1 Assignment 1: Nonlinear dynamics (due September 4, 2008)

1. Consider the ordinary differential equation $du/dt = \cos(u)$. Sketch the equilibria and indicate by arrows the increase or decrease of the solutions. Find the linearization at each equilibrium, and solve the linearized equations. Compare with your arrows.

2. Find the formula for the solution of the initial value problem $du/dt = -u + u^3$. For which initial conditions does the solution become infinite in finite time?

3. Consider the partial differential equation $\partial u/\partial t + u\partial u/\partial x = 0$. Solve it with the initial condition $u = ax + b$ at $t = 0$. For which sign of a is there a problem with the solution?

4. Write the preceding partial differential equation in conservation form. Solve it when the initial condition is $u = u_-$ for $x < 0$ and $u = u_+$ for $x > 0$ at $t = 0$. For which case is there a shock? Find the shock speed.

5. The two-dimensional analog of the conservation law is $\partial u/\partial t + \mathrm{div}(vu) = \partial u/\partial t + \partial(v_1 u)/\partial x + \partial(v_2 u)/\partial y = 0$. The integrated form of this is

$$\frac{d}{dt}\int_R u\,dx\,dy + \int_{\partial R}(v_1 u\,dy - v_2 u\,dx) = 0.$$

Here the curve ∂R in the line integral is the oriented boundary of the region R. Use Green's theorem to show that the partial differential equation implies the integrated conservation law. What is the reason for the minus sign in the line integral?

2 Assignment 2: Linear algebra (due September 11, 2008)

1. If A is a square matrix and f is a function defined by a convergent power series, then $f(A)$ is defined. Show that if A is similar to B, then $f(A)$ is similar to $f(B)$.

2. By the Jordan form theorem, A is similar to $D+N$, where D is diagonal, N is nilpotent, and D, N commute. To say that N is nilpotent is to say that for some $p \geq 1$ the power $N^p = 0$. Show that

$$f(D+N) = \sum_{m=0}^{p-1} \frac{1}{m!} f^{(m)}(D) N^m \tag{1}$$

3. Show that

$$\exp(t(D+N)) = \exp(tD) \sum_{m=0}^{p-1} \frac{1}{m!} N^m t^m. \tag{2}$$

Use this to describe the set of all solutions $\mathbf{x} = \exp(tA)\mathbf{z}$ to the differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{3}$$

with initial condition $\mathbf{x} = \mathbf{z}$ when $t = 0$.

4. Take

$$A = \begin{bmatrix} 0 & 1 \\ -k & -2c \end{bmatrix}, \tag{4}$$

where $k \geq 0$ and $c \geq 0$. The differential equation describes an oscillator with spring constant k and friction coefficient $2c$. Find the eigenvalues and sketch a typical solution in the x_1, x_2 plane in each of the following cases: overdamped $c^2 > k > 0$; critically damped $c^2 = k > 0$; underdamped $0 < c^2 < k$; undamped $0 = c^2 < k$; free motion $0 = c^2 = k$.

5. Consider the critically damped case. Find the Jordan form of the matrix A, and find a similarity transformation that transforms A to its Jordan form.

6. If $A = PDP^{-1}$, where the diagonal matrix D has diagonal entries λ_i, then $f(A)$ may be defined for an arbitrary function f by $f(A) = Pf(D)P^{-1}$, where $f(D)$ is the diagonal matrix with entries $f(\lambda_i)$. Thus, for instance, if each $\lambda_i \geq 0$, then $\sqrt{A}$ is defined. Find the square root of

$$A = \begin{bmatrix} 20 & 40 \\ -8 & -16 \end{bmatrix}. \tag{5}$$

7. Give an example of a matrix A with each eigenvalue $\lambda_i \geq 0$, but for which no square root $\sqrt{A}$ can be defined? Why does the formula in the second problem not work?

3 Assignment 3: Vector fields (due September 18, 2008)

1. Straightening out. A vector field that is non-zero at a point can be transformed into a constant vector field near that point by a change of coordinate system. Pick a point away from the origin, and find coordinates u, v so that
$$-\frac{y}{x^2+y^2}\frac{\partial}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial}{\partial y} = \frac{\partial}{\partial u}. \tag{6}$$

2. Linearization. Consider the vector field
$$\mathbf{u} = xy\frac{\partial}{\partial x} + (y+2)(2-x-y)\frac{\partial}{\partial y}. \tag{7}$$
Find its zeros. At each zero, find its linearization. For each linearization, find its eigenvalues. Use this information to sketch the vector field and some integral curves.

3. Linearization. Consider the system
$$\frac{du}{dt} = v \tag{8}$$
$$\frac{dv}{dt} = \frac{u^2-1}{(1+u^2)^2}. \tag{9}$$
Find its zeros. At each zero, find its linearization. For each linearization, find its eigenvalues. Use this information to sketch the vector field and some integral curves.

4. Nonlinearity. Consider the vector field
$$\mathbf{v} = (1+x^2+y^2)y\frac{\partial}{\partial x} - (1+x^2+y^2)x\frac{\partial}{\partial y}. \tag{10}$$
Find its linearization at zero. Show that there is no coordinate system near 0 in which the vector field is expressed by its linearization. Hint: Solve the associated system of ordinary differential equations, both for $\mathbf{v}$ and for its linearization. Find the period of a solution is both cases.

5. Nonlinear instability. Here is an example of a fixed point where the linear stability analysis gives an elliptic fixed point, but changing to polar coordinates shows the unstable nature of the fixed point:
$$\frac{dx}{dt} = -y + x(x^2+y^2) \tag{11}$$
$$\frac{dy}{dt} = x + y(x^2+y^2). \tag{12}$$
Change the vector field to the polar coordinate representation, and solve the corresponding system of ordinary differential equations.

4 Assignment 4: Differential forms (due September 25, 2008)

1. (a) Exact differentials. Is $(x^2+y^2)\,dx+2xy\,dy$ an exact differential form? If so, write it as the differential of a scalar.

   (b) Exact differentials. Is $(1+e^x)\,dy+e^x(y-x)\,dx$ an exact differential? If so, write it as the differential of a scalar.

   (c) Exact differentials. Is $e^y\,dx + x(e^y+1)\,dy$ an exact differential? If so, write it as the differential of a scalar.

2. Constant differential forms. A differential form usually cannot be transformed into a constant differential form, but there are special circumstances when that can occur. Is it possible to find coordinates u and v near a given point (not the origin) such that

$$-y\,dx + x\,dy = du? \tag{13}$$

3. Constant differential forms. A differential form usually cannot be transformed into a constant differential form, but there are special circumstances when that can occur. Is it possible to find coordinates u and v near a given point (not the origin) such that

$$-\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = du? \tag{14}$$

4. Consider a two-dimensional system with coordinates q, p. Here q represents position and p represents momentum. The area in this case is not ordinary area; in fact area $= dq\,dp$ has the dimensions of angular momentum. Let H be a scalar function (the Hamiltonian function). Find the corresponding Hamiltonian vector $\mathbf{v}$ such that

$$\mathbf{v}\rfloor\mathrm{area} = dH. \tag{15}$$

5. This is a special case of the preceding problem. Suppose that

$$H = \frac{1}{2m}p^2 + V(q). \tag{16}$$

   The two terms represent kinetic energy and potential energy. Find the corresponding Hamiltonian vector field and the corresponding Hamiltonian equations of motion.

5 Assignment 5: Integration on curves (due October 2, 2008)

1. Let C be the curve $x^2 + y^2 = 1$ in the first quadrant from $(1,0)$ to $(0,1)$. Evaluate
$$\int_C xy\,dx + (x^2 + y^2)\,dy. \tag{17}$$

2. Let C be a curve from $(2,0)$ to $(0,3)$. Evaluate
$$\int_C 2xy\,dx + (x^2 + y^2)\,dy. \tag{18}$$

3. Consider the problem of integrating the differential form
$$\alpha = -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \tag{19}$$
from $(1,0)$ to $(-1,0)$ along some curve avoiding the origin. There is are an infinite set of possible answers, depending on the curve. Describe all such answers.

4. Let R be the region $x^2 + y^2 \leq 1$ with $x \geq 0$ and $y \geq 0$. Let ∂R be its boundary (oriented counterclockwise). Evaluate directly
$$\int_{\partial R} xy\,dx + (x^2 + y^2)\,dy. \tag{20}$$

5. This continues the previous problem. Verify Green's theorem in this special case, by explicitly calculating the appropriate integral over the region R.

6 Assignment 6: Reaction-diffusion equation (due October 9, 2008)

1. Consider the initial value problem for the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2.$$

   We are interested in solutions u with $0 \leq u \leq 1$. Find the constant solutions of this equation. Find the linear equations that are the linearizations about each of these constant solutions.

2. Find the dispersion relations for each of these linear equations. Find the range of wave numbers (if any) that are responsible for any instability of these linear equations.

3. Let $z = x - ct$ and $s = t$. Find $\partial/\partial x$ and $\partial/\partial t$ in terms of $\partial/\partial z$ and $\partial/\partial s$. Write the partial differential equation in these new variables. A traveling wave solution is a solution u for which $\partial u/\partial s = 0$. Write the ordinary differential equation for a traveling wave solution (in terms of du/dz).

4. Write this ordinary differential equation as a first order system. Take $c > 0$. Find the fixed points and classify them.

5. Look for a traveling wave solution that goes from 1 at $z = -\infty$ to 0 at $z = +\infty$. For which values of c are there solutions that remain in the interval from 0 to 1?

7 Assignment 7: Complex Variable 1 (due October 16, 2008)

1. (a) Find the form of the Cauchy-Riemann equations in polar coordinates. (b) Show that $\log(z) = \ln(r) + i\theta$ satisfies the Cauchy-Riemann equations near any point away from the origin.

2. (a) Prove that
$$\frac{1}{(1-x)^{k+1}} = \sum_{n=k}^{\infty} \binom{n}{k} x^{n-k}. \tag{21}$$
(b) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < R$. Consider a point z_0 with $|z_0| < R$. The function $f(z)$ has a power series expansion
$$f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k \tag{22}$$
in powers of $z - z_0$. Find a formula (in the form of an infinite series) for the coefficients b_k. Hint: Use the binomial theorem for $z = (z - z_0) + z_0$. (c) Show directly that this series has a non-zero radius of convergence. Hint: Use the fact that $|a_n|r^n$ is bounded for every $r < R$. Prove that b_k grows at most exponentially fast.

3. Let $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ for $|z| < 1$. Show that $f(z)$ cannot be continued analytically beyond the unit disc. That is, show that for every point on the unit circle $|z| = 1$ it is impossible to find an analytic function $g(z)$ defined in an open set surrounding this point such that $g(z) = f(z)$ for $|z| < 1$. Hint: Let $z = re^{i\theta}$, where $\theta = \frac{2\pi m}{2^k}$.

4. (a) Find the singularities of $1/\sin(\pi z)$. (b) Find the residue at each singularity.

5. Evaluate
$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 81}\, dx \tag{23}$$
using contour integration.

8 Assignment 8: Complex Variable 2 (due October 23, 2008)

1. Evaluate
$$\int_0^\infty \frac{1}{\sqrt{x}(4+x^2)}\,dx$$
by contour integration. Show all steps, including estimation of integrals that vanish in the limit of large contours.

2. In the following problems $f(z)$ is analytic in some region. We say that $f(z)$ has a root of multiplicity m at z_0 if $f(z) = (z-z_0)^m h(z)$, where $h(z)$ is analytic with $h(z_0) \neq 0$. Find the residue of $f'(z)/f(z)$ at such a z_0.

3. Say that $f(z)$ has several roots inside the contour C. Evaluate
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz.$$

4. Say that
$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$
is a polynomial. Furthermore, suppose that C is a contour surrounding the origin on which
$$|a_k z^k| > |f(z) - a_k z^k|.$$
Show that on this contour
$$f(z) = a_k z^k g(z)$$
where
$$|g(z) - 1| < 1$$
on the contour. Use the result of the previous problem to show that the number of roots (counting multiplicity) inside C is k.

5. Find the number of roots (counting multiplicity) of $z^6 + 3z^5 + 1$ inside the unit circle.

9 Assignment 9: Fourier series (due October 30, 2008)

1. Let $f(x) = x$ defined for $-\pi \leq x < \pi$. Find the $L^1(T)$, $L^2(T)$, and $L^\infty(T)$ norms of f, and compare them.

2. Find the Fourier coefficients c_n of f for all n in Z.

3. Find the ℓ^∞, ℓ^2, and ℓ^1 norms of these Fourier coefficients, and compare them.

4. Use the equality of L^2 and ℓ^2 norms to compute
$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. Compare the ℓ^∞ and L^1 norms for this problem. Compare the L^∞ and ℓ^1 norms for this problem.

6. Use the pointwise convergence at $x = \pi/2$ to evaluate the infinite sum
$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1},$$
regarded as a limit of partial sums. Does this sum converge absolutely?

7. Let $F(x) = \frac{1}{2}x^2$ defined for $-\pi \leq x < \pi$. Find the Fourier coefficients of this function.

8. Use the equality of L^2 and ℓ^2 norms to compute
$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

9. Compare the ℓ^∞ and L^1 norms for this problem. Compare the L^∞ and ℓ^1 norms for this problem.

10. At which points x of T is $F(x)$ continuous? Differentiable? At which points x of T is $f(x)$ continuous? Differentiable? At which x does $F'(x) = f(x)$? Can the Fourier series of $f(x)$ be obtained by differentiating the Fourier series of $F(x)$ pointwise? (This last question can be answered by inspecting the explicit form of the Fourier series for the two problems.)

10 Assignment 9: Fourier series and PDE (due November 6, 2008)

1. Fix $L > 0$ and let $k_n = 2\pi n/L$. Let $f(x)$ and $g(x)$ be L-periodic functions with $f(x) = \sum_n c_n e^{ik_n x}$ and $g(x) = \sum_m d_m e^{ik_m x}$.

   (a) Define the convolution integral $f * g$ so that

   $$(f * g)(x) = \sum_n c_n d_n e^{ik_n x}$$

   and prove this formula.

   (b) Define the convolution $c * d$ so that

   $$f(x)g(x) = \sum_n (c * d)_n e^{ik_n x}$$

   and prove this formula.

2. Define the Fourier transform of $f(x)$ by

   $$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx.$$

   Fix L and define the wave numbers $k_n = 2\pi n/L$. Prove the Poisson summation formula in the form

   $$\sum_m f(x + mL) = \frac{1}{L}\sum_n \hat{f}(k_n)e^{ik_n x}.$$

   Hint: Expand the L-periodic function on the left hand side in a Fourier series, and evaluate the Fourier coefficients.

3. For fixed $t > 0$ consider the fundamental solution

   $$g(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}}.$$

   Recall that this function is normalized so that integral of $g(x, t)\,dx$ over the line has the value 1.

   (a) Let $\hat{g}(k, t)$ be the Fourier transform of $g(x, t)$. The time $t > 0$ remains fixed. Find a constant a such that

   $$\left(\frac{d}{dk} + ak\right)\hat{g}(k, t) = 0.$$

   (b) Solve this differential equation. The solution should involve a constant of integration. Use the normalization to find the value of this constant. Thus obtain a formula for the Fourier transform $\hat{g}(k, t)$.

4. Let $\sigma^2 > 0$ and consider the diffusion equation (heat equation)

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}$$

for x on a circle of circumference L. Take the initial condition $u = f(x)$ when $t = 0$, where f is L periodic with Fourier series $f(x) = \sum_n c_n e^{ik_n x}$.

(a) Write the solution

$$u(x,t) = \sum_n h_n(t) c_n e^{ik_n x}$$

with an explicit expression for $h_n(t)$.

(b) Define the Jacobi theta function $h(x,t) = \sum_n h_n(t) e^{ik_n x}$. Write the explicit expression for this Fourier series. Prove that the solution $u(x,t)$ is the convolution of $h(x,t)$ with $f(x)$. Write the convolution integral explicitly.

5. Prove that $h(x,t)$ is a constant multiple of $\sum_m g(x + mL, t)$. Write this with an explicit value for the constant.

6. Let $\sigma^2 > 0$, $a > 0$, and $b > 0$. Consider the nonlinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + au - bu^3. \tag{24}$$

for x on a circle of circumference L. Write the solution as a Fourier series $u = \sum_n c_n e^{ik_n x}$, where the coefficients c_n depend on time. Find a system of ordinary differential equation in time for the coefficients c_n. Be explicit about how the system involves convolutions with the coefficients c_n.

11 Assignment 10: Fourier transforms (due November 13, 2008)

1. Let $f(x) = 1/(2a)$ for $-a \leq x \leq a$ and be zero elsewhere. Find the $L^1(R)$, $L^2(R)$, and $L^\infty(R)$ norms of f, and compare them.

2. Find the Fourier transform of f.

3. Find the $L^\infty(R')$, $L^2(R')$, and $L^1(R')$ norms of the Fourier transform, and compare them.

4. Compare the $L^\infty(R')$ and $L^1(R)$ norms for this problem. Compare the $L^\infty(R)$ and $L^1(R')$ norms for this problem.

5. Use the pointwise convergence at $x = 0$ to evaluate an improper integral.

6. Calculate the convolution of f with itself.

7. Find the Fourier transform of the convolution of f with itself. Verify in this case that the Fourier transform of the convolution is the product of the Fourier transforms.

12 Assignment 11: Fourier and Radon transforms (due November 20, 2008)

1. In this problem the Fourier transform is

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ixk} f(x)\, dx$$

and the inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} e^{ixk} f(k)\, \frac{dk}{2\pi}.$$

These provide an isomorphism between the Hilbert spaces $L^2(\mathbf{R}, dx)$ and $L^2(\mathbf{R}, \frac{dk}{2\pi})$. The norm of f in the first space is equal to the norm of $\hat{f}$ in the second space. We will be interested in the situation where the Fourier transform is band-limited, that is, only waves with $|k| \leq a$ have non-zero amplitude.

Make the assumption that $|k| > a$ implies $\hat{f}(k) = 0$. That is, the Fourier transform of f vanishes outside of the interval $[-a, a]$.

Let

$$g(x) = \frac{\sin(ax)}{ax}.$$

The problem is to prove that

$$f(x) = \sum_{m=-\infty}^{\infty} f(\frac{m\pi}{a}) g(x - \frac{m\pi}{a}).$$

This says that if you know f at multiples of π/a, then you know f at all points.

Hint: Let $g_m(x) = g(x - m\pi/a)$. The task is to prove that $f(x) = \sum_m c_m g_m(x)$ with $c_m = f(m\pi/a)$. It helps to use the Fourier transform of these functions. First prove that the Fourier transform of $g(x)$ is given by $\hat{g}(k) = \pi/a$ for $|k| \leq a$ and $\hat{g}(k) = 0$ for $|k| > a$. (Actually, it may be easier to deal with the inverse Fourier transform.) Then prove that $\hat{g}_m(k) = \exp(-im\pi k/a)\hat{g}(k)$. Finally, note that the functions $\hat{g}_m(k)$ are orthogonal.

2. In this problem the Fourier transform is

$$\hat{f}(\mathbf{k}) = \int_{\mathbf{R}^n} e^{-i\mathbf{x}\cdot\mathbf{k}} f(\mathbf{x})\, d^n\mathbf{x}$$

and the inverse Fourier transform is

$$f(\mathbf{x}) = \int_{\mathbf{R}^n} e^{i\mathbf{x}\cdot\mathbf{k}} f(\mathbf{k})\, \frac{d^n\mathbf{k}}{(2\pi)^n}.$$

These provide an isomorphism between the Hilbert spaces $L^2(\mathbf{R}^n, d^n\mathbf{x})$ and $L^2(\mathbf{R}^n, \frac{d^n\mathbf{k}}{(2\pi)^n})$. Notice that we can write the vector $\mathbf{x} = r\omega$, where $r = |\mathbf{x}|$ and ω is a unit vector. Then the Lebesgue measure $d^n\mathbf{x} = r^{n-1}\, dr\, d^{n-1}\omega$, where $d^{n-1}\omega$ denotes the angular integral. For instance, in three dimensions $d^3\mathbf{x} = r^2\, dr \sin(\theta)\, d\theta\, d\phi$ and $d^2\omega = \sin(\theta)\, d\theta\, d\phi$. There is a similar polar representation $\mathbf{k} = \rho\omega$, where $\rho = |\mathbf{k}|$.

The Radon transform is defined by

$$(Rf)(s, \omega) = \int_{\omega\cdot\mathbf{x}=s} f(\mathbf{x})\, d^{n-1}\mathbf{x}.$$

Here the integral is over the plane $\{\mathbf{x}\cdot\omega = s\}$ orthogonal to ω. Note that $Rf(s, \omega) = Rf(-s, -\omega)$.

(a) Prove that

$$\int_{-\infty}^{\infty} \exp(-i\rho s)(Rf)(s, \omega)\, ds = \hat{f}(\rho\omega).$$

(b) Prove that the Radon transform Rf uniquely determines the function f.

(c) Prove that for $n > 1$ there is a constant γ_n such that the Fourier transform of $1/|\mathbf{x}|$ is $\gamma_n/|\mathbf{k}|^{n-1}$. Hint: Use a scaling argument. (Note: It may be shown that $\gamma_n = (4\pi)^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})$.)

(d) Prove that

$$\int\int_0^{\infty} |(Rf)(s, \omega)|^2\, ds\, d^{n-1}\omega = C \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \overline{f(\mathbf{x})} \frac{1}{\gamma_n|\mathbf{x}-\mathbf{y}|} f(\mathbf{y})\, d^n\mathbf{y}\, d^n\mathbf{x}.$$

Do the computation carefully to determine the value of the constant C.

Hint: First show that for each unit vector ω we have

$$\int_{-\infty}^{\infty} |Rf(s, \omega)|^2\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\rho\omega)|^2\, d\rho.$$

Then argue that

$$\int\int_0^{\infty} |Rf(s, \omega)|^2\, ds\, d^{n-1}\omega = \frac{1}{2\pi}\int\int_0^{\infty} |\hat{f}(\rho\omega)|^2\, d\rho\, d^{n-1}\omega.$$

Continue by showing that

$$\int\int_0^{\infty} |Rf(s, \omega)|^2\, ds\, d^{n-1}\omega = \frac{1}{2\pi}\int_{\mathbf{R}^n} \overline{\hat{f}(\mathbf{k})} \frac{1}{|\mathbf{k}|^{n-1}} \hat{f}(\mathbf{k})\, d^n\mathbf{k}.$$

13 Assignment 12: Transform problems (due December 4, 2008)

1. Define the Mellin transform by
$$(Mf)(p) = \int_0^\infty u^p f(u) \, \frac{du}{u}.$$
Suppose that the multiplicative convolution is defined by
$$(f * g)(u) = \int_0^\infty f(u/v) g(v) \, dv/v.$$
Find the Mellin transform of such a multiplicative convolution.

2. Define the Gamma function $\Gamma(p)$ to be the Mellin transform of $\exp(-u)$. Define the Beta function $B(\alpha, p)$ to be the Mellin transform of the function that is $(1-u)^{\alpha-1}$ on the interval from 0 to 1 and is zero elsewhere. Show that
$$B(\alpha, p) = \frac{\Gamma(\alpha)\Gamma(p)}{\Gamma(\alpha + p)}.$$
Hint: Write $\Gamma(\alpha)\Gamma(p) = \int_0^\infty \int_0^\infty u^\alpha v^p e^{-(u+v)} \frac{du\,dv}{uv}$. Make the change of variable $u = t(1-r), v = tr$. Express the 2-form $du\,dv$ in terms of $dt\,dr$.

3. Fix $\alpha > 0$. For $x \geq 0$ let
$$(Wf)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} f(y) \, dy.$$
(a) Show that $(Wf)(x)$ is the multiplicative convolution of the function that is $1/\Gamma(\alpha)(1-u)^{\alpha-1}$ on the interval from 0 to 1 with $h(y) = y^\alpha f(y)$.

   (b) Find the Mellin transform of h in terms of the Mellin transform of f.

   (c) Find the Mellin transform of Wf in terms of Gamma functions and the Mellin transform of f.

4. Suppose that $F(x) = f(e^x)$. Show that $(Mf)(-ik) = \hat{F}(k)$. So the Mellin transform reduces to the Fourier transform via a change of variable.

5. Consider the wave equation in three space and one time dimension for $t > 0$, so
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla_{\mathbf{x}}^2 u.$$
Look for a solution with $u = 0$ and $\partial u/\partial t = h(x)$ when $t = 0$. Find the spatial Fourier transform of such a solution.

6. Consider the distribution $\delta(\mathbf{x}^2 - c^2 t^2)$. Find the three-dimensional Fourier transform of this distribution. Hint: Spherical polar coordinates.

7. Show that the solution of the wave equation is the convolution of a constant multiple of this distribution with the initial h. Find the constant.

8. Expanding spheres. Find this solution at $\mathbf{x}$ as an explicit integral over the sphere with center $\mathbf{x}$ and radius ct of the function h, where the integral is respect to the area on the sphere.

9. Expanding spheres. Find this solution at $\mathbf{x}$ as an explicit integral over the sphere with center $\mathbf{x}$ and radius ct of the function h, where the integral is respect to the uniform probability distribution on the sphere. Use this to show directly that the initial condition is satisfied.