**PROBLEM SET 11**

**Problem 1**

Let \( f(x) \in L^1(\mathbb{R}^n) \) and \( g(x) \in L^1(\mathbb{R}^n) \). Prove that \( H(f + g)(x) \leq Hf(x) + Hg(x) \) for every point \( x \in \mathbb{R}^n \). Here \( Hf \) is the Hardy–Littlewood maximal function of \( f \).

**Problem 2**

Let \( f(x) \in L^1(\mathbb{R}^n) \), let \( a > 0 \), and let \( f_a(x) = f(ax) \). Prove that \( Hf_a(0) = Hf(0) \).

**Problem 3**

Let \( \psi(r) \) be a non-negative, non-increasing, right continuous function on \([0, \infty)\) such that \( \lim_{r \to \infty} \psi(r) = 0 \), and let \( f(x) \in L^1_{\text{loc}}(\mathbb{R}^n) \). Assume that either \( f(x) \geq 0 \) or \( f(x)\psi(|x|) \in L^1(\mathbb{R}^n) \). Prove that

\[
\int_{\mathbb{R}^n} f(x)\psi(|x|)dx = v_n \int_{(0,\infty)} r^{n-1}A_r f(0)d(-\psi(r)).
\]

The integral on the right in (1) is the Lebesgue–Stieltjes integral, \( v_n \) is the volume of the unit ball in \( \mathbb{R}^n \), and \( A_r f(0) \) is the average of \( f(x) \) over the ball of radius \( r \) centered at 0.

**Hint**. You may find it useful that both sides of (1) remain unchanged if one replaces the function \( f(x) \) by a radially-symmetric function \( f_av(|x|) \), which is the average of \( f(x) \) over the sphere of radius \(|x|\).

**Solution.** 1. First, let us prove (1) in the case when the function \( f \) is radially symmetric, \( f = f(|x|) \). I will use spherical co-ordinates; by \( d\omega \) I denote the measure on the unit sphere in \( \mathbb{R}^n \) that is induced by the Lebesgue measure. Then

\[
\int_{\mathbb{R}^n} f(x)\psi(|x|)dx = \omega_{n-1} \int_{(0,\infty)} r^{n-1}\psi(r)f(r)dr
\]

where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \). Then,

\[
A_r f(0) = \frac{1}{v_n r^n} \int_{B(r,0)} f(|x|)dx = \frac{\omega_{n-1}}{v_n r^n} \int_{[0,r]} \rho^{n-1} f(\rho)d\rho,
\]

so the right hand side of (1) equals

\[
\omega_{n-1} \int_{(0,\infty)} \left( \int_{[0,r]} \rho^{n-1} f(\rho)d\rho \right)d(-\psi(r)).
\]

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To see that the integral (1B) equals the right hand side of (1A), we take the set
\[ D = \{(\rho, r) \in [0, \infty)^2 : \rho \leq r \} \]
and consider the integral
\[ \omega_{n-1} \int_D \rho^{n-1} f(\rho) d\rho \times d(-\psi(r)). \]
If \( f \geq 0 \) then, by Tonelli’s theorem, it equals (1B), and it also equals
\[ \omega_{n-1} \int_{(0, \infty)} \rho^{n-1} f(\rho) \left( \int_{|\rho, \infty|} d(-\psi(r)) \right) d\rho = \omega_{n-1} \int_{(0, \infty)} \rho^{n-1} \psi(\rho) f(\rho) d\rho. \]
Here, I used the fact that \( \lim_{r \to \infty} \psi(r) = 0 \). The last integral coincides with the right hand side of (1A). In the case when \( f(x) \psi(|x|) \in L^1(\mathbb{R}^n) \), we apply Tonelli’s theorem to \( |f(x)| \) to see that the function \( \rho^{n-1} f(\rho) \) belongs to \( L^1(D, d\rho \times d(-\psi(r))) \); then we apply Fubini’s theorem.

2. Now, let us treat the general case. In spherical co-ordinates, the function \( f \) can be written as \( f(r, \omega) \); here \( \omega \) is a point on the unit sphere, \( S^{n-1} \), in \( \mathbb{R}^n \). By Fubini–Tonelli’s theorem, the function \( f_r(\omega) = f(r, \omega) \) is measurable as a function of \( \omega \) for almost all values of \( r \), and, if \( f(x) \psi(|x|) \in L^1(\mathbb{R}^n) \), then \( f_r \in L^1(S^{n-1}, d\omega) \) for almost all values of \( r \). We define a function
\[ f_{av}(r) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(r, \omega) d\omega. \]
This is the average value of the function \( f \) over a sphere of radius \( r \). Then \( f_{av}(|x|) \) is a radially symmetric function. Clearly \( f_{av} \geq 0 \) if \( f \geq 0 \). In the case \( f(x) \psi(|x|) \in L^1(\mathbb{R}^n) \), we notice that
\[ |f_{av}(r)| \leq \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |f(r, \omega)| d\omega, \]
and
\[ \int |f_{av}(|x|) \psi(|x|)| dx \leq \int_0^{\infty} \int_{S^{n-1}} |f(r, \omega)| \psi(r) d\omega dr = \int |f(|x|) \psi(|x|)| dx. \]
Therefore, the function \( f_{av}(|x|) \psi(|x|) \) belongs to \( L^1(\mathbb{R}^n) \).

I claim that both sides of (1) do not change if one replaces \( f(x) \) by \( f_{av}(|x|) \). Let us start from the expression on the right in (1). One applies the Fubini–Tonelli theorem to get
\[ A_r f(0) = \frac{1}{v_n r^n} \int_0^r \rho^{n-1} d\rho \int_{S^{n-1}} f(\rho, \omega) d\omega = \frac{\omega_{n-1}}{v_n r^n} \int_0^r f_{av}(\rho) \rho^{n-1} d\rho = A_r f_{av}(0). \]
The same argument leads to
\[ \int_{\mathbb{R}^n} f(x) \psi(|x|) dx = \int_0^{\infty} \psi(r) r^{n-1} dr \int_{S^{n-1}} f(r, \omega) d\omega = \omega_{n-1} \int_0^\infty f_{av}(r) \psi(r) r^{n-1} dr = \int_{\mathbb{R}^n} f_{av}(|x|) \psi(|x|) dx. \]
We have seen that neither side of (1) changes if a function \( f(x) \) is replaced by \( f_{av}(|x|) \), so the general case follows from the case of \( f(x) \) being a radially symmetric function.
Problem 4

Let \( \psi(r) \) be a function from problem 3. In addition, we assume that

\[
\int_{\mathbb{R}^n} \psi(|x|)dx = \omega_{n-1} \int_0^\infty r^{n-1} \psi(r) dr = 1;
\]

here \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \). Let \( f(x) \in L^1(\mathbb{R}^n) \). Prove that

\[
\lim_{\delta \to 0} \delta^{-n} \int_{\mathbb{R}^n} f(x - y) \psi(|y|/\delta) dy = \lim_{\delta \to 0} \int_{\mathbb{R}^n} f(x - \delta z) \psi(|z|) dz = f(x)
\]

for almost all \( x \).

**Hint.** Theorem 3.18 is a special case of problem 4; the corresponding function \( \psi \) equals the constant \( 1/\nu_n \) for \( 0 \leq r < 1 \), and it vanishes for \( r \geq 1 \).

**Solution.** 1. Let us prove the statement in the case when the function \( f(x) \) is continuous and bounded. Fix a point \( x \), fix a number \( \epsilon > 0 \), and let \(|f(y) - f(x)| < \epsilon/2 \) when \(|y - x| < \eta \). Here \( \eta \) is a positive number, the existence of which is guaranteed by continuity of \( f(x) \). We break the integral

\[
I = \delta^{-n} \int_{\mathbb{R}^n} f(x - y) \psi(|y|/\delta) dy - f(x) = \delta^{-n} \int_{\mathbb{R}^n} [f(x - y) - f(x)] \psi(|y|/\delta) dy
\]

into the sum

\[
I_1 + I_2 = \delta^{-n} \int_{|y| < \eta} [f(x - y) - f(x)] \psi(|y|/\delta) dy + \delta^{-n} \int_{|y| \geq \eta} [f(x - y) - f(x)] \psi(|y|/\delta) dy.
\]

One has

\[
|I_1| < \frac{\epsilon}{2} \delta^{-n} \int_{|y| < \eta} \psi(|y|/\delta) dy \leq \frac{\epsilon}{2} \delta^{-n} \int_{\mathbb{R}^n} \psi(|y|/\delta) dy = \frac{\epsilon}{2}.
\]

To estimate the second integral, \( I_2 \), let us assume that \(|f(z)| \leq M \) (the function \( f(x) \) is bounded!) Then

\[
|I_2| \leq 2M \delta^{-n} \int_{|y| \geq \eta} \psi(|y|/\delta) dy = 2M \int_{|y| \geq \eta / \delta} \psi(|y|) dy.
\]

The function \( \psi(x) \) belongs to \( L^1(\mathbb{R}^n) \), so the last integral converges to 0 when \( \delta \to 0 \) (the number \( \eta \) is fixed;) therefore it can be made smaller than \( \epsilon/4M \) is \( \delta \) if small enough. Then \(|I| < \epsilon \).

2. For a positive number \( \eta \), we find a bounded, continuous function \( h(x) \) such that the \( L^1 \) norm of the difference \( g(x) = f(x) - h(x) \) is smaller than \( \eta \). We introduce the notations

\[
f_\delta(x) = \delta^{-n} \int_{\mathbb{R}^n} f(x - y) \psi(|y|/\delta) dy = \int_{\mathbb{R}^n} f(x - \delta z) \psi(|z|) dz;
\]

\( g_\delta \) and \( h_\delta \) are similar integrals. One has

\[
|f_\delta(x) - f(x)| \leq |h_\delta(x) - h(x)| + |g_\delta(x)| + |g(x)|,
\]
and

\[(2) \limsup_{\delta \to 0} |f_\delta(x) - f(x)| \leq \limsup_{\delta \to 0} |g_\delta(x)| + \limsup_{\delta \to 0} |g(x)|.\]

For a positive number \(\epsilon\), we will estimate the measure of the set of all points \(x\) for which the right hand side of (2) is bigger than \(\epsilon\). By Chebyshev’s inequality,

\[m(\{x : |g(x)| > \epsilon/2\}) \leq \frac{2\eta}{\epsilon}.\]

We apply the result of problem 3 to the function \(g_{x,\delta}(z) = g(x - \delta z)\):

\[\int_{\mathbb{R}^n} g(x - \delta z) \psi(|z|) dz = v_n \int_{[0, \infty)} r^n A_r g_{x,\delta}(0) d(-\psi(r)).\]

It follows from the result of problem 2 that

\[|A_r g_{x,\delta}(0)| \leq H g_{x,\delta}(0) = H g(x);\]

here \(H\) denotes the Hardy–Littlewood maximal function. Therefore,

\[|g_\delta(x)| \leq H g(x) v_n \int_{[0, \infty)} r^n d(-\psi(r)) = H g(x) \omega_{n-1} \int_0^\infty r^{n-1} \psi(r) dr = H g(x).\]

By the maximal theorem,

\[m(\{x : |g_\delta(x)| > \epsilon/2\}) < \frac{C \eta}{\epsilon} \]

where \(C\) is an absolute constant. Finally,

\[m(\{x : \limsup_{\delta \to 0} |f_\delta(x) - f(x)| > \epsilon\}) < \frac{(C + 2) \eta}{\epsilon}.\]

The last inequality is valid for every \(\eta > 0\). The result follows from that.

From Folland’s book: 22–25, p. 100