

# Mapping the delta function and other Radon measures

Notes for Math583A, Fall 2008

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## 1 Radon measures

Consider a continuous function  $f$  on the real line with scalar values. It is said to have bounded support if there is a bounded interval  $[a, b]$  of real numbers such that  $f(x) \neq 0$  implies  $a \leq x \leq b$ . An alternative terminology that is often used is that  $f$  is said to have compact support.

Let  $C_c$  denote the vector space of continuous scalar valued functions on the real line, each of which has compact support. This will be called the space of (continuous) test functions.

A Radon measure is a positivity preserving linear function  $\mu$  from  $C_c$  to the scalars. The value of the Radon measure  $\mu$  on the test function  $f$  is denoted  $\langle \mu, f \rangle$ . The positivity preserving condition says that if for all  $x$  we have  $f(x) \geq 0$ , then  $\langle \mu, f \rangle \geq 0$ . Technical note: In measure theory a Radon measure would generate a measure defined on the Borel subsets of the line that is finite on compact subsets.

Example 1. Let  $h \geq 0$  be a positive locally integrable function. This means that for each point  $p$  there is a constant  $c > 0$  such that the integral

$$\int_{p-c}^{p+c} h(x) dx < \infty. \quad (1)$$

It follows that for every bounded interval  $[a, b]$  of real numbers we have

$$\int_a^b h(x) dx < \infty. \quad (2)$$

Then there is a Radon measure  $\mu_h$  given by

$$\langle \mu_h, f \rangle = \int_{-\infty}^{\infty} h(x)f(x) dx. \quad (3)$$

Example 2. Let  $a$  be a real number. The point mass at  $a$  is the Radon measure  $\delta_a$  defined by

$$\langle \delta_a, f \rangle = f(a). \quad (4)$$

One can also take linear combinations  $\sum_j c_j \delta_{a_j}$  with  $c_j \geq 0$ . The value of this on a function is of course

$$\langle \sum_j c_j \delta_{a_j}, f \rangle = \sum_j c_j f(a_j) \quad (5)$$

The intuitive interpretation of a Radon measure is as mass spread out on the line. The most general Radon measure is defined as follows. Let  $F$  be an increasing right-continuous function. To say that  $F$  is increasing is to say that  $x \leq y$  implies  $F(x) \leq F(y)$ . To say that  $F$  is right continuous is to say that  $\lim_{\epsilon \downarrow 0} F(x + \epsilon) = F(x)$ . Then the most general Radon measure is given by the Riemann-Stieltjes integral

$$\langle \mu^F, f \rangle = \int_{-\infty}^{\infty} f(x) dF(x). \quad (6)$$

Given the Radon measure, the function  $F$  is determined up to a constant of integration. The part corresponding to the jump discontinuities of  $F$  is the part corresponding to point masses. The part given by a density function  $h \geq 0$  is called the absolutely continuous part. For this part the function  $F'(x) = h(x)$  almost everywhere, and, furthermore, the function  $F$  may be recovered from  $h$  as an indefinite integral. (There can be a third part to the measure called the singular continuous part.)

## 2 Signed Radon measures

One can also define signed Radon measures. The simplest definition is to take this as the difference of two Radon measures. (In a similar way, one can define complex Radon measures.) Technical note: A signed Radon measure is not necessarily a signed measure in the sense of measure theory, since it could have both infinite positive part and infinite negative part.

Example 1. Let  $h$  be a locally integrable function. This means that for each point  $p$  there is a constant  $c > 0$  such that the integral

$$\int_{p-c}^{p+c} |h(x)| dx < \infty. \quad (7)$$

It follows that for every bounded interval  $[a, b]$  of real numbers we have

$$\int_a^b |h(x)| dx < \infty. \quad (8)$$

Then there is a signed Radon measure  $\mu_h$  given by

$$\langle \mu_h, f \rangle = \int_{-\infty}^{\infty} h(x) f(x) dx. \quad (9)$$

Example 2. Let  $a$  be a real number. The point mass at  $a$  is the Radon measure  $\delta_a$  defined by

$$\langle \delta_a, f \rangle = f(a). \quad (10)$$

One can also take linear combinations  $\sum_j c_j \delta_{a_j}$  with  $c_j$  real. The value of this on a function is of course

$$\langle \sum_j c_j \delta_{a_j}, f \rangle = \sum_j c_j f(a_j) \quad (11)$$

The intuitive interpretation of a signed Radon measure is as a charge spread out on the line.

### 3 Schwartz distributions

Schwartz distributions are more general than signed Radon measures. For Schwartz distributions the test functions are restricted to be in  $C_c^\infty$ , the space of smooth functions each of which have compact support. Each Schwartz distribution  $T$  defines a linear function  $T$  from  $C_c^\infty$  to the scalars. The value of  $T$  on  $f$  is denoted  $\langle T, f \rangle$ .

Example 1. The principal value  $1/x$  integral is defined by

$$\langle PV \frac{1}{x}, f \rangle = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{x}{x^2 + \epsilon^2} f(x) dx. \quad (12)$$

Example 2. The derivative of the point mass at  $a$  is defined by

$$\langle \delta'_a, f \rangle = -f'(a). \quad (13)$$

The important new feature is that a Schwartz distribution need not have a decomposition into a positive part and a negative part. In fact, the only time when this decomposition exists is when the Schwartz distribution is actually a signed Radon measure.

### 4 Functions naturally map backward

Functions are naturally covariant and map backward. Say that  $u = \phi(x)$ . If  $g(u)$  is a function, then the function  $g(\phi(x))$  is the pullback. This is the natural operation on functions. Similarly, if  $g(u) du$  is a differential form, then  $g(\phi(x))\phi'(x) dx$  is the natural covariant pullback. Then we have that

$$\int_a^b g(\phi(x))\phi'(x) dx = \int_{\phi(a)}^{\phi(b)} g(u) du. \quad (14)$$

Here is a consequence of this equation that will be important in the following. Suppose that  $\phi(x)$  is either increasing or decreasing on the interval  $(a, b)$  with

$a < b$ . Then applying the above formula  $g(\phi(x))$  replaced by  $h(\phi(x))f(x)$  we get

$$\int_a^b h(\phi(x))f(x) dx = \int_{\phi(a)}^{\phi(b)} h(u)f(\phi^{-1}(u))\frac{1}{\phi'(\phi^{-1}(u))} du. \quad (15)$$

Now break up the real axis into such intervals  $(a, b)$ . On those intervals where  $\phi(x)$  is decreasing we interchange  $\phi(a)$  and  $\phi(b)$  and replace  $\phi'(x)$  by  $|\phi'(x)|$ . Since there can be several intervals on which  $\phi(x) = u$ , the final result is that

$$\int_{-\infty}^{\infty} h(\phi(x))f(x) dx = \int_{-\infty}^{\infty} h(u)\left[\sum_{\phi(x)=u} f(x)\frac{1}{|\phi'(x)|}\right] du. \quad (16)$$

It is possible in particular that there may be no  $x$  with  $g(x) = u$ , in which case we interpret the sum as zero.

## 5 Functions can map forward

Sometimes a function is taken to map forward. In that case it is often called a density. Thus densities are contravariant and map forward. The function  $\rho(x)$  might represent a probability density or a mass density or a charge density. Say that  $\phi(x)$  is a function whose derivative only vanishes at isolated points. The expectation of  $g(\phi(x))$  with respect to this density is

$$\int_{-\infty}^{\infty} g(\phi(x))\rho(x) dx = \int_{-\infty}^{\infty} g(u)\left[\sum_{\phi(x)=u} \rho(x)\frac{1}{|\phi'(x)|}\right] du. \quad (17)$$

Thus the pushforward of  $\rho(x)$  is

$$\phi_*[\rho](u) = \sum_{\phi(x)=u} \rho(x)\frac{1}{|\phi'(x)|}. \quad (18)$$

So when a function is interpreted as a density, it is a contravariant object.

## 6 Signed Radon measures naturally map forward

A signed Radon measure assigns to each continuous function  $f$  with compact support a number  $\langle \mu, f \rangle$ . Signed Radon measures are naturally contravariant and map forward. If  $g \circ \phi$  is the composite function defined by  $(g \circ \phi)(x) = g(\phi(x))$ , then the pushforward measure  $\phi_*[\mu]$  is defined by

$$\langle \phi_*[\mu], g \rangle = \langle \mu, g \circ \phi \rangle.$$

Thus, for instance, if the measure is given by a density  $\rho(x)$ , then the push forward of  $\mu$  is given by  $\sum_{\phi(x)=u} \rho(x)/|\phi'(x)|$ .

Similarly, the pushforward of  $\delta_a$  is  $\delta_{\phi(a)}$ .

Here is an interesting example. Say that  $\phi(x) = a$  is a constant function. Then the pushforward of a measure with density  $\rho(x)$  is the measure  $\int_{-\infty}^{\infty} \rho(x) dx \delta_a$ . So a measure given with a density goes into a point measure.

## 7 Signed Radon measures can also map backward

A signed Radon measure (or more generally, a Schwartz distribution) can also be interpreted as a generalized function. Generalized functions are covariant and map backward. It is obvious how to do this with a density: the density  $\rho(u)$  is mapped to the density  $\rho(\phi(x))$ . Notice that this may not preserve the total mass or the total charge. The general rule is the pullback of the signed Radon measure  $\mu$  is the signed Radon measure  $\phi^*[\mu]$  defined by

$$\langle \phi^*[\mu], f \rangle = \langle \mu, \phi_*[f] \rangle. \quad (19)$$

Example. A point mass may be mapped backward. The general formula is that

$$\langle \phi^*[\delta_b], f \rangle = \sum_{\phi(a)=b} \frac{1}{|\phi'(a)|} \delta_a. \quad (20)$$

The motivation for this formula comes if we write the point mass as if it were a function. Thus we define an object  $\delta(u - b)$  that is really a point mass, so

$$\int_{-\infty}^{\infty} \delta(u - b)g(u) du = g(b). \quad (21)$$

Then the pullback is determined by

$$\int_{-\infty}^{\infty} \delta(\phi(x) - b)f(x) dx = \int_{-\infty}^{\infty} \delta(u - b) \left[ \sum_{\phi(a)=u} f(a) \frac{1}{|\phi'(a)|} \right] du. \quad (22)$$

This is equal to

$$\sum_{\phi(a)=b} f(a) \frac{1}{|\phi'(a)|} = \int_{-\infty}^{\infty} \left[ \sum_{\phi(a)=b} \delta(x - a) \frac{1}{|\phi'(a)|} \right] f(x) dx. \quad (23)$$

In other words, the pullback of  $\delta(u - b)$  is

$$\delta(\phi(x) - b) = \sum_{\phi(a)=b} \delta(x - a) \frac{1}{|\phi'(a)|}. \quad (24)$$

Schwartz distributions are a generalization of signed Radon measures. Distributions naturally map forward, but ordinarily they are considered as generalized functions, so in practice we usually do the unnatural thing and map them backwards. Such is life.