Radial functions and the Fourier transform

Notes for Math 583A, Fall 2008

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1 Area of a sphere

The volume in \( n \) dimensions is

\[
\text{vol} = d^n x = dx_1 \cdots dx_n = r^{n-1} dr d^{n-1} \omega.
\] (1)

Here \( r = |x| \) is the radius, and \( \omega = x/r \) is a radial unit vector. Also \( d^{n-1} \omega \) denotes the angular integral. For instance, when \( n = 2 \) it is \( d\theta \) for \( 0 \leq \theta \leq 2\pi \), while for \( n = 3 \) it is \( \sin(\theta) d\theta d\phi \) for \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \).

The radial component of the volume gives the area of the sphere. The radial directional derivative along the unit vector \( \omega = x/r \) may be denoted

\[
\omega \mathcal{d} = \frac{1}{r} \left( x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \right) = \frac{\partial}{\partial r}.
\] (2)

The corresponding spherical area is

\[
\omega \mathcal{d} \text{vol} = r^{n-1} d^{n-1} \omega.
\] (3)

Thus when \( n = 2 \) it is \( (1/r)(x \ dy - y \ dx) = r \ d\theta \), while for \( n = 3 \) it is \( (1/r)(x \ dy \ dz + y \ dz \ dx + x \ dx \ dy) = r^2 \sin(\theta) \ d\theta \ d\phi \).

The divergence theorem for the ball \( B_r \) of radius \( r \) is thus

\[
\int_{B_r} \text{div} \ \mathbf{v} \ d^n x = \int_{S_r} \mathbf{v} \cdot \omega r^{n-1} d^{n-1} \omega.
\] (4)

Notice that if one takes \( \mathbf{v} = x \), then \( \text{div} \ x = n \), while \( x \cdot \omega = r \). This shows that \( n \) times the volume of the ball is \( r^n \) times the surface area of the sphere.

Recall that the Gamma function is defined by \( \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \). It is easy to show that \( \Gamma(z+1) = z \Gamma(z) \). Since \( \Gamma(1) = 1 \), it follows that \( \Gamma(n) = (n-1)! \).

The result \( \Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2} \) follows reduction to a Gaussian integral. It follows that \( \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \pi^{3/2} \).

**Theorem 1** The area of the unit sphere \( S_{n-1} \subseteq \mathbb{R}^n \) is

\[
\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.
\] (5)
Thus in 3 dimensions the area of the sphere is $\omega_2 = 4\pi$, while in 2 dimensions the circumference of the circle is $\omega_1 = 2\pi$. In 1 dimension the two points get count $\omega_0 = 2$.

To prove this theorem, consider the Gaussian integral

$$\int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{2}} d^n x = 1. \quad \text{(6)}$$

In polar coordinates this is

$$\omega_{n-1} (2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-\frac{r^2}{2}} r^{n-1} dr = 1. \quad \text{(7)}$$

Let $u = r^2/2$. Then this is

$$\omega_{n-1} (2\pi)^{-\frac{n}{2}} 2^{-\frac{n-2}{2}} \int_0^\infty e^{-u} u^{\frac{n-2}{2}} du = 1. \quad \text{(8)}$$

That is

$$\omega_{n-1} \pi^{-\frac{n}{2}} 2^{-1} \Gamma\left(\frac{n}{2}\right) = 1. \quad \text{(9)}$$

This gives the result.

## 2 Fourier transform of a power

**Theorem 2** Let $1 < a < n$. The Fourier transform of $1/|x|^a$ is $C_a/|k|^{n-a}$, where

$$C_a = (2\pi)^{\frac{n-a}{2}} \frac{\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \quad \text{(10)}$$

This is not too difficult. It is clear from scaling that the Fourier transform of $1/|x|^a$ is $C/|k|^{n-a}$. It remains to evaluate the constant $C$.

Take the inner product with the Gaussian. This gives

$$\int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{2}} \frac{1}{|x|^a} d^n x = \int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{2}} C \frac{1}{|k|^{n-a}} d^n k. \quad \text{(11)}$$

Writing this in polar coordinates gives

$$(2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-\frac{r^2}{2}} r^{-n-1-a} dr = C(2\pi)^{-n} \int_0^\infty e^{-\frac{r^2}{2}} r^{1-a} dr. \quad \text{(12)}$$

This in turn gives

$$(2\pi)^{-\frac{n}{2}} 2^{-\frac{n-a}{2}} \Gamma\left(\frac{n-a}{2}\right) = C(2\pi)^{-n} 2^{-\frac{n-2}{2}} \Gamma\left(\frac{a}{2}\right). \quad \text{(13)}$$
3 The Hankel transform

Define the Bessel function

\[ J_\nu(t) = \frac{t^n}{(2\pi)^{\nu+1}\omega_{2\nu}} \int_0^\pi e^{-it\cos(\theta)} \sin(\theta)^{2\nu} d\theta. \]  

(14)

This makes sense for all real numbers \( \nu \geq 0 \), but we shall be interested mainly in the cases when \( \nu \) is an integer or \( \nu \) is a half-integer. In the case when \( \nu \) is a half-integer the exponent \( 2\nu \) is odd, and so it is possible to evaluate the integral in terms of elementary functions. Thus, for example,

\[ J_{\frac{1}{2}}(t) = \frac{t^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{2\pi^{\frac{1}{2}}}{2} \int_0^\pi e^{-it\sin(\theta)} \sin(\theta) d\theta = \frac{t^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{2\sin(t)}{t}. \]  

(15)

This is not possible when \( \nu \) is an integer. Thus for \( \nu = 0 \) we have the relatively mysterious expression

\[ J_0(t) = \frac{1}{\pi} \int_0^\pi e^{it\cos(\theta)} d\theta. \]  

(16)

Fix a value of \( \nu \). If we consider a function \( g(r) \), its Hankel transform is the function \( \hat{g}_\nu(s) \) given by

\[ \hat{g}_\nu(s) = \int_0^\infty J_\nu(sr)g(r)r \, dr. \]  

(17)

We shall see that the Hankel transform is related to the Fourier transform.

4 The radial Fourier transform

The first result is that the radial Fourier transform is given by a Hankel transform. Suppose \( f \) is a function on \( \mathbb{R}^n \). Its Fourier transform is

\[ \hat{f}(k) = \int e^{-ikx}f(x) \, dx. \]  

(18)

Let \( r = |x| \) and \( s = |k| \). Write \( f(x) = F(r) \) and \( \hat{f}(k) = F_n(s) \).

**Theorem 3** The radial Fourier transform in \( n \) dimensions is given in terms of the Hankel transform by

\[ s^{\frac{n-2}{2}} \hat{F}_n(s) = (2\pi)^n \int_0^\infty J_{\frac{n-2}{2}}(sr)r^{\frac{n-2}{2}}F(r)r \, dr. \]  

(19)

Here is the proof of the theorem. Introduce polar coordinates with the \( z \) axis along \( k \), so that \( k \cdot x = sr \cos(\theta) \). Suppose that the function is radial, that is, \( f(x) = F(r) \).

\[ \hat{f}(k) = \hat{F}_n(s) = \int_0^\pi \int_0^\infty e^{-isr\cos(\theta)}F(r)\omega_{n-2}\sin(\theta)^{n-2} \, d\theta \, dr^{n-1} \, dr. \]  

(20)
Use

\[ J_{n/2}(t) = \frac{i^{n/2}}{(2\pi)^{1/2}} \omega_n^{-1} \int_0^\pi e^{-it\cos(\theta)} \sin^n(\theta)^{-1} d\theta. \]  

(21)

For the case \( n = 3 \) the Bessel function has order \( 1/2 \) and has the above expression in terms of elementary functions. So

\[ \hat{F}_3(s) = 4\pi \int_0^\infty \frac{\sin(sr)}{sr} F(r) r^2 dr. \]  

(22)

For \( n = 2 \) the Bessel function has order 0. We get

\[ \hat{F}_2(s) = 2\pi \int_0^\infty J_0(sr) F(r) r dr. \]  

(23)