Ordinary Differential Equations

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August 16, 1992

Contents

Chapter 1

Growth and Decay

1.1 Linear constant coefficient equations

The simplest differential equation is the equation of *uniform motion*:

$$\frac{dy}{dt} = b, \tag{1.1}$$

where b is constant. Its solution is

$$y = y_0 + bt. \tag{1.2}$$

The next simplest is the equation of *growth or decay*:

$$\frac{dy}{dt} = ay, \tag{1.3}$$

where a is constant. This is the homogeneous linear constant coefficient equation. Its solution is

$$y = y_0 e^{at}. \tag{1.4}$$

These may be combined in the general *linear constant coefficient* equation:

$$\frac{dy}{dt} = ay + b, \tag{1.5}$$

where a and b are constants. The general solution for $a \neq 0$ is

$$y = Ce^{at} - \frac{b}{a}. \tag{1.6}$$

The solution with $y = y_0$ at $t = 0$ is

$$y = y_0 e^{at} + \frac{b}{a}(e^{at} - 1). \tag{1.7}$$

1.2 Autonomous equations

The general *autonomous* equation is

$$\frac{dy}{dt} = f(y). \tag{1.8}$$

An *equilibrium point* is a solution of $f(r) = 0$. For each equilibrium point we have a solution $y = r$.

Near an equilibrium point $f(y) \approx f'(r)(y - r)$. An equilibrium point r is attractive if $f'(r) < 0$ and repulsive if $f'(r) > 0$.

One can attempt to find the general solution of the equation by integrating

$$\int \frac{1}{f(y)}\, dy = \int dt. \tag{1.9}$$

Problems

1. If a population grows by $dp/dt = .05p$, how long does it take to double in size?

2. The velocity of a falling body (in the downward direction) is given by $dv/dt = g - kv$, where $g = 32$ and $k = 1/4$. If $v = 0$ when $t = 0$, what is the limiting velocity as $t \to \infty$?

3. Consider $dy/dt = ay + b$ where $y = y_0$ when $t = 0$. Fix t and find the limit of the solution y as $a \to 0$.

4. A population grows by $dp/dt = ap - bp^2$. Here $a > 0$, $b > 0$, and $0 < p < a/b$. Find the solution with $p = p_0$ at $t = 0$. Do this by letting $u = 1/p$ and solving the resulting differential equation for u.

5. Do the same problem by integrating $1/(ap - bp^2)\, dp = dt$. Use partial fractions.

6. In the same problem, find the limiting population as $t \to \infty$.

7. Use Phaser to explore the solutions of $dx/dt = x - x^3$. Try many different initial conditions. What pattern emerges? Discuss the limit of x as $t \to \infty$ as a function of the initial condition x_0.

Chapter 2

Oscillations

2.1 Linear constant coefficient equations

The homogeneous linear constant coefficient system is of the form

$$\frac{dx}{dt} = ax + by \tag{2.1}$$

$$\frac{dy}{dt} = cx + dy. \tag{2.2}$$

Try a solution of the form

$$x = ve^{\lambda t} \tag{2.3}$$

$$y = we^{\lambda t}. \tag{2.4}$$

We obtain the *eigenvalue* equation

$$av + bw = \lambda v \tag{2.5}$$

$$cv + dw = \lambda w. \tag{2.6}$$

This has a non-zero solution only when λ satisfies $\lambda^2-(a+d)\lambda+ad-bc = 0$.

We can express the same ideas in matrix notation. The equation is

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{2.7}$$

The trial solution is

$$\mathbf{x} = \mathbf{v}e^{\lambda t}. \tag{2.8}$$

The eigenvalue equation is

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2.9}$$

This has a non-zero solution only when $\det(\lambda I - A) = 0$.

2.1.1 Growth and Decay

The first case is real and unequal eigenvalues $\lambda_1 \neq \lambda_2$. This takes place when $(a-d)^2 + 4bc > 0$. There are two solutions corresponding to two independent eigenvectors. The general solution is a linear combination of these two. In matrix notation this is

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}. \tag{2.10}$$

When the two eigenvalues are both positive or both negative, the equilibrium is called a *node*. When one eigenvalue is positive and one is negative, it is called a *saddle*. An attractive node corresponds to an *overdamped* oscillator.

2.1.2 Oscillation

The second case is complex conjugate unequal eigenvalues $\lambda = \alpha + i\omega$ and $\bar{\lambda} = \alpha - i\omega$ with $\alpha = (a+d)/2$ and $\omega > 0$. This takes place when $(a-d)^2 + 4bc < 0$. There are two independent complex conjugate solutions. These are expressed in terms of $e^{\lambda t} = e^{\alpha t} e^{i\omega t}$ and $e^{\bar{\lambda} t} = e^{\alpha t} e^{-i\omega t}$. Their real and imaginary parts are independent real solutions. These are expressed in terms of $e^{\alpha t}\cos(\omega t)$ and $e^{\alpha t}\sin(\omega t)$.

In matrix notation we have complex eigenvectors $\mathbf{u} \pm i\mathbf{v}$ and the solutions are

$$x = (c_1 \pm i c_2) e^{\alpha t} e^{\pm i\omega t}(\mathbf{u} \pm i\mathbf{v}). \tag{2.11}$$

Taking the real part gives

$$x = c_1 e^{\alpha t}(\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{v}) - c_2 e^{\alpha t}(\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{v}). \tag{2.12}$$

If we write $c_i \pm i c_2 = c e^{\pm i\theta}$, these take the alternate forms

$$x = c e^{\alpha t} e^{\pm i(\omega t + \theta)}(\mathbf{u} \pm i\mathbf{v}). \tag{2.13}$$

and

$$x = c e^{\alpha t}(\cos(\omega t + \theta)\mathbf{u} - \sin(\omega t + \theta)\mathbf{v}). \tag{2.14}$$

From this we see that the solution is characterized by an amplitude c and a phase θ. When the two conjugate eigenvalues are pure imaginary, the equilibrium is called a *center*. When the two conjugate eigenvalues have a non-zero real part, it is called a *spiral* (or a *focus*). An center corresponds to an *undamped* oscillator. An attractive spiral corresponds to an *underdamped* oscillator.

2.1.3 Shearing

The remaining case is when there is only one eigenvalue $\lambda = (a+d)/2$. This takes place when $(a-d)^2 + 4bc = 0$. In this case we neeed to try a solution of the form

$$x = pe^{\lambda t} + vte^{\lambda t} \tag{2.15}$$
$$y = qe^{\lambda t} + wte^{\lambda t}. \tag{2.16}$$

We obtain the same eigenvalue equation together with the equation

$$ap + bq = \lambda p + v \tag{2.17}$$
$$cp + dq = \lambda q + w. \tag{2.18}$$

In practice we do not need to solve for the eigenvector: we merely take p, q determined by the initial conditions and use the last equation to solve for v, w.

Im matrix notation this becomes

$$\mathbf{x} = \mathbf{p}e^{\lambda t} + \mathbf{v}te^{\lambda t} \tag{2.19}$$

with

$$A\mathbf{p} = \lambda\mathbf{p} + \mathbf{v}. \tag{2.20}$$

2.1.4 Inhomogeneous equations

The general linear constant coefficient equation is

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{r}. \tag{2.21}$$

When A is non-singular we may rewrite this as

$$\frac{d\mathbf{x}}{dt} = A(\mathbf{x} - \mathbf{s}), \tag{2.22}$$

where $\mathbf{s} = -A^{-1}\mathbf{r}$ is constant. Thus $\mathbf{x} = \mathbf{s}$ is a particular solution. The general solution is the sum of this particular solution with the general solution of the homogeneous equation.

Problems

1. Find the general solution of the system

$$\frac{dx}{dt} = x + 3y$$
$$\frac{dy}{dt} = 5x + 3y.$$

2. Find the solution of this equation with the initial condition $x = 1$ and $y = 3$ when $t = 0$.

3. Use Phaser to sketch the direction field in the above problem. Sketch the given solution in the x, y phase space. Experiment to find a solution that passes very close to the origin, and sketch it.

4. Write the Taylor series of e^z about $z = 0$. Plug in $z = i\theta$, where $i^2 = -1$. Show that $e^{i\theta} = \cos\theta + i\sin\theta$.

5. Find the general solution of the system

$$\begin{aligned} \frac{dx}{dt} &= x + 5y \\ \frac{dy}{dt} &= -x - 3y. \end{aligned}$$

6. Find the solution of this equation with the initial condition $x = 5$ and $y = 4$ when $t = 0$.

7. User Phaser to sketch the direction field in the above problem. Use Phaser to find the given solution in phase space. Also plot x versus t and y versus t.

8. A frictionless spring has mass $m > 0$ and spring constant $k > 0$. Its displacement and velocity x and y satisfy

$$\begin{aligned} \frac{dx}{dt} &= y \\ m\frac{dy}{dt} &= -kx. \end{aligned}$$

   Describe the motion.

9. A spring has mass $m > 0$ and spring constant $k > 0$ and friction constant $f > 0$. Its displacement and velocity x and y satisfy

$$\begin{aligned} \frac{dx}{dt} &= y \\ m\frac{dy}{dt} &= -kx - fy. \end{aligned}$$

   Describe the motion in the case $f^2 - 4k < 0$ (underdamped).

10. Take $m = 1$ and $k = 1$ and $f = 0.1$. Use Phaser to sketch the direction field and the solution in the phase plane. Also sketch x as a function of t.

11. In the preceding problem, describe the motion in the case $f^2 - 4k > 0$ (overdamped). Is it possible for the oscillator displacement x to overshoot the origin? If so, how many times?

12. An object has mass $m > 0$ and its displacement and velocity x and y satisfy

$$\begin{aligned} \frac{dx}{dt} &= y \\ m\frac{dy}{dt} &= 0. \end{aligned}$$

Describe the motion.

13. Use Phaser to solve the above equation with many initial condition with $x = 0$ and with varying value of y. Run the solution with these initial conditions for a short time interval. Why can this be described as “shear”?

2.2 Autonomous Systems

The general *autonomous* system is

$$\frac{dx}{dt} = f(x, y) \quad (2.23)$$

$$\frac{dy}{dt} = g(x, y). \quad (2.24)$$

An *equilibrium point* is a solution of $f(r, s) = 0$ and $g(r, s) = 0$. For each equilibrium point we have a solution $x = r$ and $y = s$.

Near an equilibrium point

$$f(x, y) \approx a(x - r) + b(y - s) \quad (2.25)$$

$$g(x, y) \approx c(x - r) + d(y - s), \quad (2.26)$$

where $a = \partial f(x, y)/\partial x$, $b = \partial f(x, y)/\partial y$, $c = \partial g(x, y)/\partial x$, and $d = \partial g(x, y)/\partial y$, all evaluated at $x = r$ and $y = s$. So near the equilibrium point the equation looks like a linear equation.

Assume that the eigenvalues of the linear equation are real. Then the equilibrium point is attractive if they are both negative. On the other hand, assume that the eigenvalues of the linear equation are complex conjugates. Then the equilibrium point is attractive if the real part is negative. In general the equilibrium point is classified by the behavior of the linearized equation at that point.

A first example is the non-linear pendulum equation. This is

$$\frac{dx}{dt} = y \tag{2.27}$$

$$ml\frac{dy}{dt} = -mg\sin(x) - cy. \tag{2.28}$$

Here x is the angle and y is the angular velocity. The parameters are the mass $m > 0$, the length $l > 0$, and the gravitational acceleration $g > 0$. There may also be a friction coefficient $c \geq 0$. The first equation is the definition of angular velocity. The second equation is Newton's law of motion: mass times acceleration equals force.

There are two interesting equilibrium situations. One is where $x = 0$ and $y = 0$. In the case we use $\sin(x) \approx x$ to find the linear approximation The other interesting situation is when $x - \pi = 0$ and $y = 0$. In this case we use $\sin(x) \approx -(x - \pi)$. The minus sign makes a crucial difference.

A second example is the predator-prey system. This is

$$\frac{dx}{dt} = (a - by - mx)x \tag{2.29}$$

$$\frac{dy}{dt} = (cx - d - ny)y. \tag{2.30}$$

Here x is the prey and y is the predator. The prey equation says that the prey has a natural growth rate a, are eaten by the predators at rate by, and compete with themselves with rate mx. The predator equation says that the predators have a growth rate $cx - d$ at food level x and compete with themselves at rate ny. The parameters are strictly positive, except that we allow the special case $m = 0$ and $n = 0$ with no internal competition. We are only interested in the situation $x \geq 0$ and $y \geq 0$.

There are several equilibria. One corresponds to total extinction. Also when $m > 0$ one can have a situation when the predator is extinct and where $x = a/m$ is the natural prey carrying capacity. Whem $m = 0$, on the other hand, there is there is no natural limit to the size of the prey population: we interpret $a/m = +\infty$.

The most interesting equilibrium takes place when the natural predator growth rate $cx - d$ with $x = a/m$ at the prey carrying capacity is positive. This says that the predator can live off the land.

Problems

1. For the pendulum problem with no friction, find the linearization at $x = 0$, $y = 0$. Discuss the nature of the equilibrium.

2. Use Phaser for the pendulum problem. Find oscillatory solutions that are near the zero solution, but not too near. How large can the solutions be before the pendulum can no longer be used as a clock? Sketch Phaser plots that illustrate this point.

3. For the pendulum problem with no friction, find the linearization at $x = \pi$, $y = 0$. Discuss the nature of the equilibrium.

4. Use Phaser. Find at least two different kinds of oscillatory solutions that pass near $x = \pi$, $y = 0$. Sketch Phaser plots that illustrate these different kinds of solutions.

5. For the pendulum problem, describe the nature of the two equilibria when there is friction.

6. Consider the predator-prey equations with internal competition. Find the nature of the equilibrium corresponding to total extinction.

7. Find the nature of the equilibrium corresponding to extinction of the predators. There are two situations, depending on the sign of the predator natural growth rate.

8. Find the nature of the equilibrium corresponding to coexistence. Discuss its stability.

9. Use Phaser to sketch representative solutions.

10. Find the nature of the equilibrium corresponding to coexistence when there is no internal competition.

11. Use Phaser to sketch representative solutions.

2.3 Limit cycles

Now we come to an essentially non-linear effect: oscillations that are stabilized by the non-linearity.

The classic example is

$$\frac{dx}{dt} = v \tag{2.31}$$

$$\frac{dv}{dt} = -kx - g(x)v. \tag{2.32}$$

This is an oscillator in which the friction coefficient $g(x)$ is a function of position. There is a constant $r > 0$ such that $g(x) < 0$ for $|x| < r$ and $g(x) > 0$ for $|x| > r$. Thus when $|x|$ is small the oscillator gets a boost. A standard example is $g(x) = c(x^2 - r^2)$.

Change variables to $y = v + G(x)$, where $G'(x) = g(x)$. Then this same oscillator becomes

$$\frac{dx}{dt} = y - G(x) \tag{2.33}$$

$$\frac{dy}{dt} = -kx. \tag{2.34}$$

The equation is often studied in this form.

Problems

1. Take the van der Pol oscillator in x, y space with $G(x) = x^3 - ax$. Use Phaser to investigate the Hopf bifurcation. Sketch your results.

2. Take the non-linear van der Pol oscillator in x, v space with $g(x) = a(x^2 - 1)$. Take $a > 0$ increasingly large. The result is a relaxation oscillator. Use Phaser to make plots in the x, v plane. Also make x versus t and v versus t plots and interpret them.

Chapter 3

Conserved Quantities

3.1 Vector fields

Consider a system of the form

$$\frac{dx}{dt} = f(x,y) \tag{3.1}$$

$$\frac{dy}{dt} = g(x,y). \tag{3.2}$$

Let $K = k(x, y)$ be a function on phase space. Then along a solution

$$\frac{dK}{dt} = f(x,y)\frac{\partial K}{\partial x} + g(x,y)\frac{\partial K}{\partial y}. \tag{3.3}$$

We see from this that giving the differential equation is the same as giving the derivative along a vector field. This derivative along the vector field is

$$f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}. \tag{3.4}$$

Giving this derivative is the same as giving the vector field, and this is in turn the same as giving the system of differential equations.

A vector field can be expressed in various coordinate systems, and sometimes this simplifies things. Here are a couple of examples where polar coordinates are useful. The first is dilation. The corresponding differential operator is called the *Euler operator*.

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}. \tag{3.5}$$

The second is rotation:

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y} = \frac{\partial}{\partial \theta}. \tag{3.6}$$

3.2 Hamiltonian systems

Consider a system of the form

$$\frac{dx}{dt} = f(x,y) \tag{3.7}$$

$$\frac{dy}{dt} = g(x,y). \tag{3.8}$$

It is said to be *Hamiltonian* if there is a function $H = h(x,y)$ such that

$$\frac{\partial H}{\partial y} = f(x,y) \tag{3.9}$$

$$\frac{\partial H}{\partial x} = -g(x,y). \tag{3.10}$$

It follows H is constant along every orbit. Note that for a Hamiltonian system the divergence

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial y} = 0. \tag{3.11}$$

The frictionless oscillator is an example. This is

$$\frac{dx}{dt} = \frac{p}{m} \tag{3.12}$$

$$\frac{dp}{dt} = -V'(x), \tag{3.13}$$

where p is the *momentum*. The Hamiltonian function in this case is the energy

$$H = \frac{1}{2m}p^2 + V(x). \tag{3.14}$$

This is the context for *conservation of energy*.

A more interesting example is the predator-prey system with no internal competition. This is

$$\frac{du}{dt} = a - be^v \tag{3.15}$$

$$\frac{dv}{dt} = ce^u - d. \tag{3.16}$$

Here $u = \ln x$ is the prey and $v = \ln y$ is the predator, measured on a logarithmic scale. It is not difficult to find the Hamiltonian.

We say that an equation is *dissipative* in a region provided that the divergence

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial y} < 0 \tag{3.17}$$

in the region.

An example is an oscillator with friction:

$$\frac{dx}{dt} = \frac{p}{m} \tag{3.18}$$

$$\frac{dp}{dt} = -V'(x) - g(x)\frac{p}{m} \tag{3.19}$$

when $g(x) > 0$. Note that if we take $H = \frac{p^2}{2m} + V(x)$ as in the frictionless case, we get a time derivative $-g(x)(p/m)^2 < 0$.

Say that an equation is dissipative in a region. There is a result of Bendixon that says that if A is contained in the region, then the boundary ∂A of A cannot be the orbit of a solution. In particular, it cannot be be a limit cycle.

The proof of this result is to apply Green's theorem to A. We have

$$0 < \int\int_A \left(\frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial x}\right) dx\, dy = \int_{\partial A} -g(x,y)\, dx + f(x,y)\, dy. \tag{3.20}$$

If ∂A were the orbit of a solution with period T, then the right hand side would be

$$\int_0^T \left(-g(x,y)\frac{dx}{dt} + f(x,y)\frac{dy}{dt}\right) dt = \int_0^T (-g(x,y)f(x,y) + f(x,y)g(x,y))\, dt = 0. \tag{3.21}$$

This would give a contradiction.

Problems

1. Show that the Hamiltonian function H is always constant along an orbit by computing dH/dt.

2. Show that a vector field that arises from a Hamiltonian has zero divergence.

3. Find the Hamiltonian for the predator-prey system with no internal competition.

4. Model this system with Phaser in the x, y plane. Compare the results with the equation for the orbits obtained from setting the Hamiltonian to a constant.

5. Consider a homogeneous linear system given by a matrix. What condition on the trace of the matrix is needed for the system to be Hamiltonian? Find the Hamiltonian function H.

6. This corresponds to a condition on the sum of the eigenvalues. What is it?

7. A neurobiologist considers a system of equations

$$\frac{dx}{dt} = f(y) - ax \tag{3.22}$$

$$\frac{dy}{dt} = g(x) - by \tag{3.23}$$

where $a > 0$ and $b > 0$ and f is an increasing function and g is a decreasing function. The variables x and y represent conductances associated with two neurons. Can this system describe an oscillator in the brain?

3.3 Exact equations

Another perspective is to eliminate the time dependence. We can write the system of differential equations as

$$-g(x, y)\, dx + f(x, y)\, dy = 0. \tag{3.24}$$

This suggests consideration of a *differential form* of the form

$$p(x, y)\, dx + q(x, y)\, dy. \tag{3.25}$$

Such a differential form is said to be *exact* if there is a function $U = u(x, y)$ with

$$\frac{\partial U}{\partial x} = p(x, y) \tag{3.26}$$

$$\frac{\partial U}{\partial y} = q(x, y). \tag{3.27}$$

If this is the case, then it satisfies the *integrability condition*

$$\frac{\partial p(x, y)}{\partial y} = \frac{\partial q(x, y)}{\partial x}. \tag{3.28}$$

If the differential form is exact, then the solution of the differential equation

$$p(x, y)\, dx + q(x, y)\, dy = 0 \tag{3.29}$$

is given by $u(x, y) = C$.

A differential form can be expressed in various coordinate systems, and sometimes this simplifies things. Here are a couple of examples where polar coordinates are useful. The first is

$$x\, dx + y\, dy = r\, dr = \frac{1}{2}\, dr^2. \tag{3.30}$$

which is exact. The second is:

$$x\, dy - y\, dx = r^2\, d\theta \tag{3.31}$$

which is not exact. However dividing by r^2 produces an exact form in the region $r > 0$, $0 < \theta < 2\pi$.

Problems

1. Integrate $(e^y + x)\, dx + (xe^y - e^{2y})\, dy$.
2. Test $y\, dx + (2x - ye^y)\, dy$ for exactness. Integrate if possible.
3. Test $y^2\, dx + (2xy - y^2 e^y)\, dy$ for exactness. Integrate if possible.
4. Find a relation between the condition for a vector field to be Hamiltonian and the condition for the corresponding differential form to be exact.
5. Find a relation between the condition for a vector field to have zero divergence and for the differential form to satisfy the integrability condition.

3.4 Integrating factors

A non-zero function $M = m(x, y)$ is an *integrating factor* for a differential form if

$$Mp(x, y)\, dx + Mq(x, y)\, dy \tag{3.32}$$

is exact. If this is the case, then we have the integrability condition

$$\frac{\partial Mp(x, y)}{\partial y} = \frac{\partial Mq(x, y)}{\partial x}. \tag{3.33}$$

There are several situations when this equation can be used to determine the integrating factor.

One is when we have both

$$\frac{\partial Mp(x,y)}{\partial y} = 0 \tag{3.34}$$

$$\frac{\partial Mq(x,y)}{\partial x} = 0. \tag{3.35}$$

This situation is called *separation of variables.*

Another situation is when M depends only on x. Then

$$M\frac{\partial p(x,y)}{\partial y} = \frac{\partial Mq(x,y)}{\partial x} = \frac{dM}{dx}q(x,y) + M\frac{\partial q(x,y)}{\partial x} \tag{3.36}$$

and so

$$\frac{1}{M}\frac{dM}{dx} = \frac{1}{q(x,y)}(\frac{\partial p(x,y)}{\partial y} - \frac{\partial q(x,y)}{\partial x}) \tag{3.37}$$

depends only on x.

There is a similar situation when M depends only on y.

Problems

1. Show that $(e^{-x} + \sin y)\,dx + \cos y\,dy$ has an integrating factor that depends only on x. Find it. Multiply the form by this factor. Integrate the resulting form.

2. Find an integrating factor for $x\,dx - e^{-x}\,dy$. Multiply by this factor and integrate.

3. Find an integrating factor for $e^y\,dx - x^2 y\,dy$.

4. Find the solution of $x^2 y\,dy/dx = e^y$ that satisfies $y = 0$ when $x = 2$.

5. Consider a form $(r(x)y - s(x))\,dx + dy$ that is at most first order in y. Assume that $R'(x) = r(x)$. Find an integrating factor expressed in terms of $R(x)$.

6. Find the general solution of $dy/dx + r(x)y = s(x)$.

7. Find the solution of $3x\,dy/dx - y = \ln x + 1$ that satisfies $y = -2$ when $x = 1$.

3.5 Homogeneous equations

A function $f(x, y)$ is homogeneous of degree n if it is an eigenfunction of the Euler operator with eigenvalue n, that is,

$$(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})f(x, y) = nf(x, y). \tag{3.38}$$

A differential form $p(x, y)\,dx + q(x, y)\,dy$ is homogeneous of degree $n + 1$ if each of $p(x, y)$ and $q(x, y)$ are homogeneous of degree n.

For a homogeneous form it is natural to look at the function given by the expression $xp(x, y) + yq(x, y)$ formed from the coefficients of the Euler operator and the form. If the homogeneous form is exact, then it is the differential of this function. On the other hand, if the homogeneous form is not exact, then $1/(xp(x, y) + yq(x, y))$ is an integrating factor.

Problems

1. Find an integrating factor for $(x^2+3y^2)\,dx-2xy\,dy$. Multiply by this factor and integrate.

2. A Bernoulli form is of the form $(p(x)y+r(x)y^n)\,dx+dy$. If we divide it by y^n we obtain $(p(x)/y^{n-1}+r(x))\,dx+(1/y^n)\,dy$. Find an integrating factor for this form.

3. Solve $dy/dx - y = e^{-x}y^2$.

4. Consider the first order system

$$\frac{dx}{dt} = a\,x \tag{3.39}$$

$$\frac{dy}{dt} = d\,y. \tag{3.40}$$

   Find when the corresponding form is exact. Find an integral of the motion in this case.

5. When the form is not exact, find an integrating factor. Find an integral of the motion. Where in phase space does it become singular?

6. Consider the first order system

$$\frac{dx}{dt} = \alpha x + \omega y \tag{3.41}$$

$$\frac{dy}{dt} = -\omega x + \alpha y. \tag{3.42}$$

   Find when the corresponding form is exact. Find an integral of the motion in this case.

7. In the case when the corresponding form is not exact, find an integrating factor. Use this to find integral of the motion in a sectorial region.

8. Show that

$$(\alpha x + \omega y)\frac{\partial}{\partial x} + (-\omega x + \alpha y)\frac{\partial}{\partial y} = \alpha r\frac{\partial}{\partial r} - \omega\frac{\partial}{\partial \theta}. \tag{3.43}$$

9. Consider the first order system

$$\frac{dr}{dt} = \alpha r \tag{3.44}$$

$$\frac{d\theta}{dt} = -\omega. \tag{3.45}$$

   Find when the form on phase space is exact. Find an integral of the motion in this case.

10. Find an integrating factor when it is not exact. Find an integral of the motion in a sectorial region.

Chapter 4

Forcing

4.1 ISETL

ISETL is the interactive version of SETL (set language). We describe part of its syntax.

4.1.1 Expressions

We begin with *expressions*. We denote an expression by EXPR.

An *identifier* is a letter or word. It may be assigned a value. We denote an identifier expression by ID.

There are various types of constants. An integer constant is denoted in the usual way: thus 789 is an integer constant. A floating point constant must have a decimal point: thus 2. and 2.0 are floating point constants. (However .5 is not allowed.)

There are also function constants. One syntax for a function constant is :ID-LIST -> EXPR:. An identifier list ID-LIST consists of identifiers separated by commas. In the example `:x,y -> sqrt(x*x + y*y):` the identifier list is `x,y` and the expression giving the value is `sqrt(x*x + y*y)`.

Expressions that denote numbers may be combined with algebraic operations to form new expressions.

If an expression EXPR has a function as its value, EXPR(EXPR-LIST) is the value of the function on the inputs given by EXPR-LIST. Here an expression list EXPR-LIST consists of expressions separated by commas. For example `:x,y -> sqrt(x*x + y*y):(3,4)` evaluates to 5.0.

4.1.2 Statements

The commands of the language are known as *statements*. We denote one or more statements by STMTS.

The workhorse statement is ID `:=` EXPR `;` and is known as the *assignment statement*. It evaluates the expression EXPR on the right and assigns the value to the identifier ID on the left. Thus `m := (a+b)/2 ;` evaluates `a` and `b`, adds them, and divides by 2. It then assigns the result to the identifier `m`. We may read this as `m` "becomes" `(a+b)/2`.

The dogma of structured programming is that control should be systematic. In fact, one can get by with only two constructions. The first is the *iteration statement* `while` EXPR `do` STMTS `end ;` This repeats the statements as long as the expression is true.

The other is the *selection statement* `if` EXPR `then` STMTS `else` STMTS `end ;` This performs the first statements if the expression is true and the second statements if the expression is false.

We also need a statement that produces output. The command for writing a line is `writeln` EXPR-LIST `;`

Notice that each statement ends with a semicolon.

4.2 Solving equations

We begin with preliminary material on solving equations numerically.

Recall the *intermediate value theorem*: If f is a continuous function on the interval $[a, b]$ and $f(a) \leq 0$ and $f(b) \geq 0$, then there is a solution of $f(x) = 0$ in this interval.

This has an easy consequence: the *fixed point theorem*. This says that if g is a continuous function from $[a, b]$ to the same interval $[a, b]$, then there is a solution of $g(x) = x$ in this interval.

Let us see how we make these theorems into practical numerical procedures. We use the language ISETL. For example, we solve $f(x) = x^2 - 2 = 0$ by the bisection algorithm. This algorithm implements the intermediate value theorem.

The idea of the bisection algorithm is to start with an interval $[a, b]$ such that $f(a) \leq 0$ and $f(b) \geq 0$. Thus there must be a root in this interval. The mid point of the interval is $m = (a + b)/2$. If $f(m) \geq 0$ then there must be a root in $[a, m]$. Otherwise there must be a root in $[m, b]$. So depending on which is the case we can replace the original $[a, b]$ by an interval that is half as long and is guaranteed to contain a root. This process can be continued to find an arbitrarily small interval containing a root.

This algorithm is implemented in the following ISETL statements.

```
f := :x -> x*x - 2.0 :  ;
a := 1.0 ; b := 2.0 ;
while b - a > 0.0001 do
   m := (a + b)/2 ;
   if f(a) * f(m) <= 0.0 then
      b := m ;
   else
      a := m ;
   end ;
   writeln a, b ;
end ;
```

Another approach to numerical root-finding is *iteration*. Assume that g is a continuous function from $[a, b]$ to the same interval $[a, b]$. We look for a fixed point r with $g(r) = r$. Assume that $|g'(x)| \leq K < 1$ for all x in the interval. Then by the mean value theorem for each x there is a c with $g(x) - r = g(x) - g(r) = g'(c)(x - r)$, and so $|g(x) - r| = |g'(c)||x - r| \leq K|x - r|$. In other words each iteration replacing x by $g(x)$ brings us closer to r.

If we want to use this to solve $f(x) = 0$, we can try to take $g(x) = x - kf(x)$ for some suitable k.

```
f := :x -> x*x - 2.0 :  ;
g := :x -> x - 0.5 * f(x) :  ;
x := 1.5 ;
while abs(g(x) - x) > 0.0001 do
   x := g(x) ;
   writeln x ;
end ;
```

Newton's method is a clever variant where you take $g(x) = x - f(x)/f'(x)$.

Here is another idea. Assume that $g(x) < x$ and $g'(x) > 0$ for $r < x \leq b$. Then by the mean value theorem, for each x there is a c with $g(x) - r = g(x) - g(r) = g'(c)(x - r)$. It follows that $r < g(x) < x$ for $r < x \leq b$. In other words, the iterations decrease to the root.

Problems

1. Prove the fixed point theorem from the intermediate value theorem.
2. Calculate $g'(x)$ in Newton's method.
3. Show that in Newton's method $f(r) = 0$ with $f'(r) \neq 0$ implies $g'(r) = 0$.
4. Implement Newton's method in ISETL and run it numerically on an example.

5. Assume that $x < g(x)$ and $g'(x) > 0$ for $a \leq x < r$. Show that it follows that $x < g(x) < r$ for $a \leq x < r$ and that the iterations increase to the root.

6. Show that in Newton's method starting near the root one has either increase to the root from the left or decrease to the root from the right. What determines which case holds?

7. Run Newton's method numerically in such a way that one gets decrease of the iterations to the root.

4.3 Linear equations

We consider the linear equation

$$\frac{dy}{dt} + r(t)y = s(t). \tag{4.1}$$

Let $R'(t) = r(t)$. Then the integrating factor is $e^{R(t)}$ and the equation becomes

$$\frac{de^{R(t)}y}{dt} = e^{R(t)}s(t). \tag{4.2}$$

The solution is then

$$y = e^{-[R(t)-R(t_0)]}y_0 + \int_{t_0}^{t} e^{-[R(t)-R(t')]}s(t')\,dt'. \tag{4.3}$$

Problems

1. Sove the linear equation

$$\frac{dy}{dt} + ay = s(t) \tag{4.4}$$

with $y = y_0$ at $t = t_0$. Express the solution as an integral.

2. Solve the preceding equation when $a > 0$ and $s(t)$ is a bounded function and where $y = 0$ at $t = -\infty$. This is the steady-state solution.

3. Show that every solution is the sum of the steady-state solution with a transient exponential.

4. In the preceding equation show that if $s(t)$ is periodic with period T, then so is the steady state solution. Hint: Make the change of variables $t - t' = u$ in the t' integral.

5. Take $s(t) = b\cos(\omega t)$ and find the solution in terms of elementary functions.

4.4 Non-linear equations

We consider the non-linear equation

$$\frac{dy}{dt} = g(t, y). \tag{4.5}$$

Assume that $g(t, y)$ has period T, that is, $g(t + T, y) = g(t, y)$. It will not necessarily be the case that all solutions have period T. However there may be a special steady-state solution that has period T.

Here is the outline of the argument. Assume that $a < b$ and that $g(t, a) \geq 0$ for all t and $g(t, b) \leq 0$ for all t. Then no solution can leave the interval $[a, b]$. Thus if $y = \phi(t, y_0)$ is the solution with $y = y_0$ at $t = 0$, then $h(y_0) = \phi(T, y_0)$ is a continuous function from $[a, b]$ to itself. It follows that h has a fixed point. But then if we take the initial condition to be this fixed point we get a periodic solution.

We can sometimes get to this fixed point by iterations. Let y' be $\partial y/\partial y_0$. Then

$$\frac{dy'}{dt} = \frac{\partial g(t, y)}{\partial y} y'. \tag{4.6}$$

Also $y' = 1$ at $t = 0$ and $y' = h'(y_0)$ at $t = T$. It follows that $h'(y_0) > 0$.

Assume that $\frac{\partial g(t,y)}{\partial y} < 0$. Then $h'(y_0) < 1$ and so we can hope that fixed point iterations of h converge. This would say that every solution in the interval converges to the periodic solution.

Problems

1. Consider the equation

$$\frac{dy}{dt} = g(y) + s(t) \tag{4.7}$$

with periodic forcing function $s(t)$. Find conditions that guarantee that this has a periodic solution.

2. Apply this to the equation

$$\frac{dy}{dt} = ay - by^2 + c \sin(\omega t). \tag{4.8}$$

3. Experiment with Phaser. Which solutions converge to the periodic solution?

Chapter 5

Numerics

5.1 Existence

We want to explore several questions. When do solutions exist? When are they uniquely specified by the initial condition? How does one approximate them numerically?

We begin with existence. Consider the equation

$$\frac{dy}{dt} = g(t, y) \tag{5.1}$$

with initial condition $y = y_0$ when $t = t_0$. Assume that g is continuous. Then the solution always exists, at least for a short time interval near t_0. In general, however, we have only local existence.

Problems

1. Consider the differential equation

$$\frac{dy}{dt} = y^2 \tag{5.2}$$

   with initial condition $y = y_0$ when $t = 0$. Find the solution. For which t does the solution blow up?

2. Sketch the vector field in phase space (with $dx/dt = 1$). Sketch a solution that blows up.

3. Can this sort of blow up happen for linear equations? Discuss.

5.2 Uniqueness

Assume in addition that g has continuous derivatives. Then the solution with the given initial condition is unique.

Uniqueness can fail when g is continuous but when $g(t, y)$ has infinite slope as a function of y.

Problems

1. Plot the function $g(y) = \text{sign}(y)\sqrt{|y|}$. Prove that it is continuous.
2. Plot its derivative and prove that it is not continuous.
3. Solve the differential equation
$$\frac{dy}{dt} = \text{sign}(y)\sqrt{|y|} \tag{5.3}$$
with the initial condition $y = 0$ when $t = 0$. Find all solutions for $t \geq 0$.
4. Substitute the solutions back into the equation and check that they are in fact solutions.
5. Sketch the vector field in phase space (with $dx/dt = 1$).

5.3 Numerics

The simplest numerical method is Euler's method, which is the first-order Runge-Kutta method. Here is a program.

```
g := :t,y -> y + exp(t) * cos(t):  ;
t := 0.0 ; y := 0.0 ;
dt := 0.01 ;
while t < 3.14 do
   dy := g(t,y) * dt ;
   y := y + dy ;
   t := t + dt ;
   writeln t, y , exp(t) * sin(t) ;
end ;
```

This is not very accurate, since the slopes are computed only at the beginning of the time step. A better method would take the average of the slopes at the beginning and at the end. But we don't know the slope at the end. The solution is to use the Euler method to estimate the slope at the end. This is a second order Runge Kutta method.

```
g := :t,y -> y + exp(t) * cos(t):  ;
t := 0.0 ; y := 0.0 ;
dt := 0.01 ;
while t < 3.14 do
   dye := g(t,y) * dt ;
   dy := (1/2) * ( g(t,y) + g(t+dt,y+dye) ) * dt ;
   y := y + dy ;
   t := t + dt ;
   writeln t, y , exp(t) * sin(t) ;
end ;
```

Problems

1. Find the solution of
$$\frac{dy}{dt} = y + e^t \cos t \tag{5.4}$$
with $y = 0$ when $t = 0$. What is the value of the solution at $y = \pi$.

2. Solve this numerically with Euler's method and compare.

3. Solve this numerically with second order Runge Kutta and compare.

4. Compare the Euler and second order Runge Kutta method with the (left endpoint) Riemann sum and trapezoid rule methods for numerical integration.

5. Another method is to use midpoints: $dy = g(t + dt/2, y + dy_e/2)dt$ where $dy_e = g(t, y)dt$. This is another second-order Runge-Kutta method. Program this and solve the equation numerically. How does this compare in accuracy with the other methods?

Chapter 6

Resonance

6.1 Forced linear systems

We begin with the homogeneous equation

$$\frac{d\mathbf{x}}{dt} + A(t)\mathbf{x} = 0. \tag{6.1}$$

Assume that we know the general solution $\phi(t)$ of that equation. These solutions we can think of as the natural response of the system.

Now consider the inhomogeneous equation

$$\frac{d\mathbf{x}}{dt} + A(t)\mathbf{x} = s_i(t) \tag{6.2}$$

with a forcing function on one side. The general solution of this equation is

$$\mathbf{x} = \phi(t) + \phi_i(t), \tag{6.3}$$

where $\phi_i(t)$ is particular solution of the inhomogeneous equation. We think of this solution as a forced response.

More generally, consider an equation

$$\frac{d\mathbf{x}}{dt} + A(t)\mathbf{x} = s_1(t) + \cdots s_k(t) \tag{6.4}$$

with a number of forcing functions. The general solution is of the form

$$\mathbf{x} = \phi(t) + \phi_1(t) + \cdots + \phi_k(t). \tag{6.5}$$

That is, every solution is a sum of forced responses together with the natural response.

6.2 Forced constant coefficient linear systems

The most important special case is when the homogeneous system has constant coefficients and the forcing function is an exponential (or a sum of exponentials). This is

$$\frac{d\mathbf{x}}{dt} + A\mathbf{x} = \mathbf{a}e^{-\lambda t}. \tag{6.6}$$

(Note that λ may be complex, so sine and cosine forcing terms are also included.) The solutions of the homogeneous equation are given in terms of the eigenvalues of A. To find a particular solution of the inhomogeneous equation, try a solution of the form

$$\mathbf{x} = \mathbf{c}e^{-\lambda t}. \tag{6.7}$$

This is a solution provided that λ is not an eigenvalue of A and

$$(A - \lambda I)\mathbf{c} = \mathbf{a}. \tag{6.8}$$

Here is the most important example. Consider the oscillator

$$\frac{dx}{dt} = v \tag{6.9}$$

$$m\frac{dv}{dt} = -kx - cy + f\cos(at). \tag{6.10}$$

Here $k > 0$ is the spring constant, $c \geq 0$ is the friction coefficient, f is the forcing coefficient, and a is the forcing angular frequency.

We may write

$$\cos(at) = \frac{1}{2}(e^{iat} + e^{-iat}) \tag{6.11}$$

and treat the two forcing functions separately.

Problems

1. For the oscillator problem with $f = 0$, what are the eigenvalues that describe the problem?

2. Consider the equation for the particular solution of the oscillator problem with $f \neq 0$. Show that if $c > 0$ (non-zero friction) or $a^2 \neq k/m$ (off resonance), then the matrix in the equation is non-singular.

3. Solve the equation for the particular solution of the oscillator problem.

4. What happens to the solution as c approaches zero and a^2 approaches k/m? (This is near resonance.)

5. Use Phaser to study this system with $f \neq 0$ and $c > 0$ and a^2 near k/m (near resonance, but with friction).

6. Use Phaser to study this system with $f \neq 0$ and $c = 0$ and a^2 near k/m (near resonance, no friction).

6.3 Resonance

Consider the equation

$$\frac{d\mathbf{x}}{dt} + A\mathbf{x} = \mathbf{a}e^{-\lambda t} \tag{6.12}$$

where now λ is one of the eigenvalues of A. This is the situation when the forcing is on *resonance*. Now one must try a solution of the form $\mathbf{x} = e^{-\lambda t}\mathbf{c} + te^{-\lambda t}\mathbf{b}$.

Problems

1. Take the oscillator problem with $c = 0$ and $a^2 = k/m$. Find the response to the driving force at the resonant frequency.

2. Use Phaser to study the system on resonance.

6.4 General forcing terms

We now want to look at forced systems with more general forcing terms.

A homogeneous linear system is

$$\frac{d\mathbf{x}}{dt} + A(t)\mathbf{x} = 0. \tag{6.13}$$

Even though this is linear, the problem of understanding solutions of such a system is profound.

In any case, assume that we have independent solutions of the homogeneous equation. Form a square matrix $\Phi(t)$ whose colums are the independent solutions. Thus this matrix satisfies

$$\frac{d\Phi(t)}{dt} + A(t)\Phi(t) = 0 \tag{6.14}$$

and the *Wronskian* determinant

$$\det \Phi(t) \neq 0. \tag{6.15}$$

We are really interested in the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} + A(t)\mathbf{x} = \mathbf{s}(t). \tag{6.16}$$

It is not hard to check that

$$\frac{d\Phi(t)^{-1}\mathbf{x}}{dt} = \Phi(t)^{-1}\mathbf{s}(t). \tag{6.17}$$

Thus

$$\mathbf{x} = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t)\Phi(t')^{-1}\mathbf{s}(t')\,dt'. \tag{6.18}$$

When A is constant this has the simpler form

$$\mathbf{x} = \Psi(t-t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Psi(t-t')\mathbf{s}(t')\,dt'. \tag{6.19}$$

This can also be written

$$\mathbf{x} = \Psi(t-t_0)\mathbf{x}_0 + \int_{0}^{t-t_0} \Psi(u)\mathbf{s}(t-u)\,du. \tag{6.20}$$

Take the special case $\mathbf{s}(t) = \mathbf{a}e^{-\lambda t}$. Then the solution is given by

$$\mathbf{x} = \Psi(t-t_0)\mathbf{x}_0 + e^{-\lambda t}\int_{0}^{t-t_0} \Psi(u)\mathbf{a}e^{\lambda u}\,du \tag{6.21}$$

In the simplest case λ is not an eigenvalue of A and so we may solve $(A-\lambda)\mathbf{c} = \mathbf{a}$ for $\mathbf{c}$. Then the solution is

$$\mathbf{x} = \Psi(t-t_0)\mathbf{x}_0 + e^{-\lambda t}\mathbf{c} - \Psi(t-t_0)e^{-\lambda t_0}\mathbf{c}. \tag{6.22}$$

This solution is the sum of a solution of the homogeneous equation and of a particular solution of the form $e^{-\lambda t}\mathbf{c}$. This particular solution just responds to the forcing. Notice that if the forcing λ is far from or close to the eigenvalues of A (representing the natural motions of the free system), then the response $\mathbf{c}$ will be correspondingly smaller or larger then the force $\mathbf{a}$.

The other case is *resonance*, when λ is an eigenvalue of A. In this case we have non-zero solutions of $(A-\lambda)\mathbf{b} = 0$. We can then try to choose such a solution so that there is a solution of $(A-\lambda)\mathbf{c} + \mathbf{b} = \mathbf{a}$. We then get

$$\mathbf{x} = \Psi(t-t_0)\mathbf{x}_0 + e^{-\lambda t}\mathbf{c} - \Psi(t-t_0)e^{-\lambda t_0}\mathbf{c} + e^{-\lambda t}(t-t_0)\mathbf{b}. \tag{6.23}$$

The extra particular solution $e^{-\lambda t}(t-t_0)\mathbf{b}$ has a linear growth factor in the direction of the eigenvector. This says that when the forcing coincides with the natural motion, then there is an extra response factor that represents growth out of control.

Chapter 7

Music of the Spheres

7.1 The circle

The Laplace operator on the circle is

$$\Delta_C = \frac{d^2}{d\theta^2}. \tag{7.1}$$

The eigenfunctions are of the form $e^{in\theta}$ with eigenvalues $-n^2$, where n is an integer. Thus

$$\frac{d^2}{d\theta^2} e^{in\theta} = -n^2 e^{in\theta}. \tag{7.2}$$

7.2 The plane

The Laplace operator in the plane is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{7.3}$$

In other words,

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_C, \tag{7.4}$$

where Δ_C is the Laplace operator on the circle defined above.

We look for eigenfunctions w of the Laplace operator with eigenvalue -1, so that

$$\Delta w = -w. \tag{7.5}$$

If we take $w = f(r)e^{in\theta}$, we obtain

$$\Delta f(r) = (\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{n^2}{r^2})f(r) = -f(r). \tag{7.6}$$

Define the *Bessel operator* by

$$B_n = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{n^2}{r^2}. \tag{7.7}$$

Then our eigenvalue equation is

$$B_n u = -u. \tag{7.8}$$

This is the *Bessel equation.*

We can find interesting eigenfunctions as follows. For each r expand $e^{ir\sin\theta}$ in powers of $e^{i\theta}$ and call the coefficients $J_n(r)$. Thus

$$e^{iy} = e^{ir\sin\theta} = \sum_n J_n(r)e^{in\theta}, \tag{7.9}$$

where the sum is over all integers n. If we apply the Laplace operator to both sides of this equation and equate coefficients, we obtain

$$B_n J_n(r) = -J_n(r). \tag{7.10}$$

This *Bessel function* $J_n(r)$ is one solution of the Bessel equation. The multiples of this function form a one-dimensional space of solutions.

How about other solutions? A second order equation should have a two-dimensional space of solutions. Can the other solutions be ruled out?

The answer to this question is related to the fact that the Bessel equation is singular at $r = 0$. (This of course is just because polar coordinates are always somewhat strange at the origin.) Thus most solutions of the Bessel equation are singular and can thus be discarded, for some purposes at least.

The way to see this is to examine the equation $B_n u = 0$. This should have more or less the same singularities as the equation $B_n u = -1$, but it is much simpler to solve explicitly.

Problems

1. Find $J_n(0)$ for each n.
2. Take $n \neq 0$. Solve $B_n u = 0$. Describe the singularity structure of the solutions. What is the dimension of the subspace of non-singular solutions?

3. Solve $B_0 u = 0$. Describe the singularity structure of the solutions. What is the dimension of the subspace of non-singular solutions?

4. For $\lambda^2 > 0$ find a one-dimensional space of solutions of the equation $B_n u = -\lambda^2 u$.

5. The vibrations of a drum are described by $\Delta u = -\lambda^2 u$ with the boundary condition that $u = 0$ when $r = R$. Find what frequencies λ are allowed. Hint: Write the solution $u = J_n(\lambda r) e^{in\theta}$. What is the condition on λ that guarantees that u satisfies the boundary condition?

6. We see that it is important to know the zeros of the Bessel functions $J_n(r)$. Use Phaser to plot $J_n(r)$ for small values of n and describe the zeros.

7.3 The sphere

The Laplace operator in three dimensions is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S, \tag{7.11}$$

where

$$\Delta_S = \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\sin\phi\frac{\partial}{\partial\phi} + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2} \tag{7.12}$$

is the Laplace operator on the sphere.

We look for solutions of the eigenvalue equation

$$\Delta_S w = -\ell(\ell+1)w. \tag{7.13}$$

If we take $w = g(\phi)e^{im\theta}$, we obtain

$$\Delta_{Sm} g(\phi) = -\ell(\ell+1)g(\phi), \tag{7.14}$$

where

$$\Delta_{Sm} = \frac{1}{\sin\phi}\frac{d}{d\phi}\sin\phi\frac{d}{d\phi} - \frac{m^2}{\sin^2\phi}. \tag{7.15}$$

For simplicity we confine attention to the case $m = 0$. Then

$$\Delta_{S0} = \frac{1}{\sin\phi}\frac{d}{d\phi}\sin\phi\frac{d}{d\phi} \tag{7.16}$$

and the *Legendre equation* is

$$\Delta_{S0} u = -\ell(\ell+1)u. \tag{7.17}$$

We can find interesting eigenfunctions as follows. For each ϕ expand the reciprocal of the distance from the north pole in powers of the distance r from the origin:

$$(x^2+y^2+(z-1)^2)^{-\frac{1}{2}} = (1-2r\cos\phi+r^2)^{-\frac{1}{2}} = \sum_{\ell=0}^{\infty} P_\ell(\cos\phi)r^\ell. \tag{7.18}$$

The coefficients $P_\ell(\cos\phi)$ are *Legendre polynomials* in $\cos\phi$. If we apply the Laplace operator Δ to both sides of this equation and equate coefficients, we obtain

$$\Delta_{S0} P_\ell(\cos\phi) = -\ell(\ell+1)P_\ell(\cos\phi). \tag{7.19}$$

Multiples of these polynomials form a one-dimensional space of solutions.

The Legendre equation is singular at $\sin\phi = 0$, that is at $\phi = 0$ and at $\phi = \pi$ (north pole and south pole). Other solutions are singular and can thus be discarded, for some purposes at least.

One can study this equation by making the change of varibles $t = \cos\phi$. Then

$$\Delta_{S0} = \frac{d}{dt}(1-t^2)\frac{d}{dt}. \tag{7.20}$$

Now the relevant interval is $-1 < t < 1$ and the singularities are at the end points $t = \pm 1$.

Problems

1. Use Phaser to plot $P_\ell(t)$ for small values of ℓ.