PROBLEM SET 5

PROBLEM 1

Recall that the Cantor set $C$ is the set of all numbers from $[0, 1]$ having a ternary expansion that is free from the digit 1. We define the Cantor function on $[0, 1]$ in the following way. Let

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad a_j = 0, 1, 2,$$

be a ternary expansion of a number $x$. Then

$$C(x) = \begin{cases} \sum_{j=1}^{\infty} \frac{a_j}{3^j}, & \text{if all } a_j \text{'s are different from 1} \\ \sum_{j=1}^{n-1} \frac{a_j}{3^j} + \frac{a_n}{3^n}, & \text{if } a_n = 1 \text{ and } a_j \neq 1 \text{ for } j < n. \end{cases}$$

The function $C(x)$ is extended to the whole real line $\mathbb{R}$ by setting $C(x + n) = C(x) + n$, $n \in \mathbb{Z}$.

a) Prove that the Cantor function is correctly defined (this means that, if a number $x$ has two different representations (1), then the value of $C(x)$ does not depend on the choice of a representation.)

b) Prove that the function $C(x)$ is continuous, and it is non-decreasing.

c) Let $\mu_C$ be the Lebesgue–Stieltjes measure associated with the Cantor function. Show that $\mu_C([0, 1] \setminus C) = 0$.

d) Let $m = \mu_2$ be the Lebesgue measure. Prove that every Borel set $A \subset \mathbb{R}$ can be partitioned $A = A_1 \cup A_2$ in such a way that $m(A_1) = 0$ and $\mu_C(A_2) = 0$.

e) Let $F(x) = x + C(x)$. Prove that $F$ is a homeomorphism $\mathbb{R} \rightarrow \mathbb{R}$ (this means that $F$ is a bijection and both $F$ and $F^{-1}$ are continuous.)

f) Prove that $m(F(C)) = 1.$

PROBLEM 2

Let $X = C_0[0, \infty)$ be the space of continuous functions $x(t)$ on $[0, \infty)$ such that $x(0) = 0$. Let $0 < t_1 < t_2 < \cdots < t_k$ be a finite sequence, and let $I_1, \ldots, I_k$ be $h$-intervals ($I_j = (a_j, b_j]$ or $I_j = (a_j, \infty]$). I define an $h$-cylindrical set

$$C_{\vec{t}, \vec{I}} = \{x(t) : x(t_j) \in I_j, \quad j = 1, \ldots, k\}.$$

Here $\vec{t} = (t_1, \ldots, t_k)$ and $\vec{I} = (I_1, \ldots, I_k)$. Let $\mathcal{A}$ be the set of all finite unions of $h$-cylindrical sets.

a) Prove that $\mathcal{A}$ is an algebra.

Fix a number $\sigma > 0$. Define

$$\mu_0(C_{\vec{t}, \vec{I}}) = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi \sigma(t_j - t_{j-1})}} \int_{t_{j-1}}^{t_j} \cdots \int_{t_{1}} \exp \left\{- \frac{\sum_{j=1}^{k} (x_j - x_{j-1})^2}{2\sigma(t_j - t_{j-1})} \right\} dx_k \cdots dx_1.$$
In the last formula, it is assumed that \( t_0 = 0 \) and \( x_0 = 0 \). If \( A = A_1 \sqcup \cdots \sqcup A_n \), and \( A_j \) are \( h \)-cylindrical sets then \( \mu_0(A) = \mu_0(A_1) + \cdots + \mu_0(A_n) \).

b) Prove that \( \mu_0 \) is correctly defined. This means that \( \mu_0(A) \) does not depend on the way of how \( A \) is represented as a disjoint union of \( h \)-cylindrical sets.

c) Prove that \( \mu_0 \) is a premeasure.

The measure that is induced by \( \mu_0 \) is called the Wiener measure; sometimes it is denoted by \( w_\sigma \).

d) Let \( (1/2) < \alpha \leq 1 \), and let \( S_\alpha \subset X \) be the set of all functions that are Hölder at \( t = 0 \) with the parameter \( \alpha \). It means that there exists a constant \( C \) such that \( |f(t)| \leq Ct^\alpha \) for \( 0 < t \leq 1 \) (notice that \( f(0) = 0 \), so \( f(t) - f(0) = f(t) \)). Prove that \( w_\sigma(S_\alpha) = 0 \).

From Folland’s book: 30, 31 (pp. 39–40).