1 Assignments

1. Poisson process (due January 27, 2009)
   Model notes (pp.6–8): 1(i), 4, 5, 8
2. Simple random walk (due February 3, 2009)
   Model notes (p. 10): 2, 3, 4
3. Transience and recurrence 1 (due February 10, 2009)
   Model notes (pp. 21–22): 3,4,5
4. Transience and recurrence 2 (due February 17, 2009)
   Model notes (pp. 21–22): 6,7,8
5. Strong Markov property (due February 24, 2009)
   Model notes (p. 61): 1,5,6
6. Markov chains (due March 3, 2009)
   DPSP §2: 9, 13, 16, 17
7. Markov processes with general state space (due March 10, 2009)
   DPSP §3: 11, 13, 15(iv)(v)
8. Stopping times and the strong Markov property (due March 24, 2009)
   DPSP §4: 1, 3, 5
9. Transience and recurrence of Markov chains (due March 31, 2009)
   DPSP §5: 3, 6, 10, 12
10. Invariant probability for Markov chains (due April 7, 2009)
    DPSP §7: 2, 4, 6
11. Birth-death chains; Linear Markov processes (due April 14, 2009)
    DPSP §9: 12; DPSP §12: 1(i); Product space problem

Product space problem

Consider a product space $S = W^V$ with a probability π. Here V is a large finite set. Because a product space is so huge, it is impractical to sample directly from π by enumeration. So one wants to construct a Markov process with π as invariant measure and use it for sampling.

If σ is in S and i is in V, then we write σ_i for the ith coordinate, so that σ_i takes values in W. Let $\check{\mathcal{F}}_i$ be the σ-algebra generated by the σ_j with $j \neq i$. Define the transition operator T_i by

$$(T_i f)(\sigma) = E_\pi[f(\sigma) \mid \check{\mathcal{F}}_i].$$

1. Show that T_i satisfies detailed balance.

2. Let v be the number of points in V. Let

$$T = \frac{1}{v} \sum_{i \in V} T_i.$$

   Show that T satisfies detailed balance. (In many circumstances T will generate an aperiodic irreducible Markov chain.)

3. Suppose W is finite. Let $\sigma \mapsto \pi(\sigma)$ represent the density of the measure. Let $\sigma[y]^i$ have kth component equal to σ_k for $k \neq i$ and equal to y for $k = i$. Show that

$$(T_i f)(\sigma) = \frac{\sum_y f(\sigma[y]^i)\pi(\sigma[y]^i)}{\sum_y \pi(\sigma[y]^i)}.$$

4. Consider the Ising model. In this case $W = \{1, -1\}$ and V is the vertex set of a graph with edge set E. A point σ is regarded as a spin configuration on the vertex set. The energy of such a configuration is

$$H(\sigma) = -\sum_{\{i,j\} \in E} \sigma_i \sigma_j.$$

   Let $\beta \geq 0$ be the inverse temperature parameter (in energy units). Then the equilibrium probability corresponding to a fixed value of β is

$$\pi(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)},$$

   where Z is a normalization constant (depending on β). Show that the equilibrium Ising model has the following one-point spatial Markov property: The conditional expectation of $g(\sigma_i)$ given $\check{\mathcal{F}}_i$ depends only on σ_j for $j \in N(i)$, the set of neighbors of i in the graph. Find a simple explicit expression for this conditional expectation.

5. Consider the Markov chain with transition operator $T = \frac{1}{v}\sum_i T_i$ associated with the Ising model. Find a simple explicit expression for $(T_i f)(\sigma)$ similar to that found in the previous part.