

PROBLEM SET 2

PROBLEM 1

- a) Prove that every (non-empty) set X admits such a well ordering that $\text{card}(I_x) < \text{card}(X)$, and any two well orderings that satisfy this property are order isomorphic. (Reminder: $I_x = \{y \in X : y < x\}$.)
- b) Prove that the natural ordering in the set of cardinal numbers is a well ordering.

PROBLEM 2

Let us say that a linear ordering on a set X is a perfect ordering if every subset of X has a minimal element and a maximal element. Prove that perfect orderings exist only on finite sets. (Warning: I introduced the term “perfect ordering” just for this problem. You will probably not see this term anywhere else, though I can not guarantee that.)

PROBLEM 3

Let \leq be a linear ordering on the set N of natural numbers. We define a function $f_{\leq}(n) = \#\{k \in N : k \leq n, k \leq n\}$. Here \leq is the usual ordering of N .

- a) Prove that every function $f : N \rightarrow N$ such that $f(n) \leq n$ equals f_{\leq} for some linear ordering \leq .
- b) Describe linear orderings that correspond to the functions

$$f(n) = \begin{cases} n, & \text{for } n \text{ odd} \\ \frac{n}{2}, & \text{for } n \text{ even} \end{cases},$$
$$f(n) = \begin{cases} 1, & \text{for } n = 1 \\ n - 1, & \text{for } n > 1 \end{cases},$$

and $f(n) = \lfloor (n+1)/2 \rfloor$ where $\lfloor x \rfloor$ is the largest integer number that does not exceed x . Which ones of them (if any) are well orderings?

- c) Prove that if \leq and \leq' are two linear orderings on N and $f_{\leq}(n) = f_{\leq'}(n)$ for n large enough then these orderings are order isomorphic.

PROBLEM 4

a) Let (X, \leq) be an ordered set, and let $\tilde{X} = X_1 \sqcup X_2$ be another set ($\tilde{X} = X_1 \sqcup X_2$ means that $\tilde{X} = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$.) Suppose that both X_1 and X_2 have the same cardinality as X , and let $f_j : X_j \rightarrow X$, $j = 1, 2$, be bijections. Let a mapping $f : \tilde{X} \rightarrow X$ be given by

$$f(x) = \begin{cases} f_1(x), & \text{for } x \in X_1 \\ f_2(x), & \text{for } x \in X_2. \end{cases}$$

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We define an ordering $\tilde{\leq}$ on \tilde{X} in the following way: $x \tilde{\leq} y$ if either $f(x) < f(y)$ or $f(x) = f(y)$ and $x \in X_1$. Prove that $\tilde{\leq}$ is a well ordering if \leq is a well ordering.

b) Prove that if X is an infinite set then \tilde{X} has the same cardinality as X .

Hint. Use part a) of this problem, part a) of problem 1 and transfinite induction.

Solution to the problem 4b.

let \mathcal{X} be the set of infinite cardinal numbers with the natural ordering. This ordering is a well ordering (problem 1b.) Let $\alpha \in \mathcal{X}$, and suppose that for every $\beta \in \mathcal{X}$, $\beta < \alpha$, one has $\text{card}(\tilde{X}) = \beta$ whenever $\text{card}(X) = \beta$. Take a set X such that $\text{card}(X) = \alpha$. On X , we introduce a well ordering \leq that satisfies the properties of the problem 1a: $\text{card}(I_x) < \alpha$ for all $x \in X$. Let $\tilde{\leq}$ be the ordering on \tilde{X} from problem 4a; it is a well ordering. Take a point $x \in \tilde{X}$. Then either $I_x = f^{-1}(I_{f(x)})$ when $x \in X_1$ or $I_x = f^{-1}(I_{f(x)}) \cup \{y\}$ when $x \in X_2$; here $y \in X_1$ is such a point that $f(x) = f(y)$. Notice that

$$f^{-1}(I_{f(x)}) = f_1^{-1}(I_{f(x)}) \cup f_2^{-1}(I_{f(x)})$$

and $\text{card}(I_{f(x)}) < \alpha$. If the set $I_{f(x)}$ is finite then the set I_x is also finite, and

$$(1) \quad \text{card}(I_x) < \alpha.$$

Otherwise,

$$\text{card}(f_1^{-1}(I_{f(x)}) \cup f_2^{-1}(I_{f(x)})) = \text{card}(I_{f(x)}) < \alpha$$

by the induction assumption. This implies (1) in the case $x \in X_1$. In the case $x \in X_2$, one uses the following

Fact. Let Z be an infinite set, and let $\{y\}$ be a one-point set. Then $\text{card}(Z \sqcup \{y\}) = \text{card}(Z)$.

To summarize, (1) holds for every $x \in \tilde{X}$. Let us show that (1) implies $\text{card}(\tilde{X}) \leq \alpha$. Indeed, if $\text{card}(\tilde{X}) > \alpha = \text{card}(X)$ then X is order isomorphic to I_x for some $x \in \tilde{X}$, and, together with (1), that leads to contradiction. On the other hand, obviously, $\text{card}(\tilde{X}) \geq \text{card}(X)$. This completes the proof.

Proof of the Fact. Choose an infinite sequence $\{y_n\}$, $1 \leq n < \infty$, of mutually different points from Z . Let $y_0 = y$. Then a bijection $h : Z \sqcup \{y\} \rightarrow Z$ can be defined by the formula

$$h(z) = \begin{cases} y_{n+1}, & \text{when } z = y_n, n \geq 0 \\ z, & \text{otherwise.} \end{cases}$$