Problem 1.8.5

Subtract $b$ times the first equation from the second equation and $3$ times the first equation from the third equation to get
\[
\begin{align*}
\begin{cases}
x_1 + x_2 + bx_3 &= 1 \\
(3 - b)x_2 - (1 + b^2)x_3 &= -2 - b \\
x_2 + (1 - 3b)x_3 &= c - 3
\end{cases}
\end{align*}
\]

Interchange the second and the third equations:
\[
\begin{align*}
\begin{cases}
x_1 + x_2 + bx_3 &= 1 \\
x_2 + (1 - 3b)x_3 &= c - 3 \\
(3 - b)x_2 - (1 + b^2)x_3 &= -2 - b
\end{cases}
\end{align*}
\]

Subtract $(3 - b)$ times the second equation from the third equation:
\[
\begin{align*}
\begin{cases}
x_1 + x_2 + bx_3 &= 1 \\
x_2 + (1 - 3b)x_3 &= c - 3 \\
- (4b^2 - 10b + 4)x_3 &= -2 - b - (3 - b)(c - 3)
\end{cases}
\end{align*}
\]

If $4b^2 - 10b + 4 \neq 0$ then the system has a unique solution. The equation $4b^2 - 10b + 4 = 0$ has solutions $b = 2$ and $b = 1/2$ (use the quadratic formula.) If $b = 2$ or $b = 1/2$, and $-2 - b - (3 - b)(c - 3) = 0$ then the system has infinitely many solutions. Otherwise, it has no solutions. The equation $-2 - b - (3 - b)(c - 3) = 0$ has the solution
\[
c = \frac{-2 - b}{3 - b} + 3,
\]
so $c = -1$ when $b = 2$ and $c = 2$ when $b = 1/2$.

**Answer:** The system has a unique solution if $b \neq 2$ and $b \neq 1/2$; the system has infinitely many solutions if $b = 2$ and $c = -1$ or if $b = 1/2$ and $c = 2$; the system has no solutions if $b = 2$ and $c \neq -1$ or if $b = 1/2$ and $c \neq 2$.

Problem 1.8.7(c)

Subtract the first row from the second row and add the first row to the third row:
\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Now, subtract the second row from the third row to get
\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

The last matrix is in row echelon form; it has 2 non-zero pivots, so the rank of the matrix equals 2.
Problem 1.9.1(g)

Denote by $A$ the matrix from the problem. First, we perform operations Row2-Row1, Row3-2Row1, Row4-5Row1, and Row5-2Row1:

$$
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 9 & -5 & -15 & 30 \\
0 & 6 & -2 & -4 & 9 \\
\end{vmatrix}
$$

$$\det A = \det
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 9 & -5 & -15 & 30 \\
0 & 6 & -2 & -4 & 9 \\
\end{vmatrix}.$$

Next, we perform operations Row3-Row2, Row4-3Row2, and Row5-2Row2:

$$
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 4 & -2 & 5 \\
\end{vmatrix}
$$

$$\det A = \det
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 4 & -2 & 5 \\
\end{vmatrix}.$$

Now, we interchange the third and the fourth rows:

$$
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 4 & -2 & 5 \\
\end{vmatrix}
$$

$$\det A = -\det
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 4 & -2 & 5 \\
\end{vmatrix}.$$

Subtract the third row from the fifth row:

$$
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 0 & -10 & -19 \\
\end{vmatrix}
$$

$$\det A = -\det
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 0 & -10 & -19 \\
\end{vmatrix} = -3 \times 4 \times (-5) = 60.$$

Finally, add twice of the fourth row to the fifth row:

$$\det A = -\det
\begin{vmatrix}
1 & -2 & 1 & 4 & -5 \\
0 & 3 & -3 & -1 & 2 \\
0 & 0 & 0 & -5 & 10 \\
0 & 0 & 0 & 0 & 1 \\
\end{vmatrix} = -3 \times 4 \times (-5) = 60.$$

Problem 7.1.5(g)

$(1, -1, 2)^T$ is a normal vector to the plane $x - y + 2z = 0$, so the orthogonal projection of a vector $(x, y, z)^T$ on this plane is of the form

$$(1) \quad \begin{cases} x \\ y \\ z \end{cases} + c \begin{cases} 1 \\ -1 \\ 2 \end{cases} = \begin{cases} x + c \\ y - c \\ z + 2c \end{cases}.$$
One has
\[(x + c) - (y - c) + 2(z + 2c) = 0;\]
\[x - y + 2z + 6c = 0,\]
and
\[c = -\frac{1}{6}x + \frac{1}{6}y - \frac{1}{3}z.\]
Substituting this value of \(c\) into (1), we get that the projection of a vector \((x, y, z)^T\) on the plane \(x - y + 2z = 0\) is the vector
\[
\begin{pmatrix}
\frac{5}{6}x + \frac{1}{6}y - \frac{1}{3}z \\
\frac{5}{6}x + \frac{1}{6}y + \frac{1}{3}z \\
-\frac{1}{3}x + \frac{4}{3}y + \frac{1}{3}z
\end{pmatrix}.
\]
This linear function is given by the matrix
\[
\begin{pmatrix}
\frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\
\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\
-\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]

**Problem 7.1.7**

The linear function \(L\) is given by a \(2 \times 2\) matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]
The equation
\[L\begin{pmatrix}1 \\ 2\end{pmatrix} = \begin{pmatrix}2 \\ -1\end{pmatrix}\]
means
\[
\begin{cases}
a + 2b = 2 \\
c + 2d = -1
\end{cases}
\]
The equation
\[L\begin{pmatrix}2 \\ 1\end{pmatrix} = \begin{pmatrix}0 \\ -1\end{pmatrix}\]
means
\[
\begin{cases}
2a + b = 0 \\
2c + d = -1
\end{cases}
\]
We solve these equations for \(a, b, c,\) and \(d:\) \(a = -2/3, b = 4/3, c = d = -1/3.\)

**Answer:**
\[L\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}-2/3 & 4/3 \\ -1/3 & -1/3\end{pmatrix} \begin{pmatrix}x \\ y\end{pmatrix}.\]