1 Distributions (due January 22, 2009)

1. The distribution derivative of the locally integrable function \( \ln(|x|) \) is the principal value distribution \( \frac{1}{x} \). We know that

\[
\langle \frac{1}{x}, \phi \rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{1}{x} \phi(x) \, dx.
\]

Show that

\[
\langle \frac{1}{x}, \phi \rangle = \int_{0}^{\infty} \frac{1}{x} [\phi(x) - \phi(-x)] \, dx.
\]

Prove that the integral converges absolutely.

2. The distribution derivative of \( -\frac{1}{x} \) is the distribution \( \frac{1}{x^2} \). We know that it may be defined by

\[
\langle \frac{1}{x^2}, \phi \rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{1}{x^2} [\phi(x) - \phi(0)] \, dx.
\]

Show that

\[
\langle \frac{1}{x^2}, \phi \rangle = \int_{0}^{\infty} \frac{1}{x^2} [\phi(x) + \phi(-x) - 2\phi(0)] \, dx.
\]

Notice the subtraction of an infinite constant. Prove that the integral converges absolutely.

3. The locally integrable function \( x^{-\frac{1}{2}} + \) is defined to be \( x^{-\frac{1}{2}} \) for \( x > 0 \) and zero for \( x \leq 0 \). The distribution \( x^{-\frac{3}{2}} + \) defined to be the distribution derivative of \( -2x^{-\frac{1}{2}} \). Show that

\[
\langle x^{-\frac{3}{2}} + , \phi \rangle = \int_{0}^{\infty} \frac{1}{x^{\frac{3}{2}}} [\phi(x) - \phi(0)] \, dx.
\]

Notice the subtraction of an infinite constant. Hint: The problem reduces to evaluating the limit as \( \epsilon \downarrow 0 \) of \( 2 \int_{\epsilon}^{\infty} x^{-\frac{1}{2}} \phi'(x) \, dx \). Work on the integral using integration by parts. Write \( 2\epsilon^{-\frac{1}{2}} \phi(\epsilon) = 2\epsilon^{-\frac{1}{2}} [\phi(\epsilon) - \phi(0)] + 2\epsilon^{-\frac{1}{2}} \phi(0) \) and deal with each term separately.
4. We know that for $0 < r < 1$ we have

$$1 + \sum_{n=1}^{\infty} r^n 2 \cos(nx) = \frac{1 - r^2}{1 - 2r \cos(x) + r^2} = \frac{(1 + r)(1 - r)}{(1 - r)^2 + 4r \sin^2(x/2)}$$

and hence we have the Poisson summation formula

$$1 + \sum_{n=1}^{\infty} 2 \cos(nx) = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n).$$

Show that for $0 < r < 1$ we have

$$\sum_{n=1}^{\infty} r^n 2 \sin(nx) = \frac{2r \sin(x)}{1 - 2r \cos(x) + r^2} = \frac{4r \sin(x/2) \cos(x/2)}{(1 - r)^2 + 4r \sin^2(x/2)}$$

and hence we have the distribution identity

$$\sum_{n=1}^{\infty} 2 \sin(nx) = \cot(x/2),$$

where the right hand side is a principal value distribution. Where are the singularities of this distribution?

5. Evaluate $\delta(x^3 - 7x - 6)$ as a sum of three distributions, each concentrated at a single point.
Answers:

1. Since \( \phi(x) - \phi(-x) \) vanishes to first order at 0, the limit \( \epsilon \downarrow 0 \) gives a convergent integral.

2. Since \( \phi(x) + \phi(-x) - 2\phi(0) \) vanishes to second order at 0, the limit \( \epsilon \downarrow 0 \) gives a convergent integral.

3. The problem reduces to evaluating

\[
2 \int_{\epsilon}^{\infty} x^{-\frac{3}{2}} \phi'(x) \, dx = \int_{\epsilon}^{\infty} x^{-\frac{3}{2}} \phi(x) \, dx - 2\epsilon^{-\frac{3}{2}} \phi(\epsilon).
\]

Since \( \phi(\epsilon) - \phi(0) \) vanishes to first order, we may replace the boundary term by

\[
2\epsilon^{-\frac{3}{2}} \phi(0) = \int_{\epsilon}^{\infty} x^{-\frac{3}{2}} \phi(0) \, dx.
\]

Since \( \phi(x) - \phi(0) \) vanishes to first order, we can take \( \epsilon \downarrow 0 \).

4. This can be summed as the difference of geometric series

\[
\frac{1}{r} \left[ \frac{1}{1 - re^{ix}} - \frac{1}{1 - re^{-ix}} \right].
\]

The rest is algebra. The limit \( r \uparrow 1 \) gives a distribution with principal value singularities at the multiples \( 2\pi n \).

5. Factor \( x^3 - 7x - 6 = (x+2)(x+1)(x-3) \). The derivative is \( 3x^2 - 7 \) which has values 5, -4, 20 at the roots. So

\[
\delta(x^3 - 7x - 6) = \frac{1}{5} \delta(x+2) + \frac{1}{4} \delta(x+1) + \frac{1}{20} \delta(x-3).
\]
2 Fourier transforms of distributions (due January 29, 2009)

1. Show that whenever you have a Fourier transform pair $f(x), \hat{f}(k)$ you also have a Fourier transform pair $\hat{f}(x), 2\pi f(-k)$.

2. Show that multiplication by $x^m$ on $f(x)$ corresponds under the Fourier transform to $i^m d^m/dk^m$ on $\hat{f}(k)$. Show that $d^m/dx^m$ on $f(x)$ corresponds to $i^m k^m$ multiplied on $\hat{f}(k)$.

3. We know that the Fourier transform of 1 is $2\pi \delta(k)$. Use this to show that the Fourier transform of $x^m$ is $2\pi i^m \delta^{(m)}(k)$. Also, find a formula for the Fourier transform of $\delta^{(m)}(x)$.

4. We know that the Fourier transform of the principal value $1/x$ is $-i\pi \text{sign}(k)$. Show that the Fourier transform of $1/x^2$ is $-\pi |k|$. Find a formula for the Fourier transform of $1/x^m$. Also, find a formula for the Fourier transform of $x^m \text{sign}(x)$.

5. Evaluate $\delta'(x^3 - 7x - 6)$ as a sum of three distributions, each concentrated at a single point. Hint: Each distribution at a point $a$ will be a linear combination of $\delta(x - a)$ and $\delta'(x - a)$. 
Answers:

1. The inversion formula gives
\[ 2\pi f(-x) = \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) \, dk. \]

2. Take the Fourier transform formula
\[ \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx \]
and differentiate it \( m \) times with respect to \( k \). This gives the first formula. Take the Fourier inversion formula
\[ f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) \frac{dk}{2\pi} \]
and differentiate it \( m \) times with respect to \( x \). This gives the second formula.

3. By multiplying the function \( 1 \) by \( x^m \) we get \( i^m(d/dk)^m \) applied to \( 2\pi \delta(k) \) which gives \( 2\pi i^m \delta^{(m)}(k) \). By the first problem the Fourier transform of \( 2\pi i^m \delta^{(m)}(x) \) is \( 2\pi(-k)^m \). So the Fourier transform of \( \delta^{(m)}(x) \) is \( i^m k^m \).
Or one could just use the fact that the Fourier transform of \( \delta(x) \) is 1 and differentiate \( m \) times.

4. Taking \( m \) derivatives of \( 1/x \) gives \((-1)^{m-1}(m-1)!/x^m \). So the Fourier transform of \( 1/x^m \) is \( i^m k^m \) times \( -i\pi \text{sign}(k) \) divided by \((-1)^m(m-1)! \). So the answer is \( \pi(-i)^m/(m-1)! \) times \( k^{m-1} \text{sign}(k) \).
By the first problem the Fourier transform of \( \pi(-i)^m/(m-1)! \) times \( x^{m-1} \text{sign}(x) \) is \( 2\pi \) times \( 1/(-k)^m \). This says that the Fourier transform of \( x^{m-1} \text{sign}(x) \) is \( 2(m-1)!(-i)^m \) times \( 1/k^m \). In other words, the Fourier transform of \( x^m \text{sign}(x) \) is \( 2ml(-i)^{m+1}/k^{m+1} \).
This could also be done by noting that the Fourier transform of \( \text{sign}(x) \) is \(-2i/k \). Then multiplying by \( x^m \) is equivalent to taking \( i^m \) times the \( m \)th derivative.

5. We have \( (d/dx)\delta(x^3 - 7x - 6) = \delta'(x^3 - 7x - 6)(3x^2 - 7) \). Therefore we have that
\[ \delta'(x^3 - 7x - 6) = \frac{1}{3x^2 - 7} \left[ \frac{1}{5} \delta'(x + 2) + \frac{1}{4} \delta'(x + 1) + \frac{1}{20} \delta'(x - 3) \right]. \]
On the other hand, we know that \( g(x)\delta'(x - a) = g(a)\delta'(x - a) - g'(a)\delta(x - a) \). Here \( g(x) = 1/(3x^2 - 7) \) with derivative \(-6x/(3x^2 - 7)^2\) has \( g(a) = 1/5, -1/4, 1/20 \) and \( g'(a) = 12/5^2, 6/4^2, -18/(20)^2 \) in the three cases \( a = -2, -1, 3 \). So \( \delta'(x^3 - 7x - 6) \) is equal to
\[ \left[ \frac{1}{5} \delta'(x + 2) - \frac{12}{5^2} \delta(x + 2) \right] + \left[ -\frac{1}{4} \delta'(x + 1) - \frac{6}{4^2} \delta(x + 1) \right] + \left[ \frac{1}{(20)^2} \delta'(x - 3) + \frac{18}{(20)^3} \delta(x - 3) \right]. \]
3 Integral operators 1 (due February 12, 2009)

1. Let \( v, w \) be vectors in a Hilbert space. Let \( w^* \) be the linear transformation that sends \( u \) to the scalar \( \langle w, u \rangle \). Let \( K = vw^* \). Thus \( K \) is the linear transformation from the Hilbert space to itself that sends \( u \) into \( Ku = \langle w, u \rangle v \).

   (a) What is the set of eigenvalues of \( K \)?
   (b) Find all complex \( \lambda \) for which there is a non-trivial solution of the equation \( u - \lambda Ku = 0 \). Find the solutions.
   (c) Suppose that \( u - \lambda Ku = 0 \) has only the trivial zero solution. Find the solution of \( u - \lambda Ku = f \).
   (d) Suppose that \( I - \lambda K \) has an inverse. Find a general expression for \( (I - \lambda K)^{-1} \) in the form \( I + aK \) for suitable \( a \).
   (e) Suppose that \( u - \lambda Ku = 0 \) has a non-trivial solution. For which \( f \) does \( u - \lambda Ku = f \) have a solution? Find all solutions.

2. Let \( K = v_1w_1^* + v_2w_2^* \). Thus \( K \) is a linear transformation from the Hilbert space to itself.

   (a) What is the set of eigenvalues of \( K \)?
   (b) Find all complex \( \lambda \) for which there is a non-trivial solution of the equation \( u - \lambda Ku = 0 \).

3. For which \( \lambda \) does the integral equation
   \[
   u(x) + \lambda \int_{-1}^{1} (x^3 + x^2y)u(y) \, dy = f(x)
   \]
   have a unique solution? Hint: Use the result of the previous problem.

4. Consider the integral equation
   \[
   f(x) + \int_{-\infty}^{\infty} \frac{1}{2} \cos(x^2y^3)e^{-(x^2+y^2)} f(y) \, dy = \frac{\sin(x)}{x}.
   \]
   How many \( L^2 \) solutions does it have?

5. Consider the integral equation
   \[
   f(x) - \lambda \int_{-\infty}^{\infty} e^{-|x-y|} f(y) \, dy = h(x)
   \]
   for given \( h \) in \( L^2 \). Find the spectrum of the convolution operator (convolution by \( e^{-|x|} \)). For which \( \lambda \) is the solution of the integral equation guaranteed to be in \( L^2 \)?
4 Integral operators 2 (due February 24, 2009)

1. Consider functions in $L^2(-\infty, \infty)$. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} \cos(\sqrt{x^2 + y^4}) e^{-|x| - |y|} f(y) \, dy = g(x).$$

It is claimed that there exists $r > 0$ such that for every complex number $\lambda$ with $|\lambda| < r$ the equation has a unique solution. Prove or disprove. Interpret this as a statement about the spectrum of a certain linear operator.

2. Consider functions in $L^2(-\infty, \infty)$. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} \cos(\sqrt{x^2 + y^4}) e^{-|x| - |y|} f(y) \, dy = g(x).$$

It is claimed that there exists $R < \infty$ such that for every complex number $\lambda$ with $|\lambda| > R$ the equation does not have a unique solution. Prove or disprove. Interpret this as a statement about the spectrum of a certain linear operator.

3. Consider functions in $L^2(-\infty, \infty)$. Consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} e^{-|x| - |y|} f(y) \, dy = g(x).$$

Find all complex numbers $\lambda$ for which this equation has a unique solution. Interpret this as a statement about the spectrum of a certain linear operator.

4. Consider functions in $L^2(0, 1)$. Consider the integral equation

$$f(x) - \lambda \int_0^x f(y) \, dy = g(x).$$

Find all complex numbers $\lambda$ for which this equation has a unique solution. Interpret this as a statement about the spectrum of a certain linear operator. Hint: Differentiate. Solve a first order equation with a boundary condition.

5. Consider functions in $L^2(0, 1)$. Consider the integral equation

$$f(x) - \lambda \left[ \int_0^x y(1 - x) f(y) \, dy + \int_1^x x(1 - y) f(y) \, dy \right] = g(x).$$

Find all complex numbers $\lambda$ for which this equation has a unique solution. Interpret this as a statement about the spectrum of a certain linear operator. Hint: The integral operator $K$ has eigenfunctions $\sin(n\pi x)$. Verify this directly. This should also determine the eigenvalues.
5 Spectral Theory 1 (due March 5, 2009)

If $K$ is a bounded everywhere defined operator, then in particular $K$ is a closed densely defined operator.

If $K$ is a closed densely defined operator, and if both $K$ and $K^*$ have trivial nullspaces, then $L = K^{-1}$ is also a closed densely defined operator.

1. Let $K = L^{-1}$ be as above. Let $\lambda \neq 0$ and let $\mu = 1/\lambda$. Find a formula relating the resolvent $(L - \lambda)^{-1}$ to the resolvent $(K - \mu)^{-1}$.

2. Consider functions in $L^2(0,1)$. Consider the integral operator $K$ given by

$$(Kf)(x) = \int_0^x f(y) \, dy.$$  

Show that $L = K^{-1}$ exists and is closed and densely defined. Describe the domain of $L$. Be explicit about boundary conditions. Describe how $L$ acts on the elements of this domain. Show that $L^* = K^{*-1}$ is closed and densely defined. Describe the domain of $L^*$. Describe how $L^*$ acts on the elements of this domain. Hint: Differentiate.

3. In the preceding problem, find the spectrum of $L$. Also, find the resolvent $(L - \lambda)^{-1}$ of $L$. Hint: Solve a first order linear ordinary differential equation.

4. Consider functions in $L^2(0,1)$. Consider the integral operator $K$ given by

$$(Kf)(x) = \left[ \int_0^x y(1-x)f(y) \, dy + \int_x^1 x(1-y)f(y) \, dy \right].$$

Show that $L = K^{-1}$ exists and is closed and densely defined. Describe the domain of $L$. Be explicit about boundary conditions. Describe how $L$ acts on the elements of this domain. Hint: Differentiate twice.

5. In the preceding problem, find the spectrum of $L$. Find the resolvent $(L - \lambda)^{-1}$ of $L$. Hint: Use $\sin(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}(1-x))$ as a basis for the solutions of a homogeneous second order linear ordinary differential equation. Solve the inhomogeneous equation by variation of parameters.

6. Let $K$ be a compact operator. Suppose that $K$ and $K^*$ have trivial nullspaces, so that $L = K^{-1}$ is a closed densely defined operator. Prove that the spectrum of $L = K^{-1}$ consists of isolated eigenvalues of finite multiplicity. To what extent does this result apply to the examples in the previous problems?

7. Let $K$ be a compact self-adjoint operator. Suppose that $K$ has trivial null-space, so that $L = K^{-1}$ is a self-adjoint operator. Prove that there exists an orthogonal basis consisting of eigenvectors of $L$. To what extent does this result apply to the examples in the previous problems?
6 Spectral Theory 2 (due March 26, 2009)

1. Let $H = L^2(0,1)$. Let $L = -id/dx$ with periodic boundary conditions. Find an explicit formula for $(\lambda - L)^{-1}g$. Hint: Solve the first order ordinary differential equation $(\lambda - L)f = g$ with the boundary condition $f(0) = f(1)$.

2. Find the eigenvalues and eigenvectors of $L$. For each eigenvalue $\lambda_n$, find the residue $P_n$ of $(\lambda - L)^{-1}$ at $\lambda_n$.

3. Find the explicit form of the formula $g = \sum_n P_n g$.

4. Let $H = L^2(-\infty, \infty)$. Let $L = -id/dx$. Let $k$ be real and $\epsilon > 0$. Find an explicit formula for $(L - k - i\epsilon)^{-1}g$. Also, find an explicit formula for $(L - k + i\epsilon)^{-1}g$. Find the explicit form of the expression

$$\delta_\epsilon(L - k)g = \frac{1}{2\pi i}[(L - k - i\epsilon)^{-1} - (L - k + i\epsilon)^{-1}]g.$$ 

5. Find the explicit form of the formula

$$g = \int_{-\infty}^{\infty} \delta_\epsilon(L - k)g \, dk.$$ 

6. Let $\epsilon \to 0$. Find the explicit form of the formula

$$g = \int_{-\infty}^{\infty} \delta(L - k)g \, dk.$$
1. Perhaps the most beautiful self-adjoint operator is the spherical Laplacian
\[ \Delta_S = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \]
Show by explicit computation that this is a Hermitian operator acting on \( L^2 \) of the sphere with surface measure \( \sin \theta \, d\theta \, d\phi \). Pay explicit attention to what happens at the north pole and south pole when one integrates by parts.

2. Let \( r \) be the radius satisfying \( r^2 = x^2 + y^2 + z^2 \). Let \( L = r \frac{\partial}{\partial r} \) be the Euler operator. Show that the Laplace operator
\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
is related to \( L \) and \( \Delta_S \) by
\[ \Delta = \frac{1}{r^2} [L(L+1) + \Delta_S]. \]

3. Let \( p \) be a polynomial in \( x, y, z \) that is harmonic and homogeneous of degree \( \ell \). Thus \( \Delta p = 0 \) and \( Lp = \ell p \). Such a \( p \) is called a solid spherical harmonic. Show that each solid spherical harmonic is an eigenfunction of \( \Delta_S \) and find the corresponding eigenvalue as a function of \( \ell \).

4. The restriction of a solid spherical harmonic to the sphere \( r^2 = 1 \) is called a surface spherical harmonic. The surface spherical harmonics are the eigenfunctions of \( \Delta_S \). Show that surface spherical harmonics for different values of \( \ell \) are orthogonal in the Hilbert space of \( L^2 \) functions on the sphere.

5. The dimension of the eigenspace indexed by \( \ell \) is \( 2\ell + 1 \). For \( \ell = 0 \) the eigenspace is spanned by 1. For \( \ell = 1 \) it is spanned by \( z \), \( x + iy \), and \( x - iy \). For \( \ell = 2 \) it is spanned by \( 3z^2 - r^2 \), \( z(x + iy) \), \( z(x - iy) \), \( (x + iy)^2 \), and \( (x - iy)^2 \). For \( \ell = 3 \) it is spanned by \( 5z^3 - 3zr^2 \), \( 5z^2 - r^2 \), \( (5z^2 - r^2)(x + iy) \), \( z(x + iy)^2 \), \( z(x - iy)^2 \), \( (x + iy)^3 \), \( (x - iy)^3 \). Express the corresponding surface spherical harmonics in spherical coordinates.

6. In the case \( \ell = 1 \) we can write the general spherical harmonic as \( ax + by + cz \).
In the case \( \ell = 2 \) we can write it as \( ax^2 + by^2 + cz^2 + dxy + eyz + fzx \) with an additional condition on the coefficients. What is this condition? In the case \( \ell = 3 \) we can write it as \( a_1 x^3 + b_1 y^2 x + c_1 z^2 x + a_2 y^3 + b_2 z^2 y + c_2 x^2 y + a_3 z^3 + b_3 x^2 z + c_3 y^2 z + dxyz \) with additional conditions. What are they?
8 Spectral Theory 4 (due April 9, 2009)

1. A particularly fascinating self-adjoint operator is the quantum harmonic oscillator. (This operator also occurs in disguised form in other contexts.) It is

\[ N = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 - 1 \right) \]

acting in \( L^2(-\infty, \infty) \). Show that it factors as

\[ N = A^* A, \]

where

\[ A = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right) \]

and

\[ A^* = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right). \]

2. Show that \( AA^* = A^* A + I \).

3. Solve the equation \( Au_0 = 0 \). Show that \( Nu_0 = 0 \).

4. Show that if \( Nu_n = nu_n \) and \( u_{n+1} = A^* u_n \), then \( Nu_{n+1} = (n+1)u_{n+1} \). Thus the eigenvalues of \( N \) are the natural numbers. These are the standard type of point spectrum.

5. Show that each eigenfunction \( u_n \) is a polynomial in \( x \) times \( u_0(x) \). Find the polynomials for the cases of \( n = 0, 1, 2, 3 \) explicitly (up to constant factors). Verify that each \( u_n \) belongs to the Hilbert space.

6. It may be shown that \( A^* \) is a subnormal operator and so \( A \) is the adjoint of a subnormal operator. Find all eigenvalues (point spectrum) of \( A \). Find each corresponding eigenvector. Verify that it belongs to the Hilbert space.

7. Find all eigenvalues (point spectrum) of \( A^* \). Find the spectrum of \( A^* \). What kind of spectrum is it?

8. If \( A^* \) is indeed a subnormal operator, then we should have \( A^* A \leq AA^* \) as quadratic forms. Is this the case?
Let $L$ be a function of $q$ and $\dot{q}$ given by
\[ L = \frac{1}{2}m\dot{q}^2 - V(q). \]

Let
\[ S(q) = \int_{t_1}^{t_2} L\,dt, \]
with functions $q$ of $t$ satisfying $q = q_1$ at $t = t_1$ and $q = q_2$ at $t = t_2$.

1. Show that
\[ dS(q)h = (-m\frac{d}{dt}\dot{q} - V''(q), h), \]
where $h$ satisfies Dirichlet boundary conditions at $t_1$ and $t_2$.

2. Consider a $q(t)$ for which $dS(q) = 0$. Show that
\[ d^2S(q)(h, h) = \int_{t_1}^{t_2} \left[ m \left( \frac{dh}{dt}\right)^2 - V''(q)h^2 \right] dt. \]
where the functions $h$ satisfy Dirichlet boundary conditions at $t_1$ and $t_2$.

3. Consider a $q(t)$ for which $dS(q) = 0$. Show that
\[ d^2S(q)(h, h) = (h, [-m\frac{d^2}{dt^2} - V''(q)]h), \]
where the operator satisfies Dirichlet boundary conditions at $t_1$ and $t_2$.

4. Show that if $V(q)$ is concave down, then the solution $q(t)$ of the variational problem is actually a minimum.

5. Let $H = m\dot{q}^2 - L = (1/2)m\dot{q}^2 + V(q)$. Show that $H = E$ along a solution, where $E$ is a constant.

6. From now on take the example $V(q) = -(1/2)kq^2$. Here $k > 0$. Note the sign. We are interested in solutions with $E > 0$. Let $\omega = \sqrt{k/m}$. Show that $q = C \sinh(\omega t)$ is a solution, and find the constant $C$ in terms of $E$.

7. Take $t_1 = -T$ and $t_2 = T$. Take $q_1 = -a$ and $q_2 = a$. Fix $a$. Write the boundary condition $a = C \sinh(\omega T)$ as a relation between $T$ and $E$. Show that $T \to 0$ implies $E \to \infty$, while $T \to \infty$ implies $E \to 0$.

8. Interpret the result of the last problem intuitively in terms of particle motion satisfying conservation of energy.

9. Interpret the result of the same problem intuitively in terms of a minimization problem.