REAL ANALYSIS
FINAL EXAM

Problem 1

Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measure spaces. Let $\phi : X \to Y$ be a measurable mapping. For a measure μ on $(X, \mathfrak{M})$, the measure $\phi_*\mu$ on $(Y, \mathfrak{N})$ is defined by the formula

$$\phi_*\mu(A) = \mu(\phi^{-1}(A)), \quad A \in \mathfrak{N}.$$

Let μ and ν be two measures on $(X, \mathfrak{M})$.
a) Prove that $\mu \ll \nu$ implies $\phi_*\mu \ll \phi_*\nu$.
b) Does $\phi_*\mu \ll \phi_*\nu$ imply $\mu \ll \nu$? Prove that it does or give a counter-example showing that it does not.

Problem 2

Let H be an infinite dimensional Hilbert space. Prove that it is impossible to construct a measure on H such that all balls $B(x, r) = \{y : ||y - x|| < r\}$, $r > 0$, are measurable, the measure of $B(x, r)$ is finite, positive, and it depends on the radius r only.

Problem 3

Let $x = 0.a_1a_2\cdots$ be the decimal expansion of a number x, $0 < x < 1$. If two decimal expansions of x exist, the one that ends with 0's is taken. For what values of $q > 1$ the function

$$f_q(x) = \sum_{k=1}^{\infty} q^{-k} a_k$$

is of bounded variation?

Problem 4

Prove that an operator A defined by the formula

$$(Af)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1 + (x - y)^2} dy$$

is a bounded operator from $L^2(\mathbb{R})$ into itself, compute its norm, and prove that A is not invertible (that is, A^{-1} is not a bounded operator from $L^2(\mathbb{R})$ into itself.)

Problem 5

Let $\Omega \subset \mathbb{R}^n$ be a set of positive, finite Lebesgue measure. Let $\{\psi_j(x)\}$ be an orthonormal basis in $L^2(\Omega)$. Each function $\psi_j(x)$ is extended to $\mathbb{R}^n$: one sets $\psi_j(x) = 0$ when $x \notin \Omega$. Prove that

$$\sum_{j=1}^{\infty} |\hat{\psi}_j(\xi)|^2 = m(\Omega)$$

for every value of ξ. Here $\hat{\psi}_j$ is the Fourier transform of ψ_j and $m(\Omega)$ is the Lebesgue measure of Ω.

Problem 6

Let c be the set of all sequences $\{x_j\}_{j=1}^{\infty}$, $x_j \in \mathbb{C}$, for which the limit $\lim_{j\to\infty} x_j$ exists.

a) Prove that c is a closed subspace of l^∞.

b) It follows from a) that c, together with the l^∞ norm, is a Banach space. Find an explicit way of characterizing all continuous linear functionals on c.

c) Let $l \in c^*$. Is it always possible to find $x \in c$ such that $||x|| = 1$ and $||l|| = |l(x)|$?