PROBLEM SET 15

Problem 1

Prove that the sequence of functions $f_n(x) = \sin(\pi n x)$ converges to 0 weakly in every space $L^p([0,1])$, $1 \le p < \infty$.

Problem 2

a) Prove that a sequence of functions $g_n(x) \in L^p([0,1])$, $1 < p < \infty$, converges weakly to $g(x) \in L^p([0,1])$ if and only if it is bounded ($\sup ||g_n|| < \infty$) and

$$\int_0^t g_n(x)dx \to \int_0^t g(x)dx \quad \text{for every } t \in [0,1].$$

Show that the statement remains true for $p = \infty$ if one replaces the weak convergence by the weak* convergence.
Hint. Simple functions of the type

$$\sum_{j=1}^{k} a_j \chi_j(x)$$

where χ_j is are characteristic functions of intervals are dense in $L^{p'}$.
b) Give an example showing that the statement is false for $p = 1$.

Problem 3

Let $C^2([-1,1])$ be the Banach space of twice continuously differentiable functions on the interval $[-1,1]$. The norm in this space is given by

$$||f|| = \sup_{-1 \le x \le 1} [|f(x)| + |f'(x)| + |f''(x)|].$$

Let l_n be a sequence of bounded linear functionals given by the formula

$$l_n(f) = n^2 \left(f\left(\frac{1}{n}\right) + f\left(-\frac{1}{n}\right) - 2f(0) \right).$$

Prove that the sequence l_n weak* converges and find its weak* limit.

Problem 4

Prove that the weak convergence in the space l^1 is equivalent to the norm convergence.
Hint. The statement is equivalent to to the following one: there is no sequence $x_n \in l^1$ such that $||x_n|| = 1$ and $x_n \xrightarrow{w} 0$ (show that.) Then, assuming that such a

sequence does exist, try to produce $y \in l^\infty$ such that $y(x_n)$ does not converge to 0 (contradiction.) The construction of y is a bit tricky.

Remark. It is not difficult to see that the weak topology in l^1 is different from the norm topology, so this is an example of two different topologies that induce the same notion of convergence.

From Folland's book: problems 19, 20, 21, 22, page 192.