PROBLEM SET 11

Problem 1

Let $f(x) \in L^1(\mathbb{R}^n)$ and $g(x) \in L^1(\mathbb{R}^n)$. Prove that $H(f+g)(x) \le Hf(x)+Hg(x)$ for every point $x \in \mathbb{R}^n$. Here Hf is the Hardy–Littlewood maximal function of f.

Problem 2

Let $f(x) \in L^1(\mathbb{R}^n)$, let $a > 0$, and let $f_a(x) = f(ax)$. Prove that

$$Hf_a(0) = Hf(0).$$

Problem 3

Let $\psi(r)$ be a non-negative, non-increasing, right continuous function on $[0,\infty)$ such that $\lim_{r\to\infty}\psi(r) = 0$, and let $f(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$. Assume that either $f(x) \ge 0$ or $f(x)\psi(|x|) \in L^1(\mathbb{R}^n)$. Prove that

$$\int_{\mathbb{R}^n} f(x)\psi(|x|)dx = v_n \int_{[0,\infty)} r^n A_r f(0) d(-\psi(r)). \tag{1}$$

The integral on the right in (1) is the Lebesgue–Stieltjes integral, v_n is the volume of the unit ball in $\mathbb{R}^n$, and $A_r f(0)$ is the average of $f(x)$ over the ball of radius r centered at 0.
Hint. You may find it useful that both sides of (1) remain unchanged if one replaces the function $f(x)$ by a radially-symmetric function $f_{\text{av}}(x)$, which is the average of $f(x)$ over the sphere of radius $|x|$.
Solution. 1. First, let us prove (1) in the case when the function f is radially symmetric, $f = f(|x|)$. I will use spherical co-ordinates; by $d\omega$ I denote the measure on the unit sphere in $\mathbb{R}^n$ that is induced by the Lebesgue measure. Then

$$\int_{\mathbb{R}^n} f(x)\psi(|x|)dx = \omega_{n-1} \int_{[0,\infty)} r^{n-1}\psi(r)f(r)dr \tag{1A}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. Then,

$$A_r f(0) = \frac{1}{v_n r^n} \int_{B(r,0)} f(|x|)dx = \frac{\omega_{n-1}}{v_n r^n} \int_{[0,r]} \rho^{n-1} f(\rho)d\rho,$$

so the right hand side of (1) equals

$$\omega_{n-1} \int_{[0,\infty)} \left(\int_{[0,r]} \rho^{n-1} f(\rho)d\rho \right) d(-\psi(r)). \tag{1B}$$

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To see that the integral (1B) equals the right hand side of (1A), we take the set $D = \{(\rho, r) \in [0,\infty)^2 : \rho \le r\}$ and consider the integral

$$\omega_{n-1} \int_D \rho^{n-1} f(\rho) d\rho \times d(-\psi(r)).$$

If $f \ge 0$ then, by Tonelli's theorem, it equals (1B), and it also equals

$$\omega_{n-1} \int_{[0,\infty)} \rho^{n-1} f(\rho) \Big(\int_{[\rho,\infty)} d(-\psi(r)) \Big) d\rho = \omega_{n-1} \int_{[0,\infty)} \rho^{n-1} \psi(\rho) f(\rho) d\rho.$$

Here, I used the fact that $\lim_{r\to\infty} \psi(r) = 0$. The last integral coincides with the right hand side of (1A). In the case when $f(x)\psi(|x|) \in L^1(\mathbb{R}^n)$, we apply Tonelli's theorem to $|f(x)|$ to see that the function $\rho^{n-1} f(\rho)$ belongs to $L^1(D, d\rho \times d(-\psi(r))$; then we apply Fubini's theorem.

2. Now, let us treat the general case. In spherical co-ordinates, the function f can be written as $f(r,\omega)$; here ω is a point on the unit sphere, S^{n-1}, in $\mathbb{R}^n$. By Fubini–Tonelli's theorem, the function $f_r(\omega) = f(r,\omega)$ is measurable as a function of ω for almost all values of r, and, if $f(x)\psi(|x|) \in L^1(\mathbb{R}^n)$, then $f_r \in L^1(S^{n-1}, d\omega)$ for almoost all values of r. We define a function

$$f_{\mathrm{av}}(r) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(r,\omega) d\omega.$$

This is the average value of the function f over a sphere of radius r. Then $f_{\mathrm{av}}(|x|)$ is a radially symmetric function. Clearly $f_{\mathrm{av}} \ge 0$ if $f \ge 0$. In the case $f(x)\psi(|x|) \in L^1(\mathbb{R}^n)$, we notice that

$$|f_{\mathrm{av}}(r)| \le \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |f(r,\omega)| d\omega,$$

and

$$\int |f_{\mathrm{av}}(|x|)\psi(|x|) dx \le \int_0^\infty \int_{S^{n-1}} |f(r,\omega)|\psi(r) d\omega dr = \int |f(|x|)\psi(|x|) dx.$$

Therefore, the function $f_{\mathrm{av}}(|x|)\psi(|x|)$ belongs to $L^1(\mathbb{R}^n)$.

I claim that both sides of (1) do not change if one replaces $f(x)$ by $f_{\mathrm{av}}(|x|)$. Let us start from the expression on the right in (1). One applies the Fubini–Tonelli theorem to get

$$\begin{aligned} A_r f(0) &= \frac{1}{v_n r^n} \int_0^r \rho^{n-1} d\rho \int_{S^{n-1}} f(\rho,\omega) d\omega \\ &= \frac{\omega_{n-1}}{v_n r^n} \int_0^r f_{\mathrm{av}}(\rho) \rho^{n-1} d\rho = A_r f_{\mathrm{av}}(0). \end{aligned}$$

The same argument leads to

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\psi(|x|) dx &= \int_0^\infty \psi(r) r^{n-1} dr \int_{S^{n-1}} f(r,\omega) d\omega \\ &= \omega_{n-1} \int_0^\infty f_{\mathrm{av}}(r) \psi(r) r^{n-1} dr \\ &= \int_{\mathbb{R}^n} f_{\mathrm{av}}(|x|)\psi(|x|) dx. \end{aligned}$$

We have seen that neither side of (1) changes if a function $f(x)$ is replaced by $f_{\mathrm{av}}(|x|)$, so the general case follows from the case of $f(x)$ being a radially symmetric function.

PROBLEM 4

Let $\psi(r)$ be a function from problem 3. In addition, we assume that

$$\int_{\mathbb{R}^n} \psi(|x|)dx = \omega_{n-1}\int_0^\infty r^{n-1}\psi(r)dr = 1;$$

here ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. Let $f(x) \in L^1(\mathbb{R}^n)$. Prove that

$$\lim_{\delta\to 0} \delta^{-n}\int_{\mathbb{R}^n} f(x-y)\psi(|y|/\delta)dy = \lim_{\delta\to 0}\int_{\mathbb{R}^n} f(x-\delta z)\psi(|z|)dz = f(x)$$

for almost all x.

Hint. Theorem 3.18 is a special case of problem 4; the corresponding function ψ equals the constant $1/v_n$ for $0 \le r < 1$, and it vanishes for $r \ge 1$.

Solution. 1. Let us prove the statement in the case when the function $f(x)$ is continuous and bounded. Fix a point x, fix a number $\epsilon > 0$, and let $|f(y) - f(x)| < \epsilon/2$ when $|y - x| < \eta$. Here η is a positive number, the existence of which is guaranteed by continuity of $f(x)$. We break the integral

$$I = \delta^{-n}\int_{\mathbb{R}^n} f(x-y)\psi(|y|/\delta)dy - f(x) = \delta^{-n}\int_{\mathbb{R}^n}[f(x-y) - f(x)]\psi(|y|/\delta)dy$$

into the sum

$$I_1+I_2 = \delta^{-n}\int_{|y|<\eta}[f(x-y)-f(x)]\psi(|y|/\delta)dy+\delta^{-n}\int_{|y|\ge\eta}[f(x-y)-f(x)]\psi(|y|/\delta)dy.$$

One has

$$|I_1| < \frac{\epsilon}{2}\delta^{-n}\int_{|y|<\eta}\psi(|y|/\delta)dy \le \frac{\epsilon}{2}\delta^{-n}\int_{\mathbb{R}^n}\psi(|y|/\delta)dy = \frac{\epsilon}{2}.$$

To estimate the second integral, I_2, let us assume that $|f(z)| \le M$ (the function $f(x)$ is bounded!) Then

$$|I_2| \le 2M\delta^{-n}\int_{|y|\ge\eta}\psi(|y|/\delta)dy = 2M\int_{|y|\ge\eta/\delta}\psi(|y|)dy.$$

The function $\psi(x)$ belongs to $L^1(\mathbb{R}^n)$, so the last integral converges to 0 when $\delta \to 0$ (the number η is fixed;) therefore it can be made smaller than $\epsilon/4M$ is δ if small enough. Then $|I| < \epsilon$.

2. For a positive number η, we find a bounded, continuous function $h(x)$ such that the L^1 norm of the difference $g(x) = f(x) - h(x)$ is smaller than η. We introduce the notations

$$f_\delta(x) = \delta^{-n}\int_{\mathbb{R}^n} f(x-y)\psi(|y|/\delta)dy = \int_{\mathbb{R}^n} f(x-\delta z)\psi(|z|)dz;$$

g_δ and h_δ are similar integrals. One has

$$|f_\delta(x) - f(x)| \le |h_\delta(x) - h(x)| + |g_\delta(x)| + |g(x)|,$$

and

$$\limsup_{\delta\to 0} |f_\delta(x) - f(x)| \leq \limsup_{\delta\to 0} |g_\delta(x)| + \limsup_{\delta\to 0} |g(x)|. \tag{2}$$

For a positive number ϵ, we will estimate the measure of the set of all points x for which the right hand side of (2) is bigger that ϵ. By Chebyshev's inequality,

$$m(\{x : |g(x)| > \epsilon/2\}) \leq \frac{2\eta}{\epsilon}.$$

We apply the result of problem 3 to the function $g_{x,\delta}(z) = g(x - \delta z)$:

$$\int_{\mathbb{R}^n} g(x-\delta z)\psi(|z|)dz = v_n \int_{[0,\infty)} r^n A_r g_{x,\delta}(0) d(-\psi(r)).$$

It follows from the result of problem 2 that

$$|A_r g_{x,\delta}(0)| \leq H g_{x,\delta}(0) = Hg(x);$$

here H denotes the Hardy–Littlewood maximal function. Therefore,

$$|g_\delta(x)| \leq Hg(x) v_n \int_{[0,\infty)} d^n d(-\psi(r)) = Hg(x)\omega_{n-1} \int_0^\infty r^{n-1}\psi(r)dr = Hg(x).$$

By the maximal theorem,

$$m(\{x : |g_\delta(x)| > \epsilon/2\}) < \frac{C\eta}{\epsilon}$$

where C is an absolute constant. Finally,

$$m(\{x : \limsup_{\delta\to 0} |f_\delta(x) - f(x)| > \epsilon\}) < \frac{(C+2)\eta}{\epsilon}.$$

The last inequality is valid for every $\eta > 0$. The result follows from that.

From Folland's book: 22–25, p. 100