$ is its $(k-1)$-dimensional oriented boundary. The forms may be expressed in arbitrary coordinate systems.

4.8 References

A classic short but rigorous account of differential forms is given in the book of Spivak [15]. The book by Agricola and Friedrich [1] gives a more advanced treatment. Other books on differential forms include those by Cartan [2], do Carmo [3], Edelen [4], Flanders [7], Screiber [14], and Weintraub [17]. There are also advanced calculus texts by Edwards [5] and by Hubbard and Hubbard [8].

There are many sources for tensor analysis; a classical treatment may be found in Lovelock and Rund [10]. There is a particularly unusual and sophisticated treatment in the book of Nelson [12]. Differential forms are seen to be special kinds of tensors: covariant alternating tensors.

The most amazing reference that this author has encountered is an elementary book by Weinreich [16]. He presents the geometric theory of differential forms in pictures, and these pictures capture the geometrical essence of the situation. The principal results of the theory are true by inspection. However his terminology is most unusual. He treats only the case of dimension three. Thus he has the usual notion of covariant 1-form, 2-form, and 3-form. In his terminology the corresponding names for these are stack, sheaf, and scalar capacity (or swarm). There are also corresponding contravariant objects corresponding to what are typically called 1-vector, 2-vector (surface element), and 3-vector (volume element). The names in this case are arrow, thumbtack, and scalar capacity. The correspondence between his objects and the usual tensors may actually be slightly more complicated than this, but the intent is certainly to explicate the usual calculus geometrically. In particular, he gives geometric explanations of the usual algebraic and differential operations in all these various cases.

4.9 Problems

1. Let C be the curve $x^2 + y^2 = 1$ in the first quadrant from $(1, 0)$ to $(0, 1)$. Evaluate
$$\int_C xy\,dx + (x^2 + y^2)\,dy. \tag{4.26}$$

2. Let C be a curve from $(2, 0)$ to $(0, 3)$. Evaluate
$$\int_C 2xy\,dx + (x^2 + y^2)\,dy. \tag{4.27}$$

3. Consider the problem of integrating the differential form
$$\alpha = -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \tag{4.28}$$
from $(1, 0)$ to $(-1, 0)$ along some curve avoiding the origin. There is are an infinite set of possible answers, depending on the curve. Describe all such answers.

4. Let R be the region $x^2 + y^2 \leq 1$ with $x \geq 0$ and $y \geq 0$. Let ∂R be its boundary (oriented counterclockwise). Evaluate directly
$$\int_{\partial R} xy\,dx + (x^2 + y^2)\,dy. \tag{4.29}$$

5. This continues the previous problem. Verify Green's theorem in this special case, by explicitly calculating the appropriate integral over the region R.

6. Let
$$\alpha = -y\,dx + x\,dy + xy\,dz. \tag{4.30}$$
Fix $a > 0$. Consider the surface S that is the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$. Integrate α over the boundary ∂S of this surface (a counterclockwise circle in the x, y plane).

7. This continues the previous problem. Verify Stokes's theorem in this special case, by explicitly calculating the appropriate integral over the surface S.

8. Let $\sigma = xy^2z\,dy\,dz - y^3z\,dz\,dx + (x^2y + y^2z^2)\,dx\,dy$. Integrate σ over the sphere $x^2 + y^2 + z^2 = a^2$. Hint: This should be effortless.

Part II

Analysis with a volume

Chapter 5

The divergence theorem

5.1 Contraction

There is another operation called *interior product* (or contraction). In the case of interest to us, it is a way of defining the product of a vector with a k-form to get a $k-1$ form. We shall mainly be interested in the case when $k = 1, 2, 3$. When $k = 1$ this is already familiar. For a 1-form α the interior product $\mathbf{u}\rfloor\alpha$ is defined to be the scalar $\alpha \cdot \mathbf{v}$.

The interior product of a vector $\mathbf{u}$ with a 2-form σ is a 1-form $\mathbf{u}\rfloor\sigma$. It is defined by

$$(\mathbf{u}\rfloor\sigma) \cdot \mathbf{v} = \sigma(\mathbf{u}, \mathbf{v}). \tag{5.1}$$

This has a nice picture in two dimensions. The vector $\mathbf{u}$ is an arrow. In two dimensions the 2-form σ is given by a density of points. The contour lines of the interior product 1-form are parallel to the arrow. The get them, arrange the points defining the 2-form to be spaced according to the separation determined by the arrow (which may require some modification in the other direction to preserve the density). Then take the contour lines to be spaced according to the new arrangement of the points. These contour lines are the contour lines corresponding to the interior product 1-form.

In three dimensions the 2-form σ is given by lines. The arrow $\mathbf{u}$ and the lines determining σ determine a family of parallel planes. To get these contour planes, do the following. Arrange the lines that determine σ to be spaced according to the separation determined by the arrow (which may require some modification in the other direction to preserve the density). Then take the contour planes to be spaced according to the new separation between the lines. The resulting planes are the contour planes of the interior product 1-form.

The interior product $\mathbf{u}\rfloor\omega$ of a vector $\mathbf{u}$ with a 3-form ω is a 2-form $\mathbf{u}\rfloor\omega$. It is defined by

$$(\mathbf{u}\rfloor\omega)(\mathbf{v}, \mathbf{w}) = \omega(\mathbf{u}, \mathbf{v}, \mathbf{w}). \tag{5.2}$$

(The case of a general r-form is similar.)

The picture is similar. Consider three dimensions. The vector **u** is an arrow, and the associated 2-form $\mathbf{u}\rfloor\omega$ is given by lines that are parallel to the arrow. To get these contour lines, do the following. Arrange the points that determine ω to be spaced according to the separation determined by the arrow. Then take the contour lines to be spaced according to the new separation between the points.

One interesting property of the interior product is that if α is an r-form and β is an s-form, then

$$\mathbf{u}\rfloor(\alpha \wedge \beta) = (\mathbf{u}\rfloor\alpha) \wedge \beta + (-1)^r \alpha \wedge (\mathbf{u}\rfloor\beta). \tag{5.3}$$

This is a kind of triple product identity.

In particular, we may apply this when $r = 1$ and $s = n$. Since β is an n-form, it follows that $\alpha \wedge \beta = 0$. Hence we have in this special case

$$(\alpha \cdot \mathbf{u})\beta = \alpha \wedge (\mathbf{u}\rfloor\beta). \tag{5.4}$$

Another application is with two 1-forms β and γ. In this case it gives

$$\mathbf{a}\rfloor(\beta \wedge \gamma) = (\beta \cdot \mathbf{a})\gamma - (\gamma \cdot \mathbf{a})\beta. \tag{5.5}$$

So the interior product of a vector with $\beta \wedge \gamma$ is a linear combination of β and γ.

Later we shall see the connection with classical vector algebra in three dimensions. The exterior product $\beta \wedge \gamma$ is an analog of the cross product, while $\alpha \wedge \beta \wedge \gamma$ is an analog of the triple scalar product. The combination $-\mathbf{a}\rfloor(\beta \wedge \gamma)$ will turn out to be an analog of the triple vector product.

5.2 Duality

Consider an n-dimensional manifold. The new feature is a given n-form, taken to be never zero. We denote this form by vol. In coordinates it is of the form

$$\mathrm{vol} = \sqrt{g}\, du_1 \cdots du_n. \tag{5.6}$$

This coefficient $\sqrt{g}$ depends on the coordinate system. The choice of the notation $\sqrt{g}$ for the coefficient will be explained in the following chapter. (Then $\sqrt{g}$ will be the square root of the determinant of the matrix associated with the Riemannian metric for this coordinate system.)

The most common examples of volume forms are the volume in $\mathrm{vol} = dx\, dy\, dz$ in cartesian coordinates and the same volume $\mathrm{vol} = r^2 \sin(\theta)\, dr\, d\theta\, d\phi$ in spherical polar coordinates. The convention we are using for spherical polar coordinates is that θ is the co-latitude measured from the north pole, while ϕ is the longitude. We see from these coordinates that the $\sqrt{g}$ factor for cartesian coordinates is 1, while the $\sqrt{g}$ factor for spherical polar coordinates is $r^2 \sin(\theta)$.

In two dimensions it is perhaps more natural to call this area. So in cartesian coordinates $\mathrm{area} = dx\, dy$, while in polar coordinates $\mathrm{area} = r\, dr\, d\phi$.

For each scalar field s there is an associated n-form $s\,\mathrm{vol}$. The scalar field and the n-form determine each other in an obvious way. They are said to be dual to each other, in a certain special sense.

For each vector field $\mathbf{v}$ there is an associated $n-1$ form given by $\mathbf{v}\rfloor\mathrm{vol}$. The vector field and the $n-1$ form are again considered to be dual to each other, in this same sense. If $\mathbf{v}$ is a vector field, then $\mathbf{v}\rfloor\mathrm{vol}$ might be called the corresponding *flux*. It is an $n-1$ form that describes how much $\mathbf{v}$ is penetrating a given $n-1$ dimensional surface.

In two dimensions a vector field is of the form

$$\mathbf{u} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}. \tag{5.7}$$

The area form is

$$\mathrm{area} = \sqrt{g}\,du\,dv. \tag{5.8}$$

The corresponding flux is

$$\mathbf{u}\rfloor\mathrm{area} = \sqrt{g}(a\,dv - b\,du). \tag{5.9}$$

In three dimensions a vector field is of the form

$$\mathbf{u} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w}. \tag{5.10}$$

The volume form is

$$\mathrm{vol} = \sqrt{g}\,du\,dv\,dw. \tag{5.11}$$

The corresponding flux is

$$\sqrt{g}(a\,dv\,dw + b\,dw\,du + c\,du\,dv). \tag{5.12}$$

5.3 The divergence theorem

The divergence of a vector field $\mathbf{v}$ is defined to be the scalar $\mathrm{div}\mathbf{v}$ such that

$$d(\mathbf{u}\rfloor\mathrm{vol}) = \mathrm{div}\mathbf{u}\,\mathrm{vol}. \tag{5.13}$$

In other words, it is the dual of the differential of the dual.

The general divergence theorem then takes the form

$$\int_W \mathrm{div}\mathbf{u}\,\mathrm{vol} = \int_{\partial W} \mathbf{u}\rfloor\mathrm{vol}. \tag{5.14}$$

In two dimensions the divergence theorem says that

$$\int_R \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}a}{\partial u} + \frac{\partial\sqrt{g}b}{\partial v}\right)\mathrm{area} = \int_{\partial R} \sqrt{g}(a\,dv - b\,du). \tag{5.15}$$

Notice that the coefficients in the vector field are expressed with respect to a coordinate basis. We shall see in the next part of this book that this is not the only possible choice.

A marvellous application of the divergence theorem in two dimensions is the formula

$$\int_R dx\,dy = \frac{1}{2}\int_{\partial R} x\,dy - y\,dx. \tag{5.16}$$

This says that one can determine the area by walking around the boundary. It is perhaps less mysterious when one realizes that $x\,dy - y\,dx = r^2\,d\phi$.

In three dimensions the divergence theorem says that

$$\int_W \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}a}{\partial u} + \frac{\partial\sqrt{g}b}{\partial v} + \frac{\partial\sqrt{g}c}{\partial w}\right)\mathrm{vol} = \int_{\partial W}\sqrt{g}(a\,dv\,dw + b\,dw\,du + c\,du\,dv). \tag{5.17}$$

Again the coefficients a, b, c of the vector field are expressed in terms of the coordinate basis vectors $\partial/\partial u, \partial/\partial v, \partial/\partial w$. This is the the only possible kind of basis for a vector field, so in some treatments the formulas will appear differently. They will be ultimately equivalent in terms of their geometrical meaning.

The divergence theorem says that the integral of the divergence of a vector field over W with respect to the volume is the integral of the flux of the vector field across the bounding surface ∂W. A famous application in physics is when the vector field represents the electric field, and the divergence represents the density of charge. So the amount of charge in the region determines the flux of the electric field through the boundary.

5.4 Integration by parts

An important identity for differential forms is

$$d(s\omega) = ds \wedge \omega + s d\omega. \tag{5.18}$$

This gives an integration by parts formula

$$\int_W ds \wedge \omega + \int_W s d\omega = \int_{\partial W} s\omega. \tag{5.19}$$

Apply this to $\omega = \mathbf{u}\rfloor\mathrm{vol}$ and use $ds \wedge \mathbf{u}\rfloor\mathrm{vol} = ds \cdot \mathbf{u}\,\mathrm{vol}$. This gives the divergence identity

$$\mathrm{div}(s\mathbf{u}) = ds \cdot \mathbf{u} + s\mathrm{div}\mathbf{u}. \tag{5.20}$$

¿From this we get another important integration by parts identity

$$\int_W ds \cdot \mathbf{u}\,\mathrm{vol} + \int_W s\mathrm{div}\mathbf{u}\,\mathrm{vol} = \int_{\partial W} s\mathbf{u}\rfloor\mathrm{vol}. \tag{5.21}$$

5.5 A variant on curl

In this section we describe a non-standard variant of curl. The usual curl sends vector fields to vector fields. The variant considered here is almost the same, except that it sends differential forms to vector fields.

The context is 3 dimensional only. Define curl′ (which is not the usual curl) by the condition that

$$\mathrm{curl}'\alpha = \mathbf{v} \tag{5.22}$$

provided that

$$d\alpha = \mathbf{v}\rfloor\mathrm{vol}. \tag{5.23}$$

In other words, $\mathrm{curl}'\alpha$ is the vector field whose flux is $d\alpha$. So it is the dual of the differential. It is illuminating to work this out in coordinates. If

$$\alpha = p\,du + q\,dv + r\,dw, \tag{5.24}$$

then

$$\mathrm{curl}'\alpha = \frac{1}{\sqrt{g}}\left[\left(\frac{\partial r}{\partial v} - \frac{\partial q}{\partial w}\right)\frac{\partial}{\partial u} + \left(\frac{\partial r}{\partial w} - \frac{\partial p}{\partial u}\right)\frac{\partial}{\partial v} + \left(\frac{\partial q}{\partial u} - \frac{\partial p}{\partial v}\right)\frac{\partial}{\partial w}\right]. \tag{5.25}$$

Notice again that the result is expressed in terms of the coordinate basis $\partial/\partial u, \partial/\partial v, \partial/\partial w$. This is not the only possible choice of basis, as we shall see later on.

Notice that $\mathrm{curl}'ds = 0$ for every scalar s. Furthermore, $\mathrm{divcurl}'\alpha = 0$ for every 1-form α. This is because by the definition of divergence $(\mathrm{divcurl}'\alpha)\,\mathrm{vol} = d((\mathrm{curl}'\alpha)\rfloor\mathrm{vol}) = dd\alpha = 0$. An alternative explanation is that the $\sqrt{g}$ in the divergence and the $1/\sqrt{g}$ in the curl cancel.

There is a Stokes's theorem for curl′. It has the form

$$\int_S \mathrm{curl}'\alpha\rfloor\mathrm{vol} = \int_{\partial S} \alpha. \tag{5.26}$$

It says that the surface integral of the flux of the vector field $\mathrm{curl}'\alpha$ across the surface S is the line integral of α around its oriented boundary ∂S.

There is no particular good reason to worry about curl′, since nobody uses it. However, it is a useful transition to an understanding of the true curl described in the next part.

5.6 Problems

1. Let $r^2 = x^2 + y^2 + z^2$, and let

$$\mathbf{v} = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right). \tag{5.27}$$

   Let $\mathrm{vol} = dx\,dy\,dz$. Show that

$$\sigma = \mathbf{v}\rfloor \mathrm{vol} = \frac{1}{r^3}(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy). \tag{5.28}$$

2. In the preceding problem, show directly that $d\sigma = 0$ away from $r = 0$.

3. Find σ in spherical polar coordinates. Hint: This can be done by blind computation, but there is a better way. Express $\mathbf{v}$ in spherical polar coordinates, using Euler's theorem

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{5.29}$$

   Then use $\mathrm{vol} = r^2 \sin(\theta)\,dr\,d\theta\,d\phi$ to calculate $\sigma = \mathbf{v}\rfloor\mathrm{vol}$.

4. In the preceding problem, show that $d\sigma = 0$ away from $r = 0$ by a spherical polar coordinate calculation.

5. Let S be the sphere of radius $a > 0$ centered at the origin. Calculate the integral of σ over S.

6. Let Q be the six-sided cube with side lengths $2L$ centered at the origin. Calculate the integral of σ over Q. Hint: Given the result of the preceding problem, this should be effortless.

7. Consider a two-dimensional system with coordinates q, p. Here q represents position and p represents momentum. The volume (or area) in this case is not ordinary area; in fact $\mathrm{vol} = dq\,dp$ has the dimensions of angular momentum. Let H be a scalar function (the Hamiltonian function). Find the corresponding Hamiltonian vector $\mathbf{v}$ such that

$$\mathbf{v}\rfloor\mathrm{vol} = dH. \tag{5.30}$$

8. This is a special case of the preceding problem. Suppose that

$$H = \frac{1}{2m}p^2 + V(q). \tag{5.31}$$

   The two terms represent kinetic energy and potential energy. Find the corresponding Hamiltonian vector field and the corresponding Hamiltonian equations of motion.

Part III

Analysis with a Riemannian metric

Chapter 6

The metric

6.1 Inner product

An *inner product* on a vector space V is a real function g that takes a pair of input vectors $\mathbf{u}, \mathbf{v}$ and produces a number $\mathbf{gu} \cdot \mathbf{v}$. It is required to be a bilinear, symmetric, positive, non-degenerate form. That is, it satisfies the following axioms:

1. The form is bilinear: The function $\mathbf{gu} \cdot \mathbf{v}$ is linear in $\mathbf{u}$ and also linear in $\mathbf{v}$.

2. The form is symmetric: $\mathbf{gu} \cdot \mathbf{v} = \mathbf{gv} \cdot \mathbf{u}$.

3. The form is non-degenerate: $\mathbf{gu} \cdot \mathbf{u} = 0$ implies $\mathbf{u} = 0$.

4. The form is positive: $\mathbf{gu} \cdot \mathbf{u} \geq 0$,

An inner product g defines a linear transformation $\mathsf{g} : V \to V^*$. That is, the value of g on $\mathbf{u}$ in V is the linear function from V to the real numbers that sends $\mathbf{v}$ to $\mathbf{gu} \cdot \mathbf{v}$. Thus $\mathbf{gu}$ is such a function, that is, an element of the dual space V^*.

Since the form g is non-degenerate, the linear transformation g from V to V^* is an isomorphism of vector spaces. Therefore it has an inverse $\mathsf{g}^{-1} : V^* \to V$. Thus if ω is a linear form in V^*, the corresponding vector $\mathbf{u} = \mathsf{g}^{-1}\omega$ is the unique vector $\mathbf{u}$ such that $\mathbf{gu} \cdot \mathbf{v} = \omega \cdot \mathbf{v}$.

In short, once one has a given inner product, one has a tool that tends to erase the distinction between a vector space and its dual space. It is worth noting that in relativity theory there is a generalization of the notion of inner product in which the form is not required to be positive. However it still gives such an isomorphism between vector space and dual space.

6.2 Riemannian metric

A smooth assignment of an inner product for the tangent vectors at each point of a manifold is called a *Riemannian metric*. It is very convenient to choose coordinates so that the Riemannian metric is diagonal with respect to this coordinate system. In this case it has the form

$$\mathsf{g} = h_1^2\, du_1^2 + h_2^2\, du_2^2 + \cdots + h_n^2\, du_n^2. \tag{6.1}$$

Here each coefficient h_i is a function of the coordinates $u_1, \ldots, u_n$. The differentials is not interpreted in the sense of differential forms. Rather, what this means is that g takes vector fields to differential forms by

$$\mathsf{g}\left(a_1 \frac{\partial}{\partial u_1} + \cdots a_n \frac{\partial}{\partial u_n}\right) = h_1^2 a_1\, du_1 + \cdots h_n^2 a_n\, du_n \tag{6.2}$$

It is not always possible to find such a coordinate system for which the Riemannian metric is diagonal. However this can always be done when the dimension $n \leq 3$, and it is very convenient to do so. Such a coordinate system is called a system of *orthogonal coordinates*. See the book by Eisenhart [6] for a discussion of this point.

When we have orthogonal coordinates, it is tempting to make the basis vectors have length one. Thus instead of using the usual coordinate basis vectors $\frac{\partial}{\partial u_i}$ one uses the normalized basis vectors $\frac{1}{h_i}\frac{\partial}{\partial u_i}$. Similarly, instead of using the usual coordinate differential forms du_i one uses the normalized differentials $h_i\, du_i$. Then

$$\mathsf{g}\left(a_1 \frac{1}{h_1}\frac{\partial}{\partial u_1} + \cdots a_n \frac{1}{h_n}\frac{\partial}{\partial u_n}\right) = h_1 a_1\, du_1 + \cdots h_n a_n\, du_n \tag{6.3}$$

When you use the normalized basis vectors, the coefficients do not change.

In orthogonal coordinates the volume is given by

$$\mathrm{vol} = \sqrt{g}\, du_1 \cdots du_n = h_1 \cdots h_n du_1 \wedge \cdots \wedge du_n. \tag{6.4}$$

A simple example of orthogonal coordinates is that of polar coordinates r, ϕ in the plane. These are related to cartesian coordinates x, y by

$$x = r\cos(\phi) \tag{6.5}$$

$$y = r\sin(\phi) \tag{6.6}$$

The Riemannian metric is expressed as

$$\mathsf{g} = dr^2 + r^2\, d\phi^2. \tag{6.7}$$

The normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \phi}$. The normalized basis forms are dr and $r\, d\phi$. The area form is $r\, dr \wedge d\phi$.

Warning: Even though coordinate forms like $d\phi$ are closed forms, a normalized form like $r\, d\phi$ need not be a closed form. In fact, in this particular case $d(r\phi) = dr \wedge d\phi \neq 0$.

Another example of orthogonal coordinates is that of spherical polar coordinates r, θ, ϕ. These are related to cartesian coordinates x, y, z by

$$x = r\cos(\phi)\sin(\theta) \tag{6.8}$$
$$y = r\sin(\phi)\sin(\theta) \tag{6.9}$$
$$z = r\cos(\theta) \tag{6.10}$$

The Riemannian metric is expressed as

$$\mathsf{g} = dr^2 + r^2\,d\theta^2 + r^2\sin^2(\theta)\,d\phi^2. \tag{6.11}$$

The normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$ and $\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}$. The normalized basis forms are dr and $r\,d\theta$ and $r\sin(\theta)\,d\phi$. The volume form is $r^2\sin(\theta)\,dr \wedge d\theta \wedge d\phi$.

In these examples one could always use cartesian coordinates. However there are manifolds that cannot be naturally described by cartesian coordinates, but for which orthogonal coordinates are available. A simple example is the sphere of constant radius a. The Riemannian metric is expressed by

$$\mathsf{g} = a^2\,d\theta^2 + a^2\sin^2(\theta)\,d\phi^2. \tag{6.12}$$

The normalized basis vectors are $\frac{1}{a}\frac{\partial}{\partial \theta}$ and $\frac{1}{a\sin(\theta)}\frac{\partial}{\partial \phi}$. The normalized basis forms are $a\,d\theta$ and $a\sin(\theta)\,d\phi$. The area form is $a^2\sin(\theta)\,d\theta \wedge d\phi$.

6.3 Gradient and divergence

If f is a scalar field, then its *gradient* is

$$\nabla f = \mathrm{grad} f = \mathsf{g}^{-1} df. \tag{6.13}$$

Since du is a 1-form, and the inverse of the metric g^{-1} maps 1-forms to vector fields, the gradient ∇f is a vector field.

In orthogonal coordinates ∇f has the form

$$\nabla f = \sum_{i=1}^{n} \frac{1}{h_i^2}\frac{\partial f}{\partial u_i}\frac{\partial}{\partial u_i}. \tag{6.14}$$

In terms of normalized basis vectors this has the equivalent form

$$\nabla f = \sum_{i=1}^{n} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\frac{1}{h_i}\frac{\partial}{\partial u_i}. \tag{6.15}$$

If $\mathbf{u}$ is a vector field, then its divergence $\nabla \cdot \mathbf{u}$ is a scalar field given by requiring that

$$(\mathrm{div}\mathbf{u})\mathrm{vol} = (\nabla \cdot \mathbf{u})\mathrm{vol} = d(\mathbf{u}\rfloor \mathrm{vol}). \tag{6.16}$$

Here vol $= h_1 \cdots h_n \, du_1 \wedge \cdots du_n$ is the volume form. Say that $\mathbf{u}$ has an expression in terms of normalized basis vectors of the form

$$\mathbf{u} = \sum_{i=1}^{n} a_i \frac{1}{h_i} \frac{\partial}{\partial u_i}. \tag{6.17}$$

Then

$$\mathrm{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{1}{h_1 \cdots h_n} \frac{\partial}{\partial u_i} \left(\frac{h_1 \cdots h_n}{h_i} a_i \right). \tag{6.18}$$

6.4 Gradient dynamics

A scalar function f has both a differential df and a gradient $\mathtt{g}^{-1} df$. What can the gradient do that the differential cannot do? Well, the gradient is a vector field, so it has an associated system of differential equations

$$\frac{du_i}{dt} = \frac{1}{h_i^2} \frac{\partial f}{\partial u_i}. \tag{6.19}$$

Along a solution of this equation the function f satisfies

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{\partial u_i}{\partial t} = \sum_{i=1}^{n} \frac{1}{h_i^2} \left(\frac{\partial f}{\partial u_i} \right)^2 \geq 0. \tag{6.20}$$

In more geometrical language this says that

$$\frac{df}{dt} = df \cdot \mathtt{g}^{-1} df \geq 0. \tag{6.21}$$

Along every solution f is increasing in time. If instead you want decrease, you can follow the negative of the gradient.

6.5 The Laplace operator

The Laplace operator ∇^2 is defined as

$$\nabla^2 f = \nabla \cdot \nabla f. \tag{6.22}$$

This can also be written

$$\nabla^2 f = \mathrm{divgrad} f. \tag{6.23}$$

In coordinates the Laplacian has the form

$$\nabla^2 f = \frac{1}{h_1 \cdots h_n} \sum_{i=1}^{n} \frac{\partial}{\partial u_i} \left(\frac{h_1 \cdots h_n}{h_i^2} \frac{\partial f}{\partial u_i} \right) \tag{6.24}$$

For example, in three dimensions with cartesian coordinates it is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{6.25}$$

In spherical polar coordinates it is

$$\nabla^2 f = \frac{1}{r^2 \sin(\theta)} \left[\frac{\partial}{\partial r} r^2 \sin(\theta) \frac{\partial f}{\partial r} + \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial f}{\partial \theta} + \frac{\partial}{\partial \phi} \frac{1}{\sin(\theta)} \frac{\partial f}{\partial \phi} \right]. \tag{6.26}$$

This is often written

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} + \frac{1}{r^2} \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial f}{\partial \theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2} \right]. \tag{6.27}$$

6.6 Curl

The remaining objects are in three dimensions.

The cross product of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as the unique vector $\mathbf{v} \times \mathbf{w}$ such that

$$(\mathbf{v} \times \mathbf{w}) \rfloor \mathrm{vol} = \mathbf{g}\mathbf{v} \wedge \mathbf{g}\mathbf{w}. \tag{6.28}$$

In other words, it is the operation on vectors that corresponds to the exterior product on forms.

The curl of a vector field $\mathbf{v}$ is defined by

$$(\mathrm{curl}\mathbf{v}) \rfloor \mathrm{vol} = d(\mathbf{g}\mathbf{v}). \tag{6.29}$$

The curl has a rather complicated coordinate representation. Say that in a system of orthogonal coordinates

$$\mathbf{v} = a \frac{1}{h_u} \frac{\partial}{\partial u} + b \frac{1}{h_v} \frac{\partial}{\partial v} + c \frac{1}{h_w} \frac{\partial}{\partial w}. \tag{6.30}$$

Thus the vector field is expressed in terms of normalized basis vectors. Then

$$\mathbf{g}\mathbf{v} = a h_u \, du + b h_v \, dv + c h_w \, dw. \tag{6.31}$$

So

$$d(\mathbf{g}\mathbf{v}) = \left(\frac{\partial h_w c}{\partial v} - \frac{\partial h_v b}{\partial w} \right) dv \wedge dw + \left(\frac{\partial h_u a}{\partial w} - \frac{\partial h_w c}{\partial u} \right) dw \wedge du + \left(\frac{\partial h_v b}{\partial u} - \frac{\partial h_u a}{\partial v} \right) du \wedge dv. \tag{6.32}$$

It follows that

$$\mathrm{curl}\mathbf{v} = \frac{1}{h_v h_w} \left(\frac{\partial h_w c}{\partial v} - \frac{\partial h_v b}{\partial w} \right) \frac{1}{h_u} \frac{\partial}{\partial u} + \frac{1}{h_u h_w} \left(\frac{\partial h_u a}{\partial w} - \frac{\partial h_w c}{\partial u} \right) \frac{1}{h_v} \frac{\partial}{\partial h_v} + \frac{1}{h_u h_v} \left(\frac{\partial h_v b}{\partial u} - \frac{\partial h_u a}{\partial v} \right) \frac{1}{h_w} \frac{\partial}{\partial h_w}. \tag{6.33}$$

The reason for writing it this way is to express it again in terms of normalized basis vectors. Notice also that if we express the derivatives as normalized derivatives, then the expression is reasonably natural. For instance, the first term is $1/h_w$ times the derivative $(1/h_v)\partial/\partial v)$ of h_w times the coefficient. The only odd thing is that the h_w is inside the derivative, while the $1/h_w$ is outside the derivative.

It is easy to see that curlgrad$f = 0$ and that divcurl$\mathbf{v} = 0$. This first of these is simple; it only involves the observation that curlgrad$f = \mathrm{curl}'\mathbf{g}\mathbf{g}^{-1}df = \mathrm{curl}'df = 0$. The second of these is equally transparent: divcurl$\mathbf{v} = \mathrm{divcurl}'\mathbf{g}\mathbf{v} = 0$.

Stokes's theorem says that

$$\int_S \mathrm{curl}\mathbf{v}\rfloor\mathrm{vol} = \int_{\partial S} \mathbf{g}\mathbf{v}. \tag{6.34}$$

Of course, this is just saying that

$$\int_S d(\mathbf{g}\mathbf{v}) = \int_{\partial S} \mathbf{g}\mathbf{v}, \tag{6.35}$$

which is much simpler, since most of the effect of the metric has now cancelled out.

6.7 Problems

1. This problem is three dimensional. Compute the Laplacian of $1/r$ via a cartesian coordinate calculation.

2. This problem is three dimensional. Compute the Laplacian of $1/r$ via spherical polar coordinates.

Chapter 7

Applications

7.1 Conservation laws

A conservation law is an equation of the form

$$\frac{\partial R}{\partial t} = -dJ. \tag{7.1}$$

Here R is an n-form, the mass in kilograms, and J is an $n-1$ form, the mass flux in kilograms per second). The coefficients of these two forms have units kilograms per cubic meter and kilograms per second per square meter.) The integral form of such a conservation law is

$$\frac{d}{dt}\int_W R = -\int_{\partial W} J. \tag{7.2}$$

It says that rate of change of the amount of substance inside the region W is equal to the negative of the flow out through the boundary. In fluid dynamics the flux J of mass is $J = \mathbf{v}\rfloor R$, where $\mathbf{v}$ is the fluid velocity vector field. Since the coefficients of $\mathbf{v}$ are in meters per second, and the basis vectors are in inverse meters, the units of $\mathbf{v}$ itself is in inverse seconds.

Often one writes

$$R = \rho \mathrm{vol} \tag{7.3}$$

Here the coefficient ρ is a scalar density (in kilograms per cubic meter). In this case the conservation law reads

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\rho \mathbf{v}). \tag{7.4}$$

The corresponding integral form is

$$\frac{d}{dt}\int_W \rho \mathrm{vol} = -\int_{\partial W} \rho \mathbf{v}\rfloor \mathrm{vol}. \tag{7.5}$$

The units for this equation are kilograms per second. For a fluid it is the law of conservation of mass.

7.2 Maxwell's equations

Maxwell's equations are the equations for electric and magnetic fields. It is convenient to formulate them in such a way that there are two electric fields, E and D. Correspondingly, there are two magnetic fields H and B. Both E and H are 1-forms, while both D and B are 2-forms. The fields are related in the simplest case by

$$D = \epsilon E \tag{7.6}$$

and

$$B = \mu H. \tag{7.7}$$

The ϵ and μ here represent operations that change a 1-form to a 2-form. This might just be be taking the dual with respect to the volume form and then multiplying by a scalar function of position. These scalar functions are called the electric permittivity and the magnetic permeability. Sometimes E and H are called the electric and magnetic fields, while D and B are called the electric flux and magnetic flux.

Each of these forms is regarded as a linear combination with certain coefficients of 1-forms with units of meters or of 2-forms with units of square meters. The units of the 1-forms E and H are volt and ampere. (Thus the units of the coefficients are volt per meter and ampere per meter.) The units of the 2-forms D and B are coulomb and weber. (The units of the coefficients are coulomb per square meter and weber per square meter.)

There is also a 3-form R that represents charge, and a 2-form J that represents current. The units of R and J are coulomb and ampere. (The units of the coefficients are coulomb per cubic meter and ampere per square meter.) These satisfy a conservation law

$$dJ + \frac{\partial R}{\partial t} = 0. \tag{7.8}$$

In integral form this says that

$$\int_{\partial W} J + \frac{d}{dt}\int_W R = 0. \tag{7.9}$$

The amount of current that flows into the region is balanced by the increase in the amount of charge in the region.

The first two Maxwell equations are

$$dE + \frac{\partial B}{\partial t} = 0 \tag{7.10}$$

and

$$dB = 0. \tag{7.11}$$

The second two Maxwell equations are

$$dH = J + \frac{\partial D}{\partial t} \tag{7.12}$$

and

$$dD = R. \tag{7.13}$$

Here are the same equations in integral form. The first two Maxwell equations are

$$\int_{\partial S} E + \frac{d}{dt}\int_S B = 0 \tag{7.14}$$

(in volts) and

$$\int_{\partial W} B = 0 \tag{7.15}$$

(in webers).

The second two Maxwell equations are

$$\int_{\partial S} H = \int_S J + \frac{d}{dt}\int_S D \tag{7.16}$$

(in amperes)

$$\int_{\partial W} D = \int_W R \tag{7.17}$$

(in coulombs).

Each of these equations has a physical meaning. Here are the first two. One is that the voltage given by the circulation of an electric field around a boundary of a surface is given by the negative of the rate of change of the magnetic flux through the surface. This is Faraday's law of induction. The other is that there is no magnetic flux through a closed surface (since there are no magnetic monopoles).

Here are the second two. One is that the circulation of a magnetic field around the boundary of a surface is given by the current flux through the surface in amperes plus the rate of change of the electric flux through the surface. The other equation is that the electric flux through a boundary is the total amount of electric charge in coulombs inside.

Ampère's law was the equation for dH with only the current J on the right hand side. Maxwell observed that this equation would imply $dJ = 0$, which contradicts the conservation law for J and R. He solved this problem by adding the displacement current term $\partial D/\partial t$, which makes the equation consistent with the conservation law. With this modification one sees that a changing magnetic field produces an electric field, and also a changing electric field produces a magnetic field. This gives the explanation for electromagnetic waves, such as radio waves and light waves.

Sometimes the equation $dB = 0$ is written in the form $B = dA$, where A is a 1-form called the magnetic potential. This reflects the fact that an exact 2-form B is always a closed 2-form.

Also, the equation $dE + \partial B/\partial t = 0$ is often written $E + \partial A/\partial t = -d\phi$, where ϕ is a scalar, called the electric potential, measured in volts.

In most elementary treatments of Maxwell's equations the electric and magnetic fields are regarded as vector fields. Thus the 1-forms E and H are written in terms of vector fields $\vec{E}$ and $\vec{H}$ by

$$E = \mathtt{g}\vec{E} \tag{7.18}$$

and

$$H = \mathtt{g}\vec{H}. \tag{7.19}$$

Similarly, the 2-forms B and E are written in terms of vector fields $\vec{B}$ and $\vec{H}$ by

$$B = \vec{B}\rfloor\mathrm{vol} \tag{7.20}$$

and

$$D = \vec{D}\rfloor\mathrm{vol}. \tag{7.21}$$

Finally, the charge R and current J are given in terms of a a scalar charge density ρ and a vector current density $\vec{J}$ by

$$R = \rho\mathrm{vol} \tag{7.22}$$

and

$$J = \vec{J}\rfloor\mathrm{vol}. \tag{7.23}$$

There is also a 3-form R that represents charge, and a 2-form J that represents current. The units of R and J are coulomb and ampere. (The units of the coefficients are coulomb per cubic meter and ampere per square meter.) These satisfy a conservation law

$$\frac{\partial\rho}{\partial t} = -\mathrm{div}J. \tag{7.24}$$

In integral form this says that

$$\frac{d}{dt}\int_W \rho\mathrm{vol} = -\int_{\partial W} J\rfloor\mathrm{vol}. \tag{7.25}$$

The units of the coefficients of $\vec{E}$ and of $\vec{H}$ are volt per meter and ampere per meter. The units of the coefficients of $\vec{D}$ and $\vec{B}$ are coulomb per square meter and weber per square meter. The units of ρ and of the coefficients of $\vec{J}$ are coulomb per cubic meter and ampere per square meter. We are regarding each of these vectors as linear combinations with certain coefficients of basis vectors that have units of inverse meters.

The first two Maxwell equations are

$$\mathrm{curl}\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \tag{7.26}$$

and

$$\mathrm{div}\vec{B} = 0. \tag{7.27}$$

The second two Maxwell equations are

$$\mathrm{curl}\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{7.28}$$

and

$$\mathrm{div}\vec{D} = \rho. \tag{7.29}$$

Here are the same equations in integral form. The first two Maxwell equations are

$$\int_{\partial S} \mathtt{g}\vec{E} + \frac{d}{dt}\int_S \vec{B}\rfloor\mathrm{vol} = 0 \tag{7.30}$$

and

$$\int_{\partial W} \vec{B}\rfloor\mathrm{vol} = 0. \tag{7.31}$$

The second two Maxwell equations are

$$\int_{\partial S} \mathtt{g}\vec{H} = \int_S \vec{J}\rfloor\mathrm{vol} + \frac{d}{dt}\int_S \vec{D}\rfloor\mathrm{vol}. \tag{7.32}$$

and

$$\int_{\partial W} \vec{D}\rfloor\mathrm{vol} = \int_W \rho\mathrm{vol}. \tag{7.33}$$

Sometimes the equation $\mathrm{div}\vec{B} = 0$ is written in the form $\vec{B} = \mathrm{curl}\vec{A}$, where $\vec{A}$ is a vector field called the magnetic potential. Also, the equation $\mathrm{curl}\vec{E} + \partial\vec{B}/\partial t = 0$ is often written $\vec{E} + \partial\vec{A}/\partial t = -\mathrm{grad}\phi$, where ϕ is a scalar, called the electric potential, measured in volts.

7.3 Problems

1. Show that the electric field satisfies
$$d(\epsilon E) = R \tag{7.34}$$
and
$$\frac{\partial^2 E}{\partial t^2} + \epsilon^{-1} d\mu^{-1} dE = -\epsilon^{-1}\frac{\partial J}{\partial t}. \tag{7.35}$$
This is the most important physical conclusion: a static charge produces an electric field; an electric current changing in time produces an electric field that propagates in space and time.

2. Show that the magnetic field satisfies
$$dB = 0 \tag{7.36}$$
and
$$\frac{\partial^2 B}{\partial t^2} + d\epsilon^{-1} d\mu^{-1} B = d\epsilon^{-1} J. \tag{7.37}$$
The electric field changing in space and time is accompanied by a magnetic field changing in space and time.

3. The remaining problems deal with the magnetic and electric potentials A and ρ. Show that the equations $B = dA$ and $E + \partial A/\partial t = -d\phi$ imply the first two Maxwell equations.

4. There is some freedom in how to choose A and ϕ. Suppose that the choice is such that $d\epsilon A = 0$. (This is called the transverse gauge condition or Coulomb gauge condition.) Recall that ϵ may depend on space but not on time, and that it sends 1-forms to 2-forms. Use the fourth Maxwell equation to prove that
$$d\epsilon d\phi = -R. \tag{7.38}$$
This equation shows how the charge 3-form R determines the electric potential scalar ϕ.

5. Show that the third Maxwell equation gives
$$\frac{\partial^2 A}{\partial t^2} + \epsilon^{-1} d\mu^{-1} dA = \epsilon^{-1}\tilde{J}, \tag{7.39}$$
where
$$\tilde{J} = J - \epsilon d\frac{\partial \phi}{\partial t}. \tag{7.40}$$
Notice that μ^{-1} may depend on space but not on time, and that it sends 2-forms to 1-forms. The equation shows how the 2-form $\tilde{J}$ determines the magnetic potential 1-form A.

6. Suppose again that the transverse gauge condition $d\epsilon A = 0$ is satisfied. Show that the transverse current $\tilde{J}$ satisfies $d\tilde{J} = 0$.

Chapter 8

Length and area

8.1 Length

Sometimes it is useful to consider coordinates on a manifold that are not orthogonal coordinates. The simplest case is that of a two-dimensional manifold. Write the metric as

$$\mathsf{g} = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{8.1}$$

Here E, F, G are functions of u, v. They of course depend on the choice of coordinates. What is required is that $E > 0, G > 0$ and the determinant $EF - G^2 > 0$. When $F = 0$ we are in the case of orthogonal coordinates.

One way that such a metric arises is from a surface in three-dimensional space. Suppose the metric is the usual cartesian $dx^2 + dy^2 + dz^2$. The length of a curve is

$$s = \int_C \sqrt{dx^2 + dy^2 + dz^2}. \tag{8.2}$$

The meaning of this equation is that

$$s = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\,dt, \tag{8.3}$$

where t is a coordinate on the curve, and the end points are where $t = a$ and $t = b$.

Suppose that the curve is in the surface. Then the length is

$$s = \int_a^b \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt. \tag{8.4}$$

Here the coefficients are

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \tag{8.5}$$

and

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \tag{8.6}$$

and

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2. \tag{8.7}$$

So this gives the explicit formula for the metric on the surface in terms of the equations giving x, y, z in terms of u, v that define the surface. Often one writes the result for the length of a curve in the surface in the form

$$s = \int_C \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}. \tag{8.8}$$

This just means that one can use any convenient parameter.

8.2 Area

The formula for the area of a surface is

$$A = \int_S \text{area} = \int_S \sqrt{g}\,du \wedge dv = \int_S \sqrt{EG - F^2}\,du \wedge dv. \tag{8.9}$$

Here $g = EG - F^2$ is the determinant of the metric tensor. This is particularly simple in the case of orthogonal coordinates, in which case $F = 0$.

There is an alternative expression for the surface area of a surface inside Euclidean space that is sometimes convenient. This is

$$A = \int_S \text{area} = \int_S \sqrt{\left(\frac{dy \wedge dz}{du \wedge dv}\right)^2 + \left(\frac{dz \wedge dw}{du \wedge dv}\right)^2 + \left(\frac{dx \wedge dy}{du \wedge dv}\right)^2}\, du \wedge dv. \tag{8.10}$$

Here a fraction such as $dy \wedge dz$ divided by $du \wedge dv$ is a ratio of 2-forms on the surface S. As we know, such a ration is just a Jacobian determinant of y, z with respect to u, v.

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