1 Real functions and integrals

1.1 Elementary integrals

1.1.1 Stone vector lattices

A *Stone vector lattice* L is a vector lattice of real functions on the set X that satisfies the Stone condition: $f \in L$ implies $f \wedge 1 \in L$.

An *elementary integral* m is a linear function $m : L \to \mathbf{R}$ that is linear and satisfies monotone convergence for sequences of functions within L.

Example: Let $X = \mathbf{R}$ and $L = C_c(\mathbf{R})$. Then L is a Stone vector lattice. By Dini's theorem (a result that depends on compactness) every linear function m is an elementary integral.

1.1.2 Vector lattices with constants

A *vector lattice with constants* L is a vector lattice of real functions on the set X such that $1 \in L$.

Example: Let $X = [0, 1]$ and $L = C([0, 1])$. Then L is a vector lattice with constants. By Dini's theorem every linear function m is an elementary integral.

1.2 Integrals

1.2.1 σ-rings of real functions

A *monotone class* is a collection of functions that is closed under upward and downward monotone convergence of sequences.

A *σ-ring of real functions* $\mathcal{F}_0$ is a Stone vector lattice that is also a monotone class.

1.2.2 σ-algebras of real functions

A *σ-algebra of real functions* $\mathcal{F}$ is a vector lattice with constants that is also a monotone class.

An *integral* is a function $\mu : \mathcal{F}^+ \to [0, +\infty]$ that satisfies linearity (for positive scalars) and upward monotone convergence for sequences of functions.

Theorem (Daniell-Stone). Let L be a Stone vector lattice of real functions on X. Let $m : L \to \mathbf{R}$ be an elementary integral. Then there is an integral μ that agrees with m on L^+.

Theorem. The elementary integral m uniquely determines the integral μ on the σ-ring of real functions generated by L.

Example: When $X = \mathbf{R}$ and $L = C_c(\mathbf{R})$, then the σ-ring of functions generated by L is the Borel σ-algebra of functions.

2 Subsets and measures

2.1 Elementary measures

2.1.1 Rings of subsets

A *ring of subsets* A of a set X is a collection of subsets closed under finite unions and relative complements.

An *elementary measure* m is an additive function $m : A \to [0, +\infty)$ that satisfies monotone convergence for sequences of subsets within A.

Example: Let $X = \mathbf{R}$ and A consist of all finite unions of intervals $(a, b]$. Then A is a ring of subsets. Each increasing right-continuous function $F : \mathbf{R} \to \mathbf{R}$ determines an elementary measure by $m((a, b]) = F(b) - F(a)$. The proof of monotone convergence involves compactness and an additional argument using the right continuity.

2.1.2 Algebras of subsets

An *algebra of subsets* A is a ring of subsets of X such that $X \in A$.

Example: Let $X = (0, 1]$ and A be as before. Then A is an algebra of subsets. Each increasing right-continuous function $F : (0, 1] \to \mathbf{R}$ determines an elementary measure by $m((a, b]) = F(b) - F(a)$.

2.2 Measures

2.2.1 σ-rings of subsets

A *monotone class* is a collection of subsets that is closed under upward and downward monotone convergence of sequences.

A *σ-ring of subsets* $\mathcal{F}_0$ is a ring of subsets that is also a monotone class.

2.2.2 σ-algebras of subsets

A *σ-algebra of subsets* $\mathcal{F}$ is an algebra of subsets that is also a monotone class.

A *measure* is an additive function $\mu : \mathcal{F} \to [0, +\infty]$ that satisfies upward monotone convergence for sequences of subsets.

Theorem. Let A be a ring of subsets of X. Let $m : A \to [0, +\infty)$ be an elementary measure. Then there is a measure μ that agrees with m on A.

Theorem. The elementary measure m uniquely determines the measure μ on the σ-ring of subsets generated by A.

Example: When $X = \mathbf{R}$ and A consists of finite unions of intervals $(a, b]$, then the σ-ring of subsets generated by L is the Borel σ-algebra of subsets.

3 The relation between integrals and measures

3.1 Elementary integrals and elementary measures

Suppose A is a ring of subsets of X. Then there is a corresponding Stone vector lattice L consisting of all finite real linear combinations of indicator functions

$$f = \sum_{k=1}^{n} c_k 1_{E_k}$$

of subsets in A.

Suppose $m : A \to [0, +\infty)$ is an elementary measure. Then there is a corresponding elementary integral $m : L \to \mathbf{R}$ defined by

$$m(f) = \sum_{k=1}^{n} c_k m(E_k).$$

Thus the Stone-Daniell theorem on the construction of integrals implies the corresponding theorem on construction of measures. The converse is not true; the Stone vector lattice associated to a ring of subsets is only one special kind of Stone vector lattice. So the Daniell-Stone theorem is stronger.

3.2 Integrals and measures

There is a natural equivalence between σ-rings of subsets and σ-rings of real functions. The real functions in the σ-ring are those functions with the property that the inverse image of every Borel subset B of $\mathbf{R}$ with $0 \notin B$ is in the σ-ring of subsets.

There is a natural equivalence between σ-algebras of subsets and σ-algebras of real functions. The real functions in the σ-algebra are those functions with the property that the inverse image of every Borel subset B of $\mathbf{R}$ is in the σ-algebra of subsets.

There is a natural equivalence between measures on a σ-algebra of subsets and integrals on the corresponding σ-algebra of real functions.

3.3 Terminology

The terms *vector lattice* and *Stone vector lattice* are standard. There are various terms for *elementary integral*, including *pre-integral* and *Daniell integral*. Terms such as *σ-algebra of real functions* are not so common. The usual convention is to refer to functions measurable with respect to a give σ-algebra of subsets. The intrinsic characterization of *integral* in terms of its properties is also relatively rare; it is more common to think of an integral as being derived from a measure.

The terms *ring* and *algebra* are standard in the context of subsets. An *elementary measure* might also be called a *pre-measure*. However many authors would allow a pre-measure to have values in $[0, +\infty]$. This gives a somewhat more general construction of measure. Terms such as *σ-algebra of subsets* and *measure* are standard.