# 1 Distributions (due January 22, 2009)

1. The distribution derivative of the locally integrable function $\ln(|x|)$ is the principal value distribution $1/x$. We know that

$$\langle \frac{1}{x}, \phi \rangle = \lim_{\epsilon \downarrow 0} \int_{|x|>\epsilon} \frac{1}{x} \phi(x)\, dx.$$

   Show that

$$\langle \frac{1}{x}, \phi \rangle = \int_0^\infty \frac{1}{x} [\phi(x) - \phi(-x)]\, dx.$$

   Prove that the integral converges absolutely.

2. The distribution derivative of $-1/x$ is the distribution $1/x^2$. We know that it may be defined by

$$\langle \frac{1}{x^2}, \phi \rangle = \lim_{\epsilon \downarrow 0} \int_{|x|>\epsilon} \frac{1}{x^2} [\phi(x) - \phi(0)]\, dx.$$

   Show that

$$\langle \frac{1}{x^2}, \phi \rangle = \int_0^\infty \frac{1}{x^2} [\phi(x) + \phi(-x) - 2\phi(0)]\, dx.$$

   Notice the subtraction of an infinite constant. Prove that the integral converges absolutely.

3. The locally integrable function $x_+^{-\frac{1}{2}}$ is defined to be $x^{-\frac{1}{2}}$ for $x > 0$ and zero for $x \leq 0$. The distribution $x_+^{-\frac{3}{2}}$ defined to be the distribution derivative of $-2x_+^{-\frac{1}{2}}$. Show that

$$\langle x_+^{-\frac{3}{2}}, \phi \rangle = \int_0^\infty \frac{1}{x^{\frac{3}{2}}} [\phi(x) - \phi(0)]\, dx.$$

   Notice the subtraction of an infinite constant. Hint: The problem reduces to evaluating the limit as $\epsilon \downarrow 0$ of $2\int_\epsilon^\infty x^{-\frac{1}{2}} \phi'(x)\, dx$. Work on the integral using integration by parts. Write $2\epsilon^{-\frac{1}{2}}\phi(\epsilon) = 2\epsilon^{-\frac{1}{2}}[\phi(\epsilon) - \phi(0)] + 2\epsilon^{-\frac{1}{2}}\phi(0)$ and deal with each term separately.

4. We know that for $0 < r < 1$ we have

$$1 + \sum_{n=1}^{\infty} r^n 2\cos(nx) = \frac{1 - r^2}{1 - 2r\cos(x) + r^2} = \frac{(1+r)(1-r)}{(1-r)^2 + 4r\sin^2(x/2)}$$

and hence we have the Poisson summation formula

$$1 + \sum_{n=1}^{\infty} 2\cos(nx) = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n).$$

Show that for $0 < r < 1$ we have

$$\sum_{n=1}^{\infty} r^n 2\sin(nx) = \frac{2r\sin(x)}{1 - 2r\cos(x) + r^2} = \frac{4r\sin(x/2)\cos(x/2)}{(1-r)^2 + 4r\sin^2(x/2)}$$

and hence we have the distribution identity

$$\sum_{n=1}^{\infty} 2\sin(nx) = \cot(x/2),$$

where the right hand side is a principal value distribution. Where are the singularities of this distribution?

5. Evaluate $\delta(x^3 - 7x - 6)$ as a sum of three distributions, each concentrated at a single point.

Answers:

1. Since $\phi(x) - \phi(-x)$ vanishes to first order at 0, the the limit $\epsilon \downarrow 0$ gives a convergent integral.

2. Since $\phi(x) + \phi(-x) - 2\phi(0)$ vanishes to second order at 0, the the limit $\epsilon \downarrow 0$ gives a convergent integral.

3. The problem reduces to evaluating
$$2\int_{\epsilon}^{\infty} x^{-\frac{1}{2}}\phi'(x)\,dx = \int_{\epsilon}^{\infty} x^{-\frac{3}{2}}\phi(x)\,dx - 2\epsilon^{-\frac{1}{2}}\phi(\epsilon).$$
Since $\phi(\epsilon) - \phi(0)$ vanishes to first order, we may replace the boundary term by
$$2\epsilon^{-\frac{1}{2}}\phi(0) = \int_{\epsilon}^{\infty} x^{-\frac{3}{2}}\phi(0)\,dx.$$
Since $\phi(x) - \phi(0)$ vanishes to first order, we can take $\epsilon \downarrow 0$.

4. This can be summed as the difference of geometric series
$$\frac{1}{i}\left[\frac{1}{1-re^{ix}} - \frac{1}{1-re^{-ix}}\right].$$
The rest is algebra. The limit $r \uparrow 1$ gives a distribution with principal value singularities at the multiples $2\pi n$.

5. Factor $x^3 - 7x - 6 = (x+2)(x+1)(x-3)$. The derivative is $3x^2 - 7$ which has values $5, -4, 20$ at the roots. So
$$\delta(x^3 - 7x - 6) = \frac{1}{5}\delta(x+2) + \frac{1}{4}\delta(x+1) + \frac{1}{20}\delta(x-3).$$

## 2 Fourier transforms of distributions (due January 29, 2009)

1. Show that whenever you have a Fourier transform pair $f(x), \hat{f}(k)$ you also have a Fourier transform pair $\hat{f}(x), 2\pi f(-k)$.)

2. Show that multiplication by $x^m$ on $f(x)$ corresponds under the Fourier transform to $i^m d^m/dk^m$ on $\hat{f}(k)$. Show that $d^m/dx^m$ on $f(x)$ corresponds to $i^m k^m$ multiplied on $\hat{f}(k)$.

3. We know that the Fourier transform of 1 is $2\pi\delta(k)$. Use this to show that the Fourier transform of $x^m$ is $2\pi i^m \delta^{(m)}(k)$. Also, find a formula for the Fourier transform of $\delta^{(m)}(x)$.

4. We know that the Fourier transform of the principal value $1/x$ is $-i\pi\text{sign}(k)$. Show that the Fourier transform of $1/x^2$ is $-\pi|k|$. Find a formula for the Fourier transform of $1/x^m$. Also, find a formula for the Fourier transform of $x^m\text{sign}(x)$.

5. Evaluate $\delta'(x^3 - 7x - 6)$ as a sum of three distributions, each concentrated at a single point. Hint: Each distribution at a point $a$ will be a linear combination of $\delta(x - a)$ and $\delta'(x - a)$.

Answers:

1. The inversion formula gives
$$2\pi f(-x) = \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k)\, dk.$$

2. Take the Fourier transform formula
$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$
and differentiate it $m$ times with respect to $k$. This gives the first formula. Take the Fourier inversion formula
$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, \frac{dk}{2\pi}$$
and differentiate it $m$ times with respect to $x$. This gives the second formula.

3. By multiplying the function 1 by $x^m$ we get $i^m(d/dk)^m$ applied to $2\pi\delta(k)$ which gives $2\pi i^m \delta^{(m)}(k)$. By the first problem the Fourier transform of $2\pi i^m \delta^{(m)}(x)$ is $2\pi(-k)^m$. So the Fourier transform of $\delta^{(m)}(x)$ is $i^m k^m$.

   Or one could just use the fact that the Fourier transform of $\delta(x)$ is 1 and differentiate $m$ times.

4. Taking $m$ derivatives of $1/x$ gives $(-1)^{m-1}(m-1)!1/x^m$. So the Fourier transform of $1/x^m$ is $i^m k^m$ times $-i\pi\text{sign}(k)$ divided by $(-1)^m(m-1)!$. So the answer is $\pi(-i)^m/(m-1)!$ times $k^{m-1}\text{sign}(k)$.

   By the first problem the Fourier transform of $\pi(-i)^m/(m-1)!$ times $x^{m-1}\text{sign}(x)$ is $2\pi$ times $1/(-k)^m$. This says that the Fourier transform of $x^{m-1}\text{sign}(x)$ is $2(m-1)!(-i)^m$ times $1/k^m$. In other words, the Fourier transform of $x^m\text{sign}(x)$ is $2m!(-i)^{m+1}/k^{m+1}$.

   This could also be done by noting that the Fourier transform of $\text{sign}(x)$ is $-2i/k$. Then multiplying by $x^m$ is equivalent to taking $i^m$ times the $m$th derivative.

5. We have $(d/dx)\delta(x^3 - 7x - 6) = \delta'(x^3 - 7x - 6)(3x^2 - 7)$. Therefore we have that
$$\delta'(x^3 - 7x - 6) = \frac{1}{3x^2 - 7}\left[\frac{1}{5}\delta'(x+2) + \frac{1}{4}\delta'(x+1) + \frac{1}{20}\delta'(x-3)\right].$$
   On the other hand, we know that $g(x)\delta'(x-a) = g(a)\delta'(x-a) - g'(a)\delta(x-a)$. Here $g(x) = 1/(3x^2 - 7)$ with derivative $-6x/(3x^2-7)^2$ has $g(a) = 1/5, -1/4, 1/20$ and $g'(a) = 12/5^2, 6/4^2, -18/(20)^2$ in the three cases $a = -2, -1, 3$. So $\delta'(x^3 - 7x - 6$ is equal to
$$\left[\frac{1}{5^2}\delta'(x+2) - \frac{12}{5^3}\delta(x+2)\right] + \left[-\frac{1}{4^2}\delta'(x+1) - \frac{6}{4^3}\delta(x+1)\right] + \left[\frac{1}{(20)^2}\delta'(x-3) + \frac{18}{(20)^3}\delta(x-3)\right].$$

## 3 Integral operators 1 (due February 12, 2009)

1. Let $v, w$ be vectors in a Hilbert space. Let $w^*$ be the linear transformation that sends $u$ to the scalar $\langle w, u\rangle$. Let $K = vw^*$. Thus $K$ is the linear transformation from the Hilbert space to itself that sends $u$ into $Ku = \langle w, u\rangle v$.

   (a) What is the set of eigenvalues of $K$?

   (b) Find all complex $\lambda$ for which there is a non-trivial solution of the equation $u - \lambda Ku = 0$. Find the solutions.

   (c) Suppose that $u - \lambda Ku = 0$ has only the trivial zero solution. Find the solution of $u - \lambda Ku = f$.

   (d) Suppose that $I - \lambda K$ has an inverse. Find a general expression for $(I - \lambda K)^{-1}$ in the form $I + aK$ for suitable $a$.

   (e) Suppose that $u - \lambda Ku = 0$ has a non-trivial solution. For which $f$ does $u - \lambda Ku = f$ have a solution? Find all solutions.

2. Let $K = v_1 w_1^* + v_2 w_2^*$. Thus $K$ is a linear transformation from the Hilbert space to itself.

   (a) What is the set of eigenvalues of $K$?

   (b) Find all complex $\lambda$ for which there is a non-trivial solution of the equation $u - \lambda Ku = 0$.

3. For which $\lambda$ does the integral equation

$$u(x) + \lambda \int_{-1}^{1} (x^3 + x^2 y) u(y)\, dy = f(x)$$

   have a unique solution? Hint: Use the result of the previous problem.

4. Consider the integral equation

$$f(x) + \int_{-\infty}^{\infty} \frac{1}{2} \cos(x^2 y^3) e^{-(x^2+y^2)} f(y)\, dy = \frac{\sin(x)}{x}.$$

   How many $L^2$ solutions does it have?

5. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy = h(x)$$

   for given $h$ in $L^2$. Find the spectrum of the convolution operator (convolution by $e^{-|x|}$). For which $\lambda$ is the solution of the integral equation guaranteed to be in $L^2$?

# 4 Integral operators 2 (due February 24, 2009)

1. Consider functions in $L^2(-\infty,\infty)$. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} \cos(\sqrt{x^2+y^4}) e^{-|x|-|y|} f(y)\, dy = g(x).$$

It is claimed that there exists $r > 0$ such that for every complex number $\lambda$ with $|\lambda| < r$ the equation has a unique solution. Prove or disprove. Interpret this as a statement about the spectrum of a certain linear operator.

2. Consider functions in $L^2(-\infty,\infty)$. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} \cos(\sqrt{x^2+y^4}) e^{-|x|-|y|} f(y)\, dy = g(x).$$

It is claimed that there exists $R < \infty$ such that for every complex number $\lambda$ with $|\lambda| > R$ the equation does not have a unique solution. Prove or disprove. Interpret this as a statement about the spectrum of a certain linear operator.

3. Consider functions in $L^2(-\infty,\infty)$. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} e^{-|x|-|y|} f(y)\, dy = g(x).$$

Find all complex numbers $\lambda$ for which this equation has a unique solution. Find the solution. Interpret this as a statement about the spectrum of a certain linear operator.

4. Consider functions in $L^2(0,1)$. Consider the integral equation

$$f(x) - \lambda \int_{0}^{x} f(y)\, dy = g(x).$$

Find all complex numbers $\lambda$ for which this equation has a unique solution. Find the solution. Interpret this as a statement about the spectrum of a certain linear operator. Hint: Differentiate. Solve a first order equation with a boundary condition.

5. Consider functions in $L^2(0,1)$. Consider the integral equation

$$f(x) - \lambda \left[ \int_{0}^{x} y(1-x) f(y)\, dy + \int_{x}^{1} x(1-y) f(y)\, dy \right] = g(x).$$

Find all complex numbers $\lambda$ for which this equation has a unique solution. Interpret this as a statement about the spectrum of a certain linear operator. Hint: The integral operator $K$ has eigenfunctions $\sin(n\pi x)$. Verify this directly. This should also determine the eigenvalues.

## 5 Spectral Theory 1 (due March 5, 2009)

If $K$ is a bounded everywhere defined operator, then in particular $K$ is a closed densely defined operator.

If $K$ is a closed densely defined operator, and if both $K$ and $K^*$ have trivial nullspaces, then $L = K^{-1}$ is also a closed densely defined operator.

1. Let $K = L^{-1}$ be as above. Let $\lambda \neq 0$ and let $\mu = 1/\lambda$. Find a formula relating the resolvent $(L - \lambda)^{-1}$ to the resolvent $(K - \mu)^{-1}$.

2. Consider functions in $L^2(0, 1)$. Consider the integral operator $K$ given by

$$(Kf)(x) = \int_0^x f(y)\, dy.$$

   Show that $L = K^{-1}$ exists and is closed and densely defined. Describe the domain of $L$. Be explicit about boundary conditions. Describe how $L$ acts on the elements of this domain. Show that $L^* = K^{*-1}$ is closed and densely defined. Describe the domain of $L^*$. Describe how $L^*$ acts on the elements of this domain. Hint: Differentiate.

3. In the preceding problem, find the spectrum of $L$. Also, find the resolvent $(L - \lambda)^{-1}$ of $L$. Hint: Solve a first order linear ordinary differential equation.

4. Consider functions in $L^2(0, 1)$. Consider the integral operator $K$ given by

$$(Kf)(x) = \left[ \int_0^x y(1-x)f(y)\, dy + \int_x^1 x(1-y)f(y)\, dy \right].$$

   Show that $L = K^{-1}$ exists and is closed and densely defined. Describe the domain of $L$. Be explicit about boundary conditions. Describe how $L$ acts on the elements of this domain. Hint: Differentiate twice.

5. In the preceding problem, find the spectrum of $L$. Find the resolvent $(L - \lambda)^{-1}$ of $L$. Hint: Use $\sin(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}(1 - x))$ as a basis for the solutions of a homogeneous second order linear ordinary differential equation. Solve the inhomogeneous equation by variation of parameters.

6. Let $K$ be a compact operator. Suppose that $K$ and $K^*$ have trivial nullspaces, so that $L = K^{-1}$ is a closed densely defined operator. Prove that the spectrum of $L = K^{-1}$ consists of isolated eigenvalues of finite multiplicity. To what extent does this result apply to the examples in the previous problems?

7. Let $K$ be a compact self-adjoint operator. Suppose that $K$ has trivial null-space, so that $L = K^{-1}$ is a self-adjoint operator. Prove that there exists an orthogonal basis consisting of eigenvectors of $L$. To what extent does this result apply to the examples in the previous problems?

# 6 Spectral Theory 2 (due March 26, 2009)

1. Let $H = L^2(0,1)$. Let $L = -id/dx$ with periodic boundary conditions. Find an explicit formula for $(\lambda - L)^{-1}g$. Hint: Solve the first order ordinary differential equation $(\lambda - L)f = g$ with the boundary condition $f(0) = f(1)$.

2. Find the eigenvalues and eigenvectors of $L$. For each eigenvalue $\lambda_n$, find the residue $P_n$ of $(\lambda - L)^{-1}$ at $\lambda_n$.

3. Find the explicit form of the formula $g = \sum_n P_n g$.

4. Let $H = L^2(-\infty, \infty)$. Let $L = -id/dx$. Let $k$ be real and $\epsilon > 0$. Find an explicit formula for $(L - k - i\epsilon)^{-1}g$. Also, find an explicit formula for $(L - k + i\epsilon)^{-1}g$. Find the explicit form of the expression

$$\delta_\epsilon(L - k)g = \frac{1}{2\pi i}[(L - k - i\epsilon)^{-1} - (L - k + i\epsilon)^{-1}]g.$$

5. Find the explicit form of the formula

$$g = \int_{-\infty}^{\infty} \delta_\epsilon(L - k)g \, dk.$$

6. Let $\epsilon \to 0$. Find the explicit form of the formula

$$g = \int_{-\infty}^{\infty} \delta(L - k)g \, dk.$$

# 7 Spectral Theory 3 (due April 2, 2009)

1. Perhaps the most beautiful self-adjoint operator is the spherical Laplacian
$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$
Show by explicit computation that this is a Hermitian operator acting on $L^2$ of the sphere with surface measure $\sin\theta\, d\theta\, d\phi$. Pay explicit attention to what happens at the north pole and south pole when one integrates by parts.

2. Let $r$ be the radius satisfying $r^2 = x^2 + y^2 + z^2$. Let
$$L = r\frac{\partial}{\partial r}$$
be the Euler operator. Show that the Laplace operator
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
is related to $L$ and $\Delta_S$ by
$$\Delta = \frac{1}{r^2}[L(L+1) + \Delta_S].$$

3. Let $p$ be a polynomial in $x, y, z$ that is harmonic and homogeneous of degree $\ell$. Thus $\Delta p = 0$ and $Lp = \ell p$. Such a $p$ is called a solid spherical harmonic. Show that each solid spherical harmonic is an eigenfunction of $\Delta_S$ and find the corresponding eigenvalue as a function of $\ell$.

4. The restriction of a solid spherical harmonic to the sphere $r^2 = 1$ is called a surface spherical harmonic. The surface spherical harmonics are the eigenfunctions of $\Delta_S$. Show that surface spherical harmonics for different values of $\ell$ are orthogonal in the Hilbert space of $L^2$ functions on the sphere.

5. The dimension of the eigenspace indexed by $\ell$ is $2\ell + 1$. For $\ell = 0$ the eigenspace is spanned by 1. For $\ell = 1$ it is spanned by $z$, $x + iy$, and $x - iy$. For $\ell = 2$ it is spanned by $3z^2 - r^2$, $z(x + iy)$, $z(x - iy)$, $(x + iy)^2$, and $(x - iy)^2$. For $\ell = 3$ it is spanned by $5z^3 - 3zr^2$, $(5z^2 - r^2)(x + iy)$. $(5z^2 - r^2)(x - iy)$, $z(x + iy)^2$, $z(x - iy)^2$, $(x + iy)^3$, $(x - iy)^3$. Express the corresponding surface spherical harmonics in spherical coordinates.

6. In the case $\ell = 1$ we can write the general spherical harmonic as $ax+by+cz$. In the case $\ell = 2$ we can write it as $ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ with an additional condition on the coefficients. What is this condition? In the case $\ell = 3$ we can write it as $a_1x^3 + b_1y^2x + c_1z^2x + a_2y^3 + b_2z^2y + c_2x^2y + a_3z^3 + b_3x^2z + c_3y^2z + dxyz$ with additional conditions. What are they?

# 8 Spectral Theory 4 (due April 9, 2009)

1. A particularly fascinating self-adjoint operator is the quantum harmonic oscillator. (This operator also occurs in disguised form in other contexts.) It is
$$N = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right)$$
acting in $L^2(-\infty, \infty)$. Show that it factors as
$$N = A^* A,$$
where
$$A = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$$
and
$$A^* = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right).$$

2. Show that $AA^* = A^*A + I$.

3. Solve the equation $Au_0 = 0$. Show that $Nu_0 = 0$.

4. Show that if $Nu_n = nu_n$ and $u_{n+1} = A^*u_n$, then $Nu_{n+1} = (n+1)u_{n+1}$. Thus the eigenvalues of $N$ are the natural numbers. These are the standard type of point spectrum.

5. Show that each eigenfunction $u_n$ is a polynomial in $x$ times $u_0(x)$. Find the polynomials for the cases of $n = 0, 1, 2, 3$ explicitly (up to constant factors). Verify that each $u_n$ belongs to the Hilbert space.

6. It may be shown that $A^*$ is a subnormal operator and so $A$ is the adjoint of a subnormal operator. Find all eigenvalues (point spectrum) of $A$. Find each corresponding eigenvector. Verify that it belongs to the Hilbert space.

7. Find all eigenvalues (point spectrum) of $A^*$. Find the spectrum of $A^*$. What kind of spectrum is it?

8. If $A^*$ is indeed a subnormal operator, then we should have $A^*A \leq AA^*$ as quadratic forms. Is this the case?

## 9 Calculus of Variations (due April 23, 2009)

Let $L$ be a function of $q$ and $\dot{q}$ given by

$$L = \frac{1}{2}m\dot{q}^2 - V(q).$$

Let

$$S(q) = \int_{t_1}^{t_2} L dt,$$

with functions $q$ of $t$ satisfying $q = q_1$ at $t = t_1$ and $q = q_2$ at $t = t_2$.

1. Show that
$$dS(q)h = (-m\frac{d}{dt}\dot{q} - V'(q), h),$$
where $h$ satisfies Dirichlet boundary conditions at $t_1$ and $t_2$.

2. Consider a $q(t)$ for which $dS(q) = 0$. Show that
$$d^2S(q)(h, h) = \int_{t_1}^{t_2} [m\left(\frac{dh}{dt}\right)^2 - V''(q)h^2]\, dt.$$
where the functions $h$ satisfy Dirichlet boundary conditions at $t_1$ and $t_2$.

3. Consider a $q(t)$ for which $dS(q) = 0$. Show that
$$d^2S(q)(h, h) = (h, [-m\frac{d^2}{dt^2} - V''(q)]h),$$
where the operator satisfies Dirichlet boundary conditions at $t_1$ and $t_2$.

4. Show that if $V(q)$ is concave down, then the solution $q(t)$ of the variational problem is actually a minimum.

5. Let $H = m\dot{q}^2 - L = (1/2)m\dot{q}^2 + V(q)$. Show that $H = E$ along a solution, where $E$ is a constant.

6. From now on take the example $V(q) = -(1/2)kq^2$. Here $k > 0$. Note the sign. We are interested in solutions with $E > 0$. Let $\omega = \sqrt{k/m}$. Show that $q = C\sinh(\omega t)$ is a solution, and find the constant $C$ in terms of $E$.

7. Take $t_1 = -T$ and $t_2 = T$. Take $q_1 = -a$ and $q_2 = a$. Fix $a$. Write the boundary condition $a = C\sinh(\omega T)$ as a relation between $T$ and $E$. Show that $T \to 0$ implies $E \to \infty$, while $T \to \infty$ implies $E \to 0$.

8. Interpret the result of the last problem intuitively in terms of particle motion satisfying conservation of energy.

9. Interpret the result of the same problem intuitively in terms of a minimization problem.