COMPLEX ANALYSIS–FINAL EXAM

Problem 1

Let $0 < \rho < 1/2$, and

$$\Omega_\rho = \{z \in \mathbb{C} : |z| < 1,\ |z - 1/2| > \rho\}.$$

Find a conformal map from Ω_ρ onto an annulus $\{r < |z| < 1\}$. Find two terms in the asymptotic expansion of $r(\rho)$ as $\rho \to 1/2$.

Problem 2

Let

$$P(z) = z^8 + c_1 z^7 + \cdots + c_7 z + c_8$$

be a unital polynomial of degree 8. Prove that if t is a sufficiently large real number then the equation $P(z) + t = 0$ has exactly two solutions in each open quadrant.

Problem 3

Let

$$Z(z) = \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} (m^2 + n^2)^{-z/2}. \tag{1}$$

a) Prove that the series (1) converges if z belongs to the half-plane $\mathrm{Re} z > 2$ and that $Z(z)$ is holomorphic in that half-plane.
b) Prove that $Z(z)$ admits an analytic continuation to a meromorphic function in the whole complex plane. Find all poles of $Z(z)$ and compute the residues at the poles. Compute $Z(0)$.
c) Prove that

$$Z(z) = \frac{\pi^z}{\sin(\pi z/2)[\Gamma(z/2)]^2} Z(2 - z).$$

Hint. You may find useful to relate $Z(z)$ to the function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

Problem 4

Let

$$F(z) = \int_0^\infty \frac{\log x}{1 + x^z} dx. \tag{2}$$

a) Show that the integral (2) converges when z belongs to the half-plane $\mathrm{Re} z > 1$ and that $F(z)$ is holomorphic in that half-plane.
b) Evaluate the integral (2) in the case when z is a real number greater than 1.
c) Find the largest domain in $\mathbb{C}$ to which the function $F(z)$ can be continued analytically. Find all singularities of $F(z)$ and classify them (poles, essential singular points).