PROBLEM SET 6

Problem 1

Recall that the Cantor set C is the set of all numbers from $[0,1]$ having a ternary expansion that is free from the digit 1. We define the Cantor function on $[0,1]$ in the following way. Let

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j},\ a_j = 0,1,2, \tag{1}$$

be a ternary expansion of a number x. Then

$$C(x) = \begin{cases} \sum_{j=1}^{\infty} \frac{a_j}{2^{j+1}}, & \text{if all } a_j\text{'s are different from } 1 \\ \sum_{j=1}^{n-1} \frac{a_j}{2^{j+1}} + \frac{a_n}{2^n}, & \text{if } a_n = 1 \text{ and } a_j \neq 1 \text{ for } j < n. \end{cases}$$

The function $C(x)$ is extended to the whole real line $\mathbb{R}$ by setting $C(x+n) = C(x) + n$, $n \in \mathbb{Z}$.

a) Prove that the Cantor function is correctly defined (this means that, if a number x has two different representations (1), then the value of $C(x)$ does not depend on the choice of a representation.)

b) Prove that the function $C(x)$ is continuous, and it is non-decreasing.

c) Let μ_C be the Lebesgue–Stieltjes measure associated with the Cantor function. Show that $\mu_C([0,1] \setminus C) = 0$.

d) Let $m = \mu_x$ be the Lebesgue measure. Prove that every Borel set $A \subset \mathbb{R}$ can be partitioned $A = A_1 \sqcup A_2$ in such a way that $m(A_1) = 0$ and $\mu_C(A_2) = 0$.

e) Let $F(x) = x + C(x)$. Prove that F is a homeomorphism $\mathbb{R} \to \mathbb{R}$ (this means that F is a bijection and both F and F^{-1} are continuous.)

f) Prove that $m(F(C)) = 1$.

From Folland's book: 29, 30, 31 (pp. 39–40), 2,3,8 (p. 48)