

PROBLEM SET 5

PROBLEM 1

Let $f(x)$ be a bounded function on an interval $[a, b]$, $-\infty < a < b < \infty$. Let $S = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a subdivision of the interval $[a, b]$, and let

$$R_S(f) = \sum_{j=1}^n f(x_j^*)(x_j - x_{j-1})$$

be a Riemann sum: here $x_j^* \in [x_{j-1}, x_j]$. Recall that a function $f(x)$ is called Riemann integrable on $[a, b]$ if the limit $\lim_{m(S) \rightarrow 0} R_S(f)$ exists; here $m(S) = \max\{x_j - x_{j-1}\}$.

Let $\delta > 0$ be a positive number. For every point $x \in [a, b]$ we introduce a number

$$\text{osc}(f; x, \delta) = \sup_{y \in [a, b] \cap [x-\delta, x+\delta]} f(y) - \inf_{y \in [a, b] \cap [x-\delta, x+\delta]} f(y).$$

a) Prove that the limit

$$\text{osc}(f; x) = \lim_{\delta \rightarrow 0} \text{osc}(f; x, \delta)$$

exists.

b) Prove that a function f is continuous at a point x if and only if $\text{osc}(f; x) = 0$.

Let μ^* be an outer measure on \mathbb{R} defined by the formula

$$\mu^*(A) = \inf_{A \subset \cup_{j=1}^{\infty} (\alpha_j, \beta_j)} \sum_{j=1}^{\infty} (\beta_j - \alpha_j).$$

c) Define $D_\epsilon = \{x : \text{osc}(f; x) \geq \epsilon\}$; here ϵ is a positive number. Prove that if the function $f(x)$ is Riemann integrable then $\mu^*(D_\epsilon) = 0$.

d) Let D be the set of all points where the function $f(x)$ is discontinuous. Prove that if the function $f(x)$ is Riemann integrable then $\mu^*(D) = 0$.

e) Prove that the sets $D_\epsilon(f)$ are closed.

f) Suppose that $\mu^*(D) = 0$. Prove that for every $\epsilon > 0$ and for every $\eta > 0$ there exist a *finite* set of mutually disjoint intervals $[a_j, b_j] \subset [a, b]$, $j = 1, \dots, N$ such that

$$[a, b] \setminus D_\epsilon \subset \cup_{j=1}^N [a_j, b_j] \quad \text{and} \quad \sum_{j=1}^N (b_j - a_j) \geq b - a - \eta.$$

From Folland's book: 17, 18, 19, 20, 24 (pp. 32–33)